

ESPRIT, Part I: Setup

ECE 6279: Spatial Array Processing
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Lecture 21

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Sources

- **ESPRIT: “Estimation of Signal Parameters via Rotational Invariance Techniques”**
- **J&D, p. 419, Problem 7.22**
- **Journal papers (linked on class website)**
- **Lecture notes from Dan Fuhrmann**

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Setup

- **Need two identical subarrays displaced (not rotated) by a known displacement vector $\vec{\Delta}$ with magnitude Δ**
- **For simplicity, let's do displacement along the x dimension in these slides**

$$\mathbf{e}(\phi) \rightarrow \mathbf{e}(\phi) \exp\left(j \frac{2\pi}{\lambda} \Delta \sin(\phi)\right)$$

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Notation (1)

$$\mathbf{y}^{(0)}(l) = \underbrace{\mathbf{D}}_{[\mathbf{e}(\phi_1) \cdots \mathbf{e}(\phi_{N_s})]} \mathbf{s}(l) + \mathbf{n}^{(0)}(l)$$

$$\mathbf{y}^{(1)}(l) = \underbrace{\mathbf{D}\Phi}_{\begin{bmatrix} \exp(j\gamma_1) & & \\ & \ddots & \\ & & \exp(j\gamma_{N_s}) \end{bmatrix}} \mathbf{s}(l) + \mathbf{n}^{(1)}(l)$$

$$\gamma_i = \frac{2\pi}{\lambda} \Delta \sin(\phi_i)$$

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Notation (2)

$$\mathbf{y}(l) = \begin{bmatrix} \mathbf{y}^{(0)}(l) \\ \mathbf{y}^{(1)}(l) \end{bmatrix} = \begin{bmatrix} \mathbf{D} \\ \mathbf{D}\Phi \end{bmatrix} \mathbf{s}(l) + \begin{bmatrix} \mathbf{n}^{(0)}(l) \\ \mathbf{n}^{(1)}(l) \end{bmatrix}$$

$$= \bar{\mathbf{D}}\mathbf{s}(l) + \mathbf{n}(l)$$

- **Goal: Exploit structure of $\bar{\mathbf{D}}$ to estimate diagonal elements of Φ without needing to know \mathbf{D}**

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Ideal Covariance of the Data

- **Ideally:**

$$\mathbf{R}_y = \underbrace{\bar{\mathbf{D}}\mathbf{R}_s\bar{\mathbf{D}}^H}_{\begin{bmatrix} A_1^2 & & \\ & \ddots & \\ & & A_{N_s}^2 \end{bmatrix}} + \sigma_n^2 \mathbf{I}$$

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Splitting the Signal+Noise Subspace

- **Do eigendecomposition of \mathbf{R}_y**

$$\mathbf{V}_{s+n} = [\mathbf{v}_1 \cdots \mathbf{v}_{N_s}]$$

- **Since \mathbf{V}_{s+n} and $\bar{\mathbf{D}}$ span the same subspace, there exists a $\exists \mathbf{T}$ s.t.**

$$\mathbf{V}_{s+n} = \bar{\mathbf{D}}\mathbf{T}$$

$$\mathbf{V}_{s+n} \equiv \begin{bmatrix} \mathbf{E}_0 \\ \mathbf{E}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{D} \\ \mathbf{D}\Phi \end{bmatrix} \mathbf{T}$$

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Some Trickery

$$\mathbf{E}_0 = \mathbf{D}\mathbf{T}$$

$$\mathbf{E}_0\mathbf{T}^{-1} = \mathbf{D}$$

$$\mathbf{E}_1 = \mathbf{D}\Phi\mathbf{T} = \mathbf{E}_0 \underbrace{\mathbf{T}^{-1}\Phi\mathbf{T}}_{\Psi}$$

- **Fact from linear algebra: Ψ and Φ have the same eigenvalues**

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Ideal ESPRIT Procedure (1)

- Solve $\mathbf{E}_1 = \mathbf{E}_0 \Psi$ for Ψ
- Find eigenvalues of Ψ ; these are the diagonal elements of

$$\Phi = \begin{bmatrix} \exp(j\gamma_1) & & \\ & \ddots & \\ & & \exp(j\gamma_{N_s}) \end{bmatrix}$$

(possibly reordered)

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Ideal ESPRIT Procedure (2)

$$\begin{aligned} \gamma_i &= \frac{2\pi}{\lambda} \Delta \sin(\phi_i) \\ \phi_i &= \sin^{-1} \left(\gamma_i \frac{\lambda}{2\pi\Delta} \right) \\ &= \sin^{-1} \left(\arg(\lambda_i^{(\Psi)}) \frac{\lambda}{2\pi\Delta} \right) \end{aligned}$$

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ESPRIT Procedure in Reality

- In practice, compute eigendecomposition from empirical covariance matrix
- “Solve” $\hat{\mathbf{E}}_1 = \hat{\mathbf{E}}_0 \Psi$

$$\hat{\phi}_i = \sin^{-1} \left(\arg(\hat{\lambda}_i^{(\Psi)}) \frac{\lambda}{2\pi\Delta} \right)$$

- Trouble: $\hat{\mathbf{E}}_0$ and $\hat{\mathbf{E}}_1$ may not span the same subspace

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