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# ***Subspace Methods: Eigenvalue Method and MUSIC***

**ECE 6279: Spatial Array Processing  
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Lecture 19**

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# Where We Are in J&D

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- **Section 7.3**



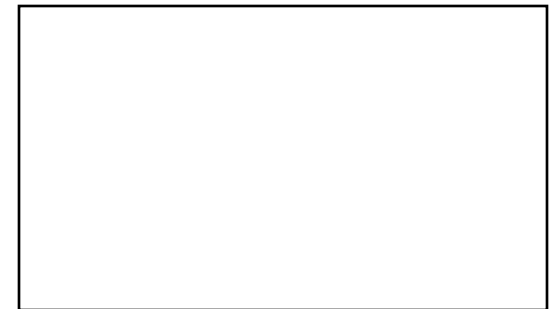
# Eigenexpansions

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- Write correlation matrix in terms of its eigensystem:

$$\mathbf{R}_y = \sum_{m=1}^M \lambda_m \mathbf{v}_m \mathbf{v}_m^H$$

$$\mathbf{R}_y^{-1} = \sum_{m=1}^M \lambda_m^{-1} \mathbf{v}_m \mathbf{v}_m^H$$



# Eigensystem for a One-Signal Case (1)

- Think about one signal in white noise:

$$\mathbf{R}_y = A_s^2 \mathbf{e}(\vec{k}^0) \mathbf{e}^H(\vec{k}^0) + \sigma_n^2 \mathbf{I}$$

- Let  $\mathbf{v}_1 = \mathbf{e}(\vec{k}^0) / \sqrt{M}$
- Let  $\mathbf{v}_2, \dots, \mathbf{v}_M$  be any set of orthogonal vectors orthogonal to  $\mathbf{v}_1$

$$\mathbf{R}_y = MA_s^2 \mathbf{v}_1 \mathbf{v}_1^H + \sum_{i=1}^M \sigma_n^2 \mathbf{v}_i \mathbf{v}_i^H$$

$$\lambda_1 = MA_s^2 + \sigma_n^2, \lambda_2, \dots, \lambda_M = \sigma_n^2$$

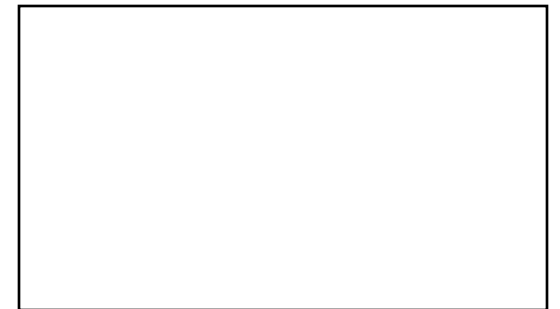


## Eigensystem for a One-Signal Case (2)

$$\begin{aligned}\mathbf{R}_y &= A_s^2 \mathbf{e}(\vec{k}^0) \mathbf{e}^H(\vec{k}^0) + \sigma_n^2 \mathbf{I} \\ &= MA_s^2 \mathbf{v}_1 \mathbf{v}_1^H + \sum_{m=1}^M \sigma_n^2 \mathbf{v}_m \mathbf{v}_m^H\end{aligned}$$

$$\lambda_1 = MA_s^2 + \sigma_n^2, \lambda_2, \dots, \lambda_M = \sigma_n^2$$

- If we had  $\mathbf{R}_y$  exactly, we could find  $\mathbf{e}(\vec{k}^0)$  without knowing anything about the array!



# Eigensystem for a Two-Signal Case

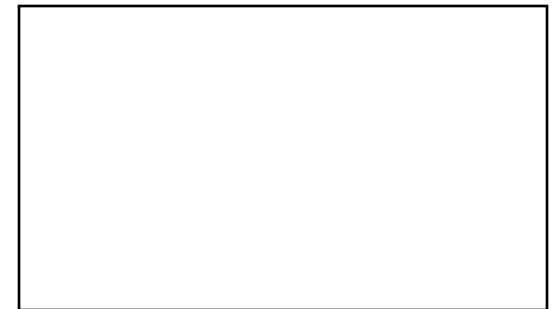
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- **Two signals in white noise:**  $\mathbf{R}_y = A_1^2 \mathbf{e}(\vec{k}_1^0) \mathbf{e}^H(\vec{k}_1^0) + A_2^2 \mathbf{e}(\vec{k}_2^0) \mathbf{e}^H(\vec{k}_2^0) + \sigma_n^2 \mathbf{I}$

- **Can express steering vectors as linear combinations of the two largest eigenvalues  $\mathbf{v}_1, \mathbf{v}_2$**

$$\mathbf{e}(\vec{k}_1^0) = t_{11} \mathbf{v}_1 + t_{21} \mathbf{v}_2$$

$$\mathbf{e}(\vec{k}_2^0) = t_{12} \mathbf{v}_1 + t_{22} \mathbf{v}_2$$



# Subspace Decompositions

$$\begin{bmatrix} \mathbf{e}(\vec{k}_1^0) & \mathbf{e}(\vec{k}_2^0) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}}_{\mathbf{V}_{s+n}} \underbrace{\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}}_{\mathbf{T}}$$

For  $N_s < M$  signals, we have

$\mathbf{V}_1, \dots, \mathbf{V}_{N_s}$  **signal+noise subspace**

$\mathbf{V}_{N_s+1}, \dots, \mathbf{V}_M$  **noise subspace**

Can never really know

$\mathbf{T}$ , so focus on **noise** subspace



# Decomposing MVDR

$$P^{MV}(\vec{k}) \equiv \left[ \mathbf{e}^H(\vec{k}) \mathbf{R}_y^{-1} \mathbf{e}(\vec{k}) \right]^{-1}$$
$$= \left[ \sum_{i=1}^M \lambda_i^{-1} \left| \mathbf{e}^H(\vec{k}) \mathbf{v}_i \right|^2 \right]^{-1}$$

$$\underbrace{\sum_{i=1}^{N_s} \lambda_i^{-1} \left| \mathbf{e}^H(\vec{k}) \mathbf{v}_i \right|^2}_{\text{signal+noise terms}} + \underbrace{\sum_{i=N_s+1}^M \lambda_i^{-1} \left| \mathbf{e}^H(\vec{k}) \mathbf{v}_i \right|^2}_{\text{noise terms}}$$

signal+noise terms

noise terms





# Considering the Signal+Noise Terms

- How can we get more “dramatic” peaks from the MVDR spectrum?

$$\sum_{i=1}^{N_s} \lambda_i^{-1} \left| \mathbf{e}^H(\vec{k}) \mathbf{v}_i \right|^2 + \sum_{i=N_s+1}^M \lambda_i^{-1} \left| \mathbf{e}^H(\vec{k}) \mathbf{v}_i \right|^2$$

- When  $\mathbf{e}(\vec{k})$  corresponds to an actual signal, inner products of it with signal +subspace eigenvectors are large
- This makes the term above not as small as it could be, so MVDR power (reciprocal above term) is not as big as it could be



# Considering the Noise Terms

- How can we get more “dramatic” peaks from the MVDR spectrum?

$$\sum_{i=1}^{N_s} \lambda_i^{-1} \left| \mathbf{e}^H(\vec{k}) \mathbf{v}_i \right|^2 + \sum_{i=N_s+1}^M \lambda_i^{-1} \left| \mathbf{e}^H(\vec{k}) \mathbf{v}_i \right|^2$$

- When  $\mathbf{e}(\vec{k})$  corresponds to an actual signal, inner products of it with noise eigenvectors are small (ideally zero)
- Reciprocals of small numbers are big!



# Eigenvalue Method

$$\sum_{i=1}^{N_s} \lambda_i^{-1} \left| \mathbf{e}^H(\vec{k}) \mathbf{v}_i \right|^2 + \sum_{i=N_s+1}^M \lambda_i^{-1} \left| \mathbf{e}^H(\vec{k}) \mathbf{v}_i \right|^2$$

$$P^{EV}(\vec{k}) \equiv \left[ \sum_{i=N_s+1}^M \lambda_i^{-1} \left| \mathbf{e}^H(\vec{k}) \mathbf{v}_i \right|^2 \right]^{-1} = \left[ \mathbf{e}^H(\vec{k}) \mathbf{R}_{EV}^{-1} \mathbf{e}(\vec{k}) \right]^{-1}$$

where  $\mathbf{R}_{EV}^{-1} = \sum_{i=N_s+1}^M \lambda_i^{-1} \mathbf{v}_i \mathbf{v}_i^H$



# MUSIC (1)

$$P^{EV}(\vec{k}) \equiv \left[ \sum_{i=N_s+1}^M \lambda_i^{-1} \left| \mathbf{e}^H(\vec{k}) \mathbf{v}_i \right|^2 \right]^{-1}$$

- If  $\mathbf{K}_n = \sigma_n^2 \mathbf{I}$ , noise subspace eigenvalues are theoretically  $\sigma_n^2$

$$P^{MUSIC}(\vec{k}) \equiv \left[ \sum_{i=N_s+1}^M \left| \mathbf{e}^H(\vec{k}) \mathbf{v}_i \right|^2 \right]^{-1} \boxed{\phantom{\text{noise subspace eigenvalues}}}$$



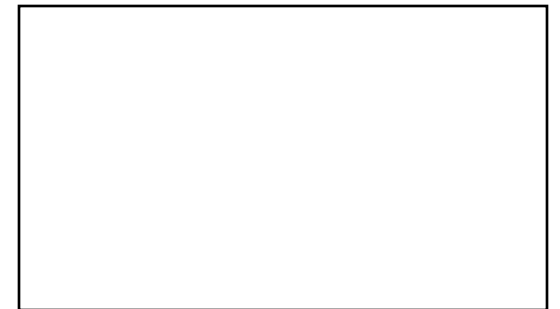
## MUSIC (2)

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$$P^{MUSIC}(\vec{k}) = \left[ \mathbf{e}^H(\vec{k}) \mathbf{R}_{MUSIC}^{-1} \mathbf{e}(\vec{k}) \right]^{-1}$$

where  $\mathbf{R}_{MUSIC}^{-1} = \sum_{i=N_s+1}^M \mathbf{v}_i \mathbf{v}_i^H$

- “Whitens” noise subspace
- “Flattens” background
- “Pulls” signals out of noisy background



# Special Cases (1)

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- **EV doesn't change the eigenvalues it keeps**
- **Hence, it reduces to MVDR if we take the number of sources to be zero**




## Special Cases (2)

- Remember Pisarenko method:

Look for valleys in:  $\left| \mathbf{e}^H(\vec{k}) \mathbf{v}_{\min} \right|$

Look for peaks in:  $\left| \mathbf{e}^H(\vec{k}) \mathbf{v}_{\min} \right|^{-1}$

- Like MUSIC method with  $N_s = M - 1$

$$P^{MUSIC}(\vec{k}) \equiv \left[ \sum_{i=N_s+1}^M \left| \mathbf{e}^H(\vec{k}) \mathbf{v}_i \right|^2 \right]^{-1}$$




# Good Properties of EV and Music

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- **Good resolution properties overall**
- **In white Gaussian noise:**
  - Resolution increases without limit with increasing SNR
  - Peak locations “asymptotically unbiased”
- **In non-white noise:**  
**Resolution still good, but has limits**
  - Peak locations usually biased





# Drawbacks of EV and MUSIC

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- **Neither EV or MUSIC are very robust to modeling assumptions**
  - Why “conventional BF” is still popular
- **MUSIC very sensitive to assumption of number of sources due to flattening the eigenvalues**
- EV also sensitive, but not as sensitive
- **Coherent sources badly mess up subspace methods (see p. 389)**

