ESPRIT, Part I: Setup

ECE 6279: Spatial Array Processing
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Lecture 21

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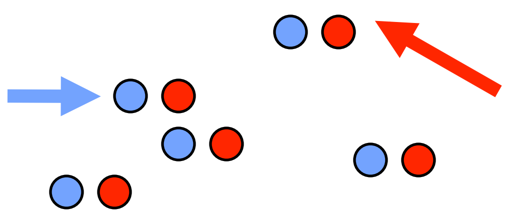
Sources

- ESPRIT: "Estimation of Signal Prameters via Rotational Invariance Techniques"
- J&D, p. 419, Problem 7.22
- Journal papers (linked on class website)
- Lecture notes from Dan Fuhrmann

Setup

- **Need two identical subarrays displaced (not rotated) by a known displacement vector $\vec{\Delta}$ with magnitude Δ**
- **For simplicity, let's do displacement along the x dimension in these slides**

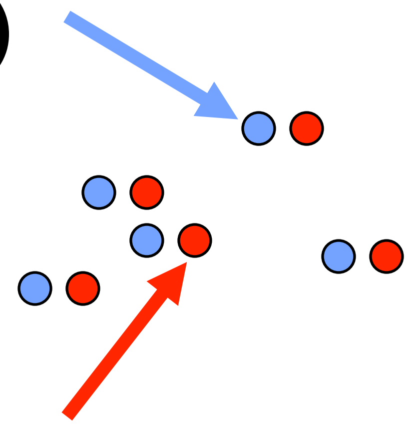
$$\mathbf{e}(\phi)$$



$$\mathbf{e}(\phi)\exp\left(j\frac{2\pi}{\lambda}\Delta\sin(\phi)\right)$$

Notation (1)

$$\mathbf{y}^{(0)}(l) = \underbrace{\mathbf{D}}_{[\mathbf{e}(\phi_1) \ \cdots \ \mathbf{e}(\phi_{N_s})]}\mathbf{s}(l) + \mathbf{n}^{(0)}(l)$$



$$\mathbf{y}^{(1)}(l) = \mathbf{D}\underbrace{\mathbf{\Phi}}_{\begin{bmatrix} \exp(j\gamma_1) & & \\ & \ddots & \\ & & \exp(j\gamma_{N_s}) \end{bmatrix}}\mathbf{s}(l) + \mathbf{n}^{(1)}(l)$$

$$\gamma_i = \frac{2\pi}{\lambda}\Delta \sin(\phi_i)$$

Notation (2)

$$\mathbf{y}(l) = \begin{bmatrix} \mathbf{y}^{(0)}(l) \\ \mathbf{y}^{(1)}(l) \end{bmatrix} = \begin{bmatrix} \mathbf{D} \\ \mathbf{D}\Phi \end{bmatrix} \mathbf{s}(l) + \begin{bmatrix} \mathbf{n}^{(0)}(l) \\ \mathbf{n}^{(1)}(l) \end{bmatrix}$$

$$= \overline{\mathbf{D}}\mathbf{s}(l) + \mathbf{n}(l)$$

- **Goal: Exploit structure of $\overline{\mathbf{D}}$ to estimate diagonal elements of Φ without needing to know $\mathbf{D}$**

Ideal Covariance of the Data

- **Ideally:**

$$\mathbf{R}_{\mathbf{y}} = \overline{\mathbf{D}}\underbrace{\mathbf{R}_{\mathbf{s}}}_{\begin{bmatrix} A_1^2 & & \\ & \ddots & \\ & & A_{N_s}^2 \end{bmatrix}}\overline{\mathbf{D}}^H + \sigma_n^2\mathbf{I}$$

Splitting the Signal+Noise Subspace

- Do eigendecomposition of $\mathbf{R}_{\mathbf{y}}$

$$\mathbf{V}_{s+n} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_{N_s} \end{bmatrix}$$

- Since $\mathbf{V}_{s+n}$ and $\bar{\mathbf{D}}$ span the same subspace, there exists a $\exists \mathbf{T}$ s.t.

$$\mathbf{V}_{s+n} = \bar{\mathbf{D}}\mathbf{T}$$

$$\mathbf{V}_{s+n} \equiv \begin{bmatrix} \mathbf{E}_0 \\ \mathbf{E}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{D} \\ \mathbf{D}\boldsymbol{\Phi} \end{bmatrix} \mathbf{T}$$

Some Trickery

$$\mathbf{E}_0 = \mathbf{D}\mathbf{T}$$

$$\mathbf{E}_0\mathbf{T}^{-1} = \mathbf{D}$$

$$\mathbf{E}_1 = \mathbf{D}\Phi\mathbf{T} = \mathbf{E}_0\underbrace{\mathbf{T}^{-1}\Phi\mathbf{T}}_{\Psi}$$

- **Fact from linear algebra: Ψ and Φ have the same eigenvalues**

Ideal ESPRIT Procedure (1)

- **Solve $\mathbf{E}_1 = \mathbf{E}_0 \Psi$ for Ψ**
- **Find eigenvalues of Ψ; these are the diagonal elements of**

$$\Phi = \begin{bmatrix} \exp(j\gamma_1) & & \\ & \ddots & \\ & & \exp(j\gamma_{N_s}) \end{bmatrix}$$

(possibly reordered)

Ideal ESPRIT Procedure (2)

$$\gamma_i = \frac{2\pi}{\lambda} \Delta \sin(\phi_i)$$

$$\phi_i = \sin^{-1}\left(\gamma_i \frac{\lambda}{2\pi\Delta} \right)$$

$$= \sin^{-1}\left(\arg(\lambda_i^{(\Psi)}) \frac{\lambda}{2\pi\Delta} \right)$$

ESPRIT Procedure in Reality

- **In practice, compute eigendecomposition from empirical covariance matrix**
- **"Solve"** $\hat{\mathbf{E}}_1 = \hat{\mathbf{E}}_0 \Psi$

$$\hat{\phi}_i = \sin^{-1}\left(\arg(\hat{\lambda}_i^{(\Psi)}) \frac{\lambda}{2\pi\Delta} \right)$$

- **Trouble:** $\hat{\mathbf{E}}_0$ **and** $\hat{\mathbf{E}}_1$ **may not span the same subspace**