
Signal to Noise

**ECE 6279: Spatial Array Processing
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Lecture 13**

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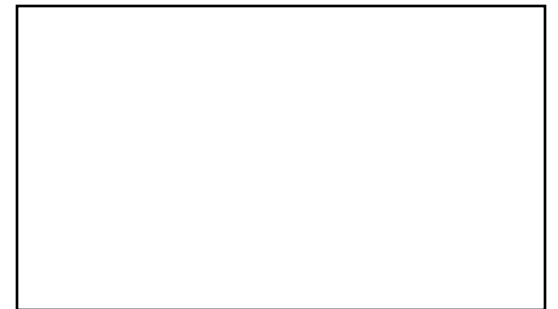
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Where We Are (and Aren't) in J&D

- **Inspired by Section 4.5**



Single Plane Wave, Stochastic Model

To keep notation compact, suppress notation of time variable:

$$\text{Data: } \underline{\mathbf{y}} = \mathbf{e}(\vec{k}^0) \underline{\mathbf{s}} + \underline{\mathbf{n}}$$

Beamformer output: $z = \mathbf{e}^H(\vec{k}) \mathbf{W} \underline{\mathbf{y}}$

$$= \mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{e}(\vec{k}^0) \underline{\mathbf{s}} + \mathbf{e}^H(\vec{k}) \mathbf{W} \underline{\mathbf{n}}$$



Signal to Noise Ratio (1)

$$SNR = \frac{E \left[\left| \mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{e}(\vec{k}^0) \underline{s} \right|^2 \right]}{E \left[\left| \mathbf{e}^H(\vec{k}) \mathbf{W} \underline{\mathbf{n}} \right|^2 \right]}$$

$$= \frac{E \left[\mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{e}(\vec{k}^0) \underline{s} \underline{s}^H \mathbf{e}^H(\vec{k}^0) \mathbf{W}^H \mathbf{e}(\vec{k}) \right]}{E \left[\mathbf{e}^H(\vec{k}) \mathbf{W} \underline{\mathbf{n}} \underline{\mathbf{n}}^H \mathbf{W}^H \mathbf{e}(\vec{k}) \right]}$$



Signal to Noise Ratio (2)

$$\begin{aligned} &= \frac{\mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{e}(\vec{k}^0) E[\underline{s} \underline{s}^H] \mathbf{e}^H(\vec{k}^0) \mathbf{W}^H \mathbf{e}(\vec{k})}{\mathbf{e}^H(\vec{k}) \mathbf{W} E[\underline{n} \underline{n}^H] \mathbf{W}^H \mathbf{e}(\vec{k})} \\ &= \frac{P_s \mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{e}(\vec{k}^0) \mathbf{e}^H(\vec{k}^0) \mathbf{W}^H \mathbf{e}(\vec{k})}{\mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{K}_n \mathbf{W}^H \mathbf{e}(\vec{k})} \end{aligned}$$



Special Case of the Numerator (1)

Suppose $\vec{k} = \vec{k}^0$

$$\mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{e}(\vec{k}^0) = \mathbf{e}^H(\vec{k}^0) \mathbf{W} \mathbf{e}(\vec{k}^0) =$$

$$\begin{bmatrix} e^{j\vec{k}^0 \cdot \vec{x}_0} & \dots & e^{j\vec{k}^0 \cdot \vec{x}_{M-1}} \end{bmatrix} \begin{bmatrix} w_0 & & \\ & \ddots & \\ & & w_{M-1} \end{bmatrix} \begin{bmatrix} e^{-j\vec{k}^0 \cdot \vec{x}_0} \\ \vdots \\ e^{-j\vec{k}^0 \cdot \vec{x}_0} \end{bmatrix}$$
$$= \sum_{m=0}^{M-1} w_m$$



Special Case of the Numerator (2)

Suppose $\vec{k} = \vec{k}^0$

$$\text{numer} = P_s \mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{e}(\vec{k}^0) \mathbf{e}^H(\vec{k}^0) \mathbf{W}^H \mathbf{e}(\vec{k})$$

$$= P_s \mathbf{e}^H(\vec{k}^0) \mathbf{W} \mathbf{e}(\vec{k}^0) \mathbf{e}^H(\vec{k}^0) \mathbf{W}^H \mathbf{e}(\vec{k}^0)$$

$$= P_s \left(\sum_{m=0}^{M-1} w_m \right) \left(\sum_{m=0}^{M-1} w_m^* \right) = P_s \left| \sum_{m=0}^{M-1} w_m \right|^2$$



Special Case of the Denominator

Special case $\mathbf{K}_n = \sigma_n^2 \mathbf{I}$

$$\textit{denom} = \mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{K}_n \mathbf{W}^H \mathbf{e}(\vec{k})$$

$$= \sigma_n^2 \sum_{m=0}^{M-1} |w_m|^2$$



Both Special Cases

Suppose $\vec{k} = \vec{k}^0$ and $\mathbf{K}_n = \sigma_n^2 \mathbf{I}$

$$SNR = \frac{P_s \left| \sum_{m=0}^{M-1} w_m \right|^2}{\sigma_n^2 \left(\sum_{m=1}^{M-1} |w_m|^2 \right)}$$

$$\text{If } w_m = w_0 : SNR = \frac{P_s M^2 w_0^2}{\sigma_n^2 M w_0^2} = \frac{P_s M}{\sigma_n^2}$$

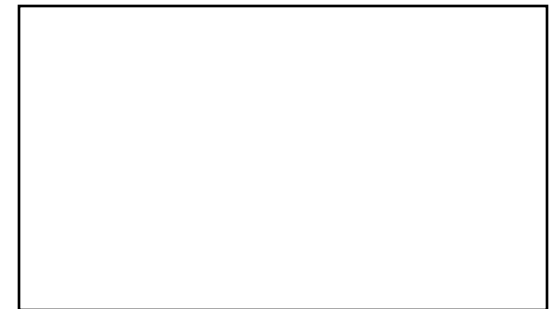


Two Signal Models

Define the **array gain**:

$$G = \frac{SNR_{array}}{SNR_{sensor}} = \frac{P_s M / \sigma_n^2}{P_s / \sigma_n^2} = M$$

**in this
special
case**



Can We Do Better?

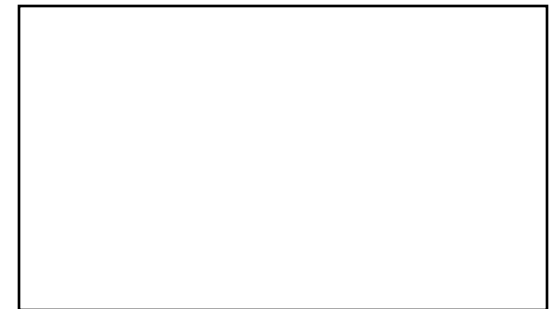
- **Schwartz Inequality:**

$$\left| \sum_{m=0}^{M-1} a_m b_m^* \right|^2 \leq \sum_{m=0}^{M-1} |a_m|^2 \sum_{m=0}^{M-1} |b_m|^2$$

with equality iff $a_m = kb_m$

- **Let** $a_m = w_m$, $b_m = 1$

$$\left| \sum_{m=0}^{M-1} w_m \right|^2 \leq M \sum_{m=0}^{M-1} |w_m|^2$$



No, That's the Best We Can Do!

$$\frac{\left| \sum_{m=0}^{M-1} w_m \right|^2}{\sum_{m=0}^{M-1} |w_m|^2} \leq M$$

**So in uniform noise,
uniform weights
maximizes the SNR
when beamforming on
true wavenumber
vector**



A Fresh Approach

To keep notation compact, suppress notation of time variable:

$$\text{Data: } \underline{\mathbf{y}} = \mathbf{e}(\vec{k}^0) \underline{s} + \underline{\mathbf{n}}$$

Beamformer output: $z = \mathbf{a}^H \underline{\mathbf{y}}$

$$= \mathbf{a}^H \mathbf{e}(\vec{k}^0) \underline{s} + \mathbf{a}^H \underline{\mathbf{n}}$$

What choice of \mathbf{a} maximizes SNR?



SNR for Colored Noise

$$SNR = \frac{P_s \left| \mathbf{a}^H \mathbf{e}(\vec{k}^0) \right|^2}{\mathbf{a}^H \mathbf{K}_n \mathbf{a}}$$

- **Want to study case of a general \mathbf{K}_n**
- **Substitute $\tilde{\mathbf{a}} = \mathbf{K}_n^{1/2} \mathbf{a}$, $\mathbf{a} = \mathbf{K}_n^{-1/2} \tilde{\mathbf{a}}$**

$$SNR = \frac{P_s \left| \tilde{\mathbf{a}}^H \mathbf{K}_n^{-1/2} \mathbf{e}(\vec{k}^0) \right|^2}{\tilde{\mathbf{a}}^H \tilde{\mathbf{a}}}$$



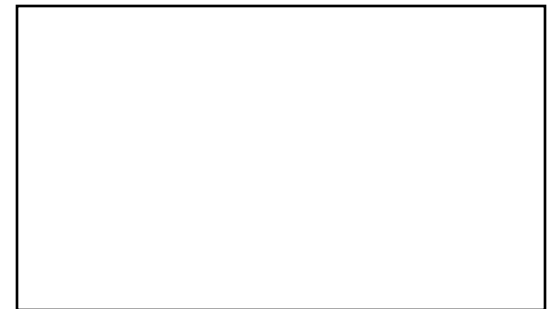
Rewriting Schwartz

- **Schwartz Inequality:**

$$|\mathbf{f}^H \mathbf{g}|^2 \leq (\mathbf{f}^H \mathbf{f})(\mathbf{g}^H \mathbf{g}) \equiv \|\mathbf{f}\|^2 \|\mathbf{g}\|^2$$

with equality iff $\mathbf{f} = k\mathbf{g}$

$$\frac{|\mathbf{f}^H \mathbf{g}|^2}{\|\mathbf{f}\|^2} \leq \|\mathbf{g}\|^2$$



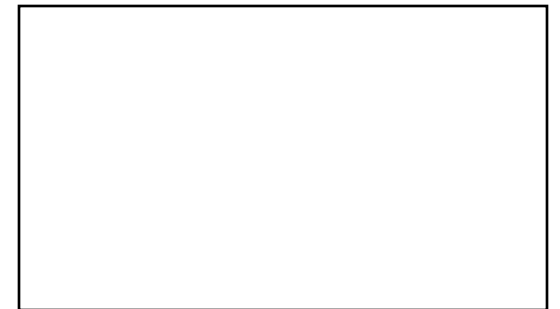
Use the Schwartz!

- Let $\mathbf{f} = \tilde{\mathbf{a}}$, $\mathbf{g} = \mathbf{K}_n^{-1/2} \mathbf{e}(\vec{k}^0)$

$$\frac{\left| \tilde{\mathbf{a}}^H \mathbf{K}_n^{-1/2} \mathbf{e}(\vec{k}^0) \right|^2}{\|\tilde{\mathbf{a}}\|^2} \leq \left\| \mathbf{K}_n^{-1/2} \mathbf{e}(\vec{k}^0) \right\|^2$$

with equality iff

$$\tilde{\mathbf{a}} = \kappa \mathbf{K}_n^{-1/2} \mathbf{e}(\vec{k}^0)$$



Maximizing SNR in Colored Noise

- **Schwartz inequality tells us that to maximize SNR, we can pick**

$$\tilde{\mathbf{a}} = \mathbf{K}_n^{-1/2} \mathbf{e}(\vec{k}^0)$$

$$\mathbf{a} = \mathbf{K}_n^{-1/2} \tilde{\mathbf{a}} = \mathbf{K}_n^{-1/2} \mathbf{K}_n^{-1/2} \mathbf{e}(\vec{k}^0)$$

$$= \mathbf{K}_n^{-1} \mathbf{e}(\vec{k}^0)$$

$$z = \mathbf{a}^H \underline{\mathbf{y}} = \mathbf{e}^H(\vec{k}^0) \mathbf{K}_n^{-1} \underline{\mathbf{y}}$$



Direction Finding in Colored Noise

- **Realistically, sweep \vec{k} as usual:**

$$z = \mathbf{e}^H(\vec{k}) \mathbf{K}_n^{-1} \underline{\mathbf{y}}$$

