
Time Averaging

**ECE 6279: Spatial Array Processing
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Lecture 14**

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Where We Are (and Aren't) in J&D

- **Inspired by Section 4.9.1**



Average Beamformer Output

- **Average power of beamformer output over time**

$$\begin{aligned}\hat{z} &= \frac{1}{L} \sum_{l=0}^{L-1} |z(l)|^2 = \frac{1}{L} \sum_{l=0}^{L-1} \left| \mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{y}(l) \right|^2 \\ &= \frac{1}{L} \sum_{l=0}^{L-1} \mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{y}(l) \mathbf{y}^H(l) \mathbf{W}^H \mathbf{e}(\vec{k}) \\ &= \mathbf{e}^H(\vec{k}) \mathbf{W} \left[\frac{1}{L} \sum_{l=0}^{L-1} \mathbf{y}(l) \mathbf{y}^H(l) \right] \mathbf{W}^H \mathbf{e}(\vec{k})\end{aligned}$$

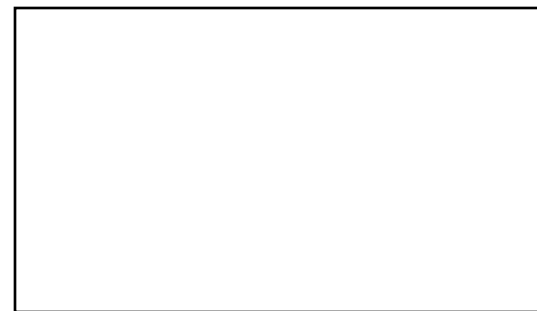


Under the Stochastic Model

$$\hat{\mathbf{z}} = \mathbf{e}^H(\vec{k}) \mathbf{W} \hat{\mathbf{R}}_y \mathbf{W}^H \mathbf{e}(\vec{k})$$

where
$$\hat{\mathbf{R}}_y = \frac{1}{L} \sum_{l=0}^{L-1} \mathbf{y}(l) \mathbf{y}^H(l)$$

- Under the stochastic signal model, $\hat{\mathbf{R}}_y$ is the unconstrained maximum-likelihood estimate of the correlation matrix of a Gaussian random vector $\underline{\mathbf{y}}$



Improvement from Time Averaging

$$\hat{\mathbf{R}}_y \rightarrow \mathbf{R}_y = E[\underline{\mathbf{y}}\underline{\mathbf{y}}^T] \text{ as } L \rightarrow \infty$$

- **If snapshots are independent, variance of estimate improves as $O(1/L)$, standard deviation improves as $O(1/\text{sqrt}(L))$**
- **Can't make L too big if the sources are moving!**



Outer Product of Steering Vectors

- **For linear array**

$$\gamma = k \sin(\phi) \cos(\theta) d$$

$$= \frac{2\pi}{\lambda} \sin(\phi) \cos(\theta) d$$

$$= \begin{bmatrix} 1 \\ e^{j\gamma} \\ \vdots \\ e^{j(M-1)\gamma} \end{bmatrix} \begin{bmatrix} 1 & e^{-j\gamma} & \dots & e^{-j(M-1)\gamma} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & e^{-j\gamma} & \dots & e^{-j(M-1)\gamma} \\ e^{j\gamma} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & e^{-j\gamma} \\ e^{j(M-1)\gamma} & \dots & e^{j\gamma} & 1 \end{bmatrix}$$



Linear Array, Single Plane Wave

- Assuming no noise

$$\mathbf{y}(l) = s(l) \exp\left(-j \frac{M-1}{2} \gamma\right) \exp \begin{bmatrix} 0 \\ j\gamma \\ \vdots \\ j(M-1)\gamma \end{bmatrix}$$

$$\mathbf{y}(l) \mathbf{y}^H(l) = |s(l)|^2 \exp \begin{bmatrix} 0 & -j\gamma & \cdots & -j(M-1)\gamma \\ j\gamma & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -j\gamma \\ j(M-1)\gamma & \cdots & j\gamma & 0 \end{bmatrix}$$



Toeplitz Structure of Est. Correlation

$$\hat{\mathbf{R}}_y = \frac{1}{L} \sum_{l=0}^{L-1} \mathbf{y}(l) \mathbf{y}^H(l)$$

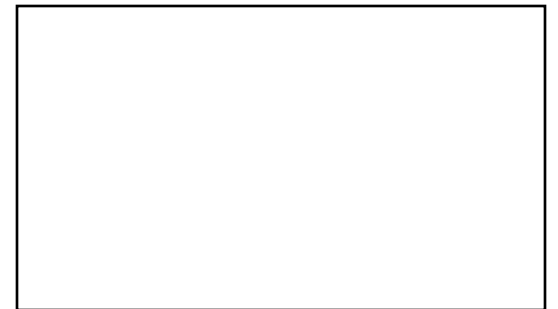
- **$\hat{\mathbf{R}}_y$ is Toeplitz since a sum of Toeplitz matrices is Toeplitz**
- **Remember we are assuming no noise**



Convergence to True Correlation

$$\hat{\mathbf{R}}_y \rightarrow E[|\underline{s}|^2] \exp \begin{bmatrix} 0 & -j\gamma & \cdots & -j(M-1)\gamma \\ j\gamma & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -j\gamma \\ j(M-1)\gamma & \cdots & j\gamma & 0 \end{bmatrix}$$

as $L \rightarrow \infty$



Linear Array, Two Plane Waves

- Again assuming no noise

$$\mathbf{y}(l) = s_1(l) \exp\left(-j \frac{M-1}{2} \gamma_1\right) \exp \begin{bmatrix} 0 \\ j\gamma_1 \\ \vdots \\ j(M-1)\gamma_1 \end{bmatrix}$$
$$+ s_2(l) \exp\left(-j \frac{M-1}{2} \gamma_2\right) \exp \begin{bmatrix} 0 \\ j\gamma_2 \\ \vdots \\ j(M-1)\gamma_2 \end{bmatrix}$$

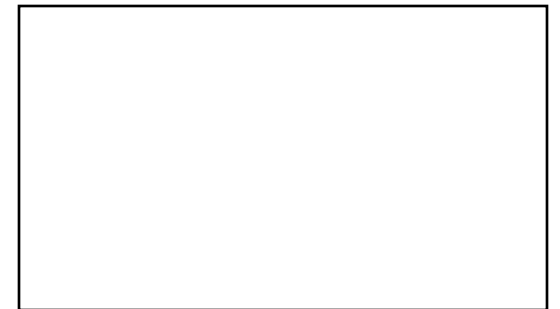


Two Signals

$$\mathbf{y}(l)\mathbf{y}^H(l) = |s_1(l)|^2 \begin{bmatrix} 1 & e^{-j\gamma_1} & \dots & e^{-j(M-1)\gamma_1} \\ e^{j\gamma_1} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & e^{-j\gamma_1} \\ e^{j(M-1)\gamma_1} & \dots & e^{j\gamma_1} & 1 \end{bmatrix}$$

$$+ |s_2(l)|^2 [\text{same but with } \gamma_2]$$

+ crossterms!!!



Crossterms (1)

$$\begin{aligned} & s_1(l) e^{-j \frac{M-1}{2} \gamma_1} \begin{bmatrix} 1 \\ e^{j\gamma_1} \\ \vdots \\ e^{j(M-1)\gamma_1} \end{bmatrix} s_2^*(l) e^{j \frac{M-1}{2} \gamma_2} \begin{bmatrix} 1 \\ e^{-j\gamma_2} \\ \vdots \\ e^{-j(M-1)\gamma_2} \end{bmatrix}^T \\ & + s_2(l) e^{-j \frac{M-1}{2} \gamma_2} \begin{bmatrix} 1 \\ e^{j\gamma_2} \\ \vdots \\ e^{j(M-1)\gamma_2} \end{bmatrix} s_1^*(l) e^{j \frac{M-1}{2} \gamma_1} \begin{bmatrix} 1 \\ e^{-j\gamma_1} \\ \vdots \\ e^{-j(M-1)\gamma_1} \end{bmatrix}^T \end{aligned}$$



Crossterms (2)

$$s_1(l)s_2^*(l)e^{-j\frac{(M-1)}{2}(\gamma_1-\gamma_2)} \times$$

$$\exp \begin{bmatrix} 0 & -j\gamma_2 & \dots & -j(M-1)\gamma_2 \\ j\gamma_1 & j(\gamma_1-\gamma_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & j[(M-2)\gamma_1-(M-2)\gamma_2] \\ j(M-1)\gamma_1 & \dots & j[(M-1)\gamma_1-(M-2)\gamma_2] & j(M-1)(\gamma_1-\gamma_2) \end{bmatrix}$$

$$+s_2(l)s_1^*(l)e^{-j\frac{(M-1)}{2}(\gamma_2-\gamma_1)} \times$$

$$\exp \begin{bmatrix} 0 & -j\gamma_1 & \dots & -j(M-1)\gamma_1 \\ j\gamma_2 & j(\gamma_2-\gamma_1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & j[(M-2)\gamma_2-(M-2)\gamma_1] \\ j(M-1)\gamma_2 & \dots & j[(M-1)\gamma_2-(M-2)\gamma_1] & j(M-1)(\gamma_2-\gamma_1) \end{bmatrix}$$



Crossterm Matrix Elements

- i th diagonal element of crossterm matrix given by

$$2\operatorname{Re}\left\{s_1(l)s_2^*(l)\exp\left(-j\frac{M-1}{2}[\gamma_1 - \gamma_2]\right)\exp(j[\gamma_1 - \gamma_2]i)\right\}$$
$$= 2|s_1(l)s_2^*(l)|\cos\left([\gamma_1 - \gamma_2]\left[i - \frac{M-1}{2}\right] + \angle\{s_1(l)s_2^*(l)\}\right)$$



Intersignal Coherence

$$\frac{1}{L} \sum_{l=0}^{L-1} s_1(l) s_2^*(l) \rightarrow \underbrace{E[s_1 s_2^*]}_{\text{Intersignal coherence}} \text{ as } L \rightarrow \infty$$

- **Incoherent** signals have an intersignal coherence of zero
- **Coherence suggests multipath effects**

