

Signal to Noise

ECE 6279: Spatial Array Processing
Spring 2011
Lecture 13

Prof. Aaron D. Lanterman

School of Electrical & Computer Engineering
Georgia Institute of Technology
AL: 404-385-2548
<lanterma@ece.gatech.edu>

Copyright 2011 Aaron Lanterman



Where We Are (and Aren't) in J&D

- Inspired by Section 4.5



Copyright 2011 Aaron Lanterman



Single Plane Wave, Stochastic Model

To keep notation compact, suppress notation of time variable:

$$\text{Data: } \underline{\mathbf{y}} = \mathbf{e}(\vec{k}^0) \underline{s} + \underline{\mathbf{n}}$$

$$\begin{aligned} \text{Beamformer output: } z &= \mathbf{e}^H(\vec{k}) \mathbf{W} \underline{\mathbf{y}} \\ &= \mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{e}(\vec{k}^0) \underline{s} + \mathbf{e}^H(\vec{k}) \mathbf{W} \underline{\mathbf{n}} \end{aligned}$$

Copyright 2011 Aaron Lanterman



Signal to Noise Ratio (1)

$$\begin{aligned} SNR &= \frac{E\left[\left| \mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{e}(\vec{k}^0) \underline{s} \right|^2 \right]}{E\left[\left| \mathbf{e}^H(\vec{k}) \mathbf{W} \underline{\mathbf{n}} \right|^2 \right]} \\ &= \frac{E\left[\mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{e}(\vec{k}^0) \underline{s} \underline{s}^H \mathbf{e}^H(\vec{k}^0) \mathbf{W}^H \mathbf{e}(\vec{k}) \right]}{E\left[\mathbf{e}^H(\vec{k}) \mathbf{W} \underline{\mathbf{n}} \underline{\mathbf{n}}^H \mathbf{W}^H \mathbf{e}(\vec{k}) \right]} \end{aligned}$$

Copyright 2011 Aaron Lanterman



Signal to Noise Ratio (2)

$$= \frac{\mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{e}(\vec{k}^0) E[\underline{s} \underline{s}^H] \mathbf{e}^H(\vec{k}^0) \mathbf{W}^H \mathbf{e}(\vec{k})}{\mathbf{e}^H(\vec{k}) \mathbf{W} E[\underline{n} \underline{n}^H] \mathbf{W}^H \mathbf{e}(\vec{k})}$$

$$= \frac{P_s \mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{e}(\vec{k}^0) \mathbf{e}^H(\vec{k}^0) \mathbf{W}^H \mathbf{e}(\vec{k})}{\mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{K}_n \mathbf{W}^H \mathbf{e}(\vec{k})}$$



Copyright 2011 Aaron Lanterman

Special Case of the Numerator (1)

Suppose $\vec{k} = \vec{k}^0$

$$\mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{e}(\vec{k}^0) = \mathbf{e}^H(\vec{k}^0) \mathbf{W} \mathbf{e}(\vec{k}^0) =$$

$$\begin{bmatrix} e^{j\vec{k}^0 \cdot \vec{x}_0} & \dots & e^{j\vec{k}^0 \cdot \vec{x}_{M-1}} \\ & & \vdots \\ & & w_{M-1} \end{bmatrix} \begin{bmatrix} e^{-j\vec{k}^0 \cdot \vec{x}_0} \\ \vdots \\ e^{-j\vec{k}^0 \cdot \vec{x}_0} \end{bmatrix}$$

$$= \sum_{m=0}^{M-1} w_m$$



Copyright 2011 Aaron Lanterman

Special Case of the Numerator (2)

Suppose $\vec{k} = \vec{k}^0$

$$\text{numer} = P_s \mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{e}(\vec{k}^0) \mathbf{e}^H(\vec{k}^0) \mathbf{W}^H \mathbf{e}(\vec{k})$$

$$= P_s \mathbf{e}^H(\vec{k}^0) \mathbf{W} \mathbf{e}(\vec{k}^0) \mathbf{e}^H(\vec{k}^0) \mathbf{W}^H \mathbf{e}(\vec{k}^0)$$

$$= P_s \left(\sum_{m=0}^{M-1} w_m \right) \left(\sum_{m=0}^{M-1} w_m^* \right) = P_s \left| \sum_{m=0}^{M-1} w_m \right|^2$$



Copyright 2011 Aaron Lanterman

Special Case of the Denominator

Special case $\mathbf{K}_n = \sigma_n^2 \mathbf{I}$

$$\text{denom} = \mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{K}_n \mathbf{W}^H \mathbf{e}(\vec{k})$$

$$= \sigma_n^2 \sum_{m=0}^{M-1} |w_m|^2$$



Copyright 2011 Aaron Lanterman

Both Special Cases

Suppose $\vec{k} = \vec{k}^0$ and $\mathbf{K}_n = \sigma_n^2 \mathbf{I}$

$$SNR = \frac{P_s \left| \sum_{m=0}^{M-1} w_m \right|^2}{\sigma_n^2 \left(\sum_{m=0}^{M-1} |w_m|^2 \right)}$$

If $w_m = w_0$: $SNR = \frac{P_s M^2 w_0^2}{\sigma_n^2 M w_0^2} = \frac{P_s M}{\sigma_n^2}$

Copyright 2011 Aaron Lanterman



Two Signal Models

Define the **array gain**:

$$G = \frac{SNR_{array}}{SNR_{sensor}} = \frac{P_s M / \sigma_n^2}{P_s / \sigma_n^2} = M$$

in this
special
case



Copyright 2011 Aaron Lanterman



Can We Do Better?

- **Schwartz Inequality:**

$$\left| \sum_{m=0}^{M-1} a_m b_m^* \right|^2 \leq \sum_{m=0}^{M-1} |a_m|^2 \sum_{m=0}^{M-1} |b_m|^2$$

with equality iff $a_m = kb_m$

- Let $a_m = w_m$, $b_m = 1$

$$\left| \sum_{m=0}^{M-1} w_m \right|^2 \leq M \sum_{m=0}^{M-1} |w_m|^2$$

Copyright 2011 Aaron Lanterman



No, That's the Best We Can Do!

$$\frac{\left| \sum_{m=0}^{M-1} w_m \right|^2}{\sum_{m=0}^{M-1} |w_m|^2} \leq M$$

So in uniform noise,
uniform weights
maximizes the SNR
when beamforming on
true wavenumber
vector



Copyright 2011 Aaron Lanterman



A Fresh Approach

To keep notation compact, suppress notation of time variable:

$$\text{Data: } \underline{\mathbf{y}} = \mathbf{e}(\vec{k}^0) \underline{s} + \underline{\mathbf{n}}$$

$$\begin{aligned} \text{Beamformer output: } z &= \mathbf{a}^H \underline{\mathbf{y}} \\ &= \mathbf{a}^H \mathbf{e}(\vec{k}^0) \underline{s} + \mathbf{a}^H \underline{\mathbf{n}} \end{aligned}$$

What choice of \mathbf{a} maximizes SNR?

Copyright 2011 Aaron Lanterman



SNR for Colored Noise

$$SNR = \frac{P_s |\mathbf{a}^H \mathbf{e}(\vec{k}^0)|^2}{\mathbf{a}^H \mathbf{K}_n \mathbf{a}}$$

- Want to study case of a general \mathbf{K}_n
- Substitute $\tilde{\mathbf{a}} = \mathbf{K}_n^{1/2} \mathbf{a}$, $\mathbf{a} = \mathbf{K}_n^{-1/2} \tilde{\mathbf{a}}$

$$SNR = \frac{P_s |\tilde{\mathbf{a}}^H \mathbf{K}_n^{-1/2} \mathbf{e}(\vec{k}^0)|^2}{\tilde{\mathbf{a}}^H \tilde{\mathbf{a}}}$$

Copyright 2011 Aaron Lanterman



Rewriting Schwartz

- Schwartz Inequality:

$$|\mathbf{f}^H \mathbf{g}|^2 \leq (\mathbf{f}^H \mathbf{f})(\mathbf{g}^H \mathbf{g}) \equiv \|\mathbf{f}\|^2 \|\mathbf{g}\|^2$$

with equality iff $\mathbf{f} = \kappa \mathbf{g}$

$$\frac{|\mathbf{f}^H \mathbf{g}|^2}{\|\mathbf{f}\|^2} \leq \|\mathbf{g}\|^2$$



Copyright 2011 Aaron Lanterman



Use the Schwartz!

- Let $\mathbf{f} = \tilde{\mathbf{a}}$, $\mathbf{g} = \mathbf{K}_n^{-1/2} \mathbf{e}(\vec{k}^0)$

$$\frac{|\tilde{\mathbf{a}}^H \mathbf{K}_n^{-1/2} \mathbf{e}(\vec{k}^0)|^2}{\|\tilde{\mathbf{a}}\|^2} \leq \|\mathbf{K}_n^{-1/2} \mathbf{e}(\vec{k}^0)\|^2$$

with equality iff

$$\tilde{\mathbf{a}} = \kappa \mathbf{K}_n^{-1/2} \mathbf{e}(\vec{k}^0)$$



Copyright 2011 Aaron Lanterman



Maximizing SNR in Colored Noise

- Schwartz inequality tells us that to maximize SNR, we can pick

$$\tilde{\mathbf{a}} = \mathbf{K}_n^{-1/2} \mathbf{e}(\vec{k}^0)$$

$$\mathbf{a} = \mathbf{K}_n^{-1/2} \tilde{\mathbf{a}} = \mathbf{K}_n^{-1/2} \mathbf{K}_n^{-1/2} \mathbf{e}(\vec{k}^0)$$

$$= \mathbf{K}_n^{-1} \mathbf{e}(\vec{k}^0)$$

$$z = \mathbf{a}^H \underline{\mathbf{y}} = \mathbf{e}^H(\vec{k}^0) \mathbf{K}_n^{-1} \underline{\mathbf{y}}$$

Copyright 2011 Aaron Lanterman



Direction Finding in Colored Noise

- Realistically, sweep \vec{k} as usual:

$$z = \mathbf{e}^H(\vec{k}) \mathbf{K}_n^{-1} \underline{\mathbf{y}}$$

Copyright 2011 Aaron Lanterman

