Spatial Averaging and Co-arrays

ECE 6279: Spatial Array Processing
Spring 2011
Lecture 15

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Where We Are in J&D

- **Material from:**
  - Sec. 4.9.2 on spatial averaging
  - Sec. 3.3.4 on co-arrays
    - **We will use uniform weights**
  - Notes by Doug Williams

Why Spatial Averaging?

- **Last lecture, we looked at temporal averaging**
- **Sometimes the source is moving too fast to employ a lot of temporal averaging**
- **If the SNR is still too low, it helps to have some redundancy in the array**

Ex: Uniform Linear Array

- For a ULA, with multiple incoherent sources, and no noise:

$$\mathbf{R_y} = \begin{bmatrix} R_0 & R_1 & R_2 & \cdots & R_{M-1} \\ R_1^* & R_0 & R_1 & \cdots & R_{M-2} \\ R_2^* & R_1^* & R_0 & \cdots & R_{M-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{M-1}^* & R_{M-2}^* & R_{M-3}^* & \cdots & R_0 \end{bmatrix}$$

←Toeplitz structure

Forcing a Toeplitz Structure (1)

$$\hat{\mathbf{R}}_{\mathbf{y}} = \frac{1}{L}\sum_{l=0}^{L-1}\mathbf{y}(l)\mathbf{y}^{H}(l)$$

$$\hat{\mathbf{R}}_{\mathbf{y}} = \begin{bmatrix} \hat{R}_{0,0} & \hat{R}_{0,1} & \hat{R}_{0,2} & \cdots & \hat{R}_{0,M-1} \\ \hat{R}^*_{0,1} & \hat{R}_{1,1} & \hat{R}_{1,2} & \cdots & \hat{R}_{1,M-1} \\ \hat{R}^*_{0,2} & \hat{R}^*_{1,2} & \hat{R}_{2,2} & \cdots & \hat{R}_{2,M-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{R}^*_{0,M-1} & \hat{R}^*_{1,M-1} & \hat{R}^*_{2,M-1} & \cdots & \hat{R}_{M-1,M-1} \end{bmatrix}$$

$\hat{\hat{R}}_{M-1}$ $\quad \hat{\hat{R}}_0$ $\quad \hat{\hat{R}}_1$

- **One idea: Average along diagonals**
- **Elements near main diagonal get larger variability reduction**

Forcing a Toeplitz Structure (2)

- **Now we're guaranteed a Toeplitz structure:**

$$\hat{\hat{\mathbf{R}}}_{\mathbf{y}} = \begin{bmatrix} \hat{\hat{R}}_0 & \hat{\hat{R}}_1 & \hat{\hat{R}}_2 & \cdots & \hat{\hat{R}}_{M-1} \\ \hat{\hat{R}}_1^* & \hat{\hat{R}}_0 & \hat{\hat{R}}_1 & \cdots & \hat{\hat{R}}_{M-2} \\ \hat{\hat{R}}_2^* & \hat{\hat{R}}_1^* & \hat{\hat{R}}_0 & \cdots & \hat{\hat{R}}_{M-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{\hat{R}}_{M-1}^* & \hat{\hat{R}}_{M-2}^* & \hat{\hat{R}}_{M-3}^* & \cdots & \hat{\hat{R}}_0 \end{bmatrix}$$

- **…but we're not guaranteed a nonnegative definite one!**
  - i.e, at least one eigenvalue may becomes negative

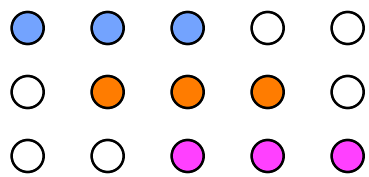
What Can Go Wrong

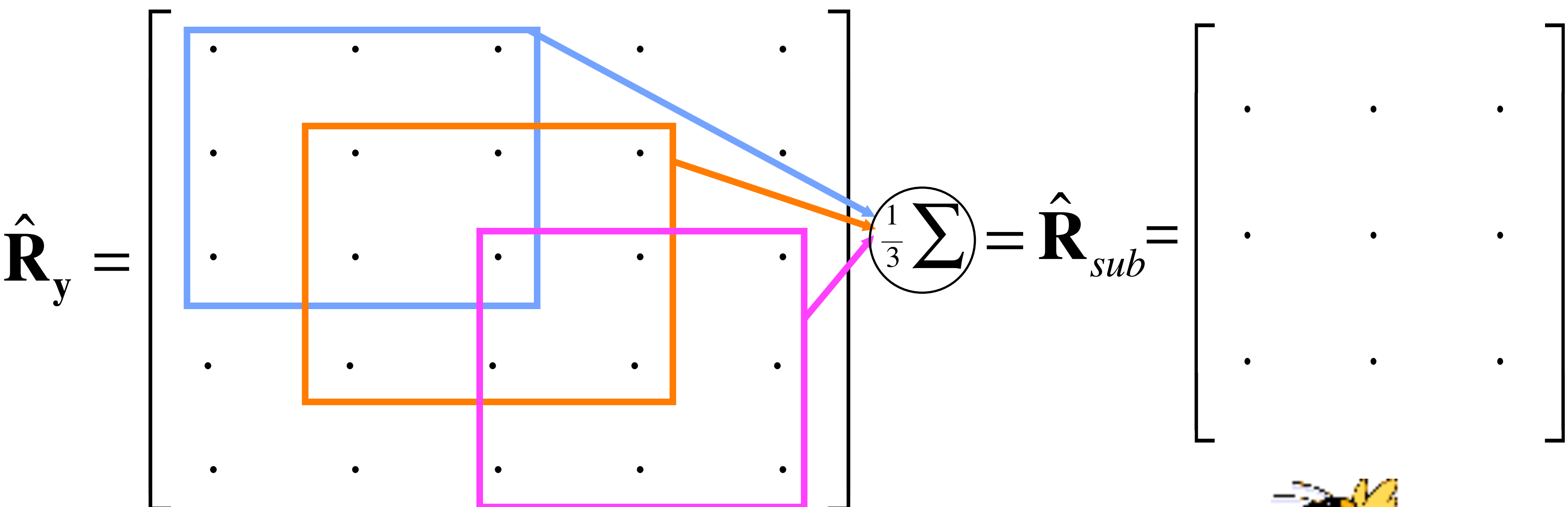
- **Resulting matrix not guaranteed to be nonnegative definite!**
- **Also, procedure can introduce some bias into angle estimates**
  - Usually only a major problem if coherent signals are present
- **One solution: maximum-likelihood structured covariance estimation**
  - Expectation-Maximization (EM) algorithm (computationally intensive)
    - **M.I. Miller and D.L. Snyder, "The Role of Likelihood and Entropy in Incomplete Data Problems: Application to Estimating Point Process Intensity and Toeplitz Constrained Covariances," Proc. IEEE, Vol. 75, No. 7, pp. 892-907, July 1987.**

Subaperture Concept

- **Suppose we have a 5 element ULA:**

- **Form 3 subarrays:**





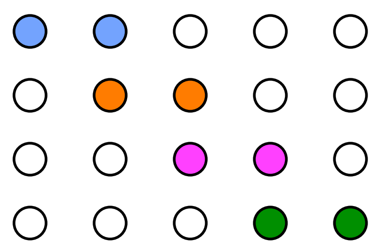
Subaperture Averaging: Pros and Cons

- **Advantages**
  - All elements of estimated spatial correlation matrix (SCM) get the same improvement
  - Resulting SCM estimate is guaranteed to be nonnegative definite
  - Helps reduce problems with coherent sources
    - **Sinusoidal components along diagonals get smoothed out**
- **Disadvantage**
  - Individual subapertures have lower resolution

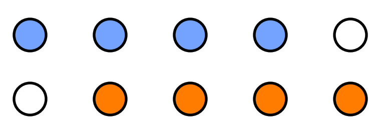
The Tradeoff

- Lower variability in covariance estimate comes at the expense of lower resolution

- 4 small subarrays (bad resolution, low/good variability):



- 2 big subarrays (good resolution, high/bad variability)



Forward/Backward Averaging

- From Prob. 7.11 on p. 418 of J&D
- Consider the data for a uniform linear array:

$$\mathbf{y} = [y_0 \quad y_1 \quad \cdots \quad y_{M-2} \quad y_{M-1}]^T$$

- Define:

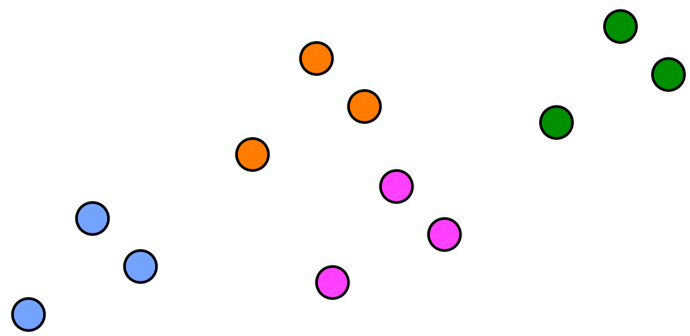
$$\mathbf{y}_r^* = [y_{M-1}^* \quad y_{M-2}^* \quad \cdots \quad y_1^* \quad y_0^*]^T$$

- Describes same signal and noise situation, but coherence effects differ

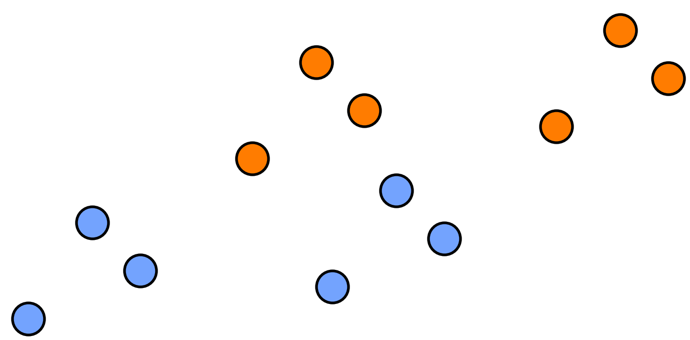
$$\hat{\mathbf{R}}_{fb} = \frac{1}{2L} \sum_{l=0}^{L-1} \left\{ \mathbf{y}(l)\mathbf{y}^H(l) + \mathbf{y}_r^*(l)[\mathbf{y}_r^*(l)]^H \right\}$$

General Subarrays

- 12 element array
- 4, 3-element subarrays



- 2, 6-element subarrays



Interpreting the SCM

- **SCM can be viewed as samples of the correlation function for the entire field**
- **Samples are taken at differences between sensor locations**
- **For a single plane wave:**

$$[\mathbf{R}]_{m_1,m_2} = P_s \exp\{jk^0 \cdot (\vec{x}_{m_1} - \vec{x}_{m_2})\}$$

- **For a general WSS field :**

$$[\mathbf{R}]_{m_1,m_2} = R_f(\vec{x}_{m_1} - \vec{x}_{m_2})$$

where $R_f(\chi) = E[\underline{f}(0)\underline{f}(\chi)]$

The Coarray

- **Set of differences between all pairs of sensor locations is called the coarray**

$$\bigcup_{m_1, m_2} \{\vec{x}_{m_1} - \vec{x}_{m_2}\}$$

- **Differences are known as lags**

$$\vec{z}_{m_1, m_2} \equiv \vec{x}_{m_1} - \vec{x}_{m_2}$$

- **Elements of the coarray have associated coarray values, which is the number of distinct baselines (pairs of actual sensors) with the same vector difference**

Properties of Co-arrays

- **Dimension of coarray is same as dimension of array**
  - Linear <-> Linear
  - Planar <-> Planar

- **Certain lags must exist**

$\vec{z}_{m,m} = 0$ **(coarray value of *M*)**

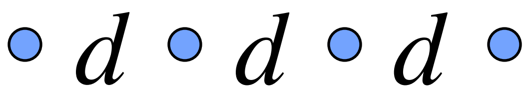
$$\vec{z}_{m_1,m_2} = -\vec{z}_{m_2,m_1}$$

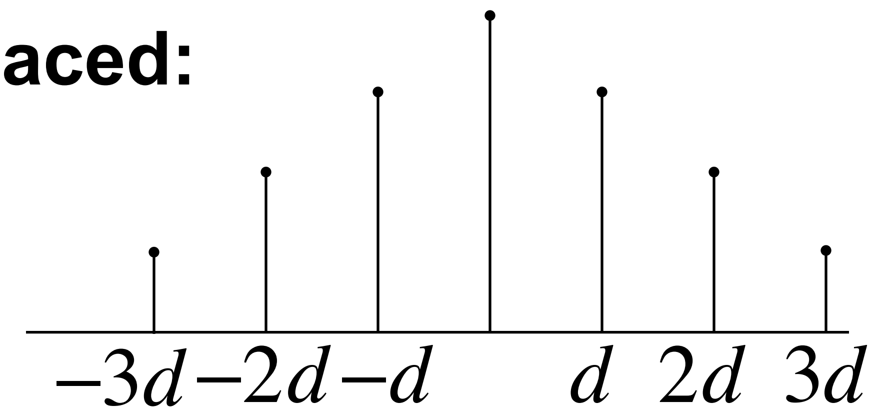
Redundancies

- **Repeated lags (i.e. co-array values > 1) are called redundancies**
- **Zero-lag redundancies occur along main diagonal of SCM**
  - Ideally equal if only incoherent signals are present
  - Usually not equal due to noise or coherent signals
- **Minimum no. of distinct positive lags: M**
- **Maximum no. of distinct positive lags: $M(M-1)/2$ (not achievable if $M>4$)**

Linear Co-array Examples

- **4-sensor uniformly spaced:**





- **4-sensor "perfect" array:**