
ESPRIT, Part II: Total Least Squares

**ECE 6279: Spatial Array Processing
Spring 2011
Lecture 22**

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References

- **ESPRIT journal papers (posted on website)**
- **On total least squares: see Section 12.3 in Matrix Computations by Golub & Van Loan**



Ideal ESPRIT Algorithm

- **Do eigendecomposition of \mathbf{R}_y**

$$\mathbf{V}_{s+n} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_{N_s} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{E}_0 \\ \mathbf{E}_1 \end{bmatrix}$$

- **Solve $\mathbf{E}_1 = \mathbf{E}_0 \Psi$ for Ψ**

$$\phi_i = \sin^{-1} \left(\arg(\lambda_i^{(\Psi)}) \frac{\lambda}{2\pi\Delta} \right)$$



Total Least Squares for Practical ESPIRT

- **Goal: “Solve” $\hat{\mathbf{E}}_1 = \hat{\mathbf{E}}_0 \Psi$ when both sides have errors**

- **Formulation:**

$$(\hat{\mathbf{E}}_1 + \Delta\mathbf{E}_1) = (\hat{\mathbf{E}}_0 + \Delta\mathbf{E}_0)\Psi$$

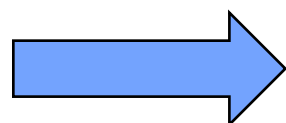
- **Find $\hat{\mathbf{E}}_0$, $\hat{\mathbf{E}}_1$, and Ψ that minimizes**

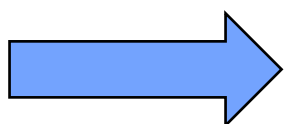
$$\left\| [\Delta\mathbf{E}_0 \quad \Delta\mathbf{E}_1] \right\|_F^2 \equiv \text{tr}([\Delta\mathbf{E}_0 \quad \Delta\mathbf{E}_1][\Delta\mathbf{E}_0 \quad \Delta\mathbf{E}_1]^H)$$



The Null Subspace - Ideally

- Ideally $\mathbf{E}_1 = \mathbf{E}_0 \Psi$, so \mathbf{E}_0 and \mathbf{E}_1 share the same subspace

 $\overbrace{[\mathbf{E}_0 \ \mathbf{E}_1]}^{2N_s}$ has rank N_s

 $\exists \mathbf{F} = \left. \begin{bmatrix} \mathbf{F}_0 \\ \mathbf{F}_1 \end{bmatrix} \right\}^{2N_s} \quad 0 = [\mathbf{E}_0 \ \mathbf{E}_1] \mathbf{F}$
 $= \mathbf{E}_0 \mathbf{F}_0 + \mathbf{E}_1 \mathbf{F}_1$

spans null subspace of $[\mathbf{E}_0 \ \mathbf{E}_1]$



Exploiting the Null Subspace

- Ideally $\mathbf{E}_1 = \mathbf{E}_0 \Psi$, so \mathbf{E}_0 and \mathbf{E}_1 share the same subspace

$$0 = \mathbf{E}_0 \mathbf{F}_0 + \mathbf{E}_1 \mathbf{F}_1$$

$$\mathbf{E}_1 \mathbf{F}_1 = -\mathbf{E}_0 \mathbf{F}_0$$

$$\mathbf{E}_1 = \mathbf{E}_0 (-\mathbf{F}_0 \mathbf{F}_1^{-1})$$

$$\longrightarrow \Psi = -\mathbf{F}_0 \mathbf{F}_1^{-1}$$



The Null Subspace – in Reality

- In reality, won't have $\begin{bmatrix} \hat{\mathbf{E}}_0 & \hat{\mathbf{E}}_1 \end{bmatrix} \mathbf{F} = \mathbf{0}$
- It “is easily shown” we should replace it with

$$\hat{\mathbf{E}}_{01} \mathbf{F} = \begin{bmatrix} \Delta \mathbf{E}_0 & \Delta \mathbf{E}_1 \end{bmatrix}$$

where $\hat{\mathbf{E}}_{01} = \begin{bmatrix} \hat{\mathbf{E}}_0 & \hat{\mathbf{E}}_1 \end{bmatrix}$

- Seek \mathbf{F} that minimizes $\left\| \hat{\mathbf{E}}_{01} \mathbf{F} \right\|_F$
s.t. $\mathbf{F}^H \mathbf{F} = \mathbf{I}$



Magic

- “applying standard Lagrange techniques leads to a solution”
- Compute eigenvectors of $\hat{\mathbf{E}}_{01}^H \hat{\mathbf{E}}_{01}$

$$\begin{bmatrix} \mathbf{F}_0 \\ \mathbf{F}_1 \end{bmatrix} = \mathbf{F} = \underbrace{\begin{bmatrix} \tilde{\mathbf{v}}_{N_s+1} & \cdots & \tilde{\mathbf{v}}_{2N_s} \end{bmatrix}}_{N_s \text{ smallest eigenvectors of } \hat{\mathbf{E}}_{01}^H \hat{\mathbf{E}}_{01}}$$

N_s smallest eigenvectors of $\hat{\mathbf{E}}_{01}^H \hat{\mathbf{E}}_{01}$

- In general cases, total least squares uses a SVD



Practical ESPRIT Algorithm

- Do eigendecomposition of $\hat{\mathbf{R}}_y$

$$\hat{\mathbf{V}}_{s+n} = \begin{bmatrix} \hat{\mathbf{v}}_1 & \cdots & \hat{\mathbf{v}}_{N_s} \end{bmatrix} \equiv \begin{bmatrix} \hat{\mathbf{E}}_0 \\ \hat{\mathbf{E}}_1 \end{bmatrix}$$

- Let $\mathbf{F} = N_s$ smallest eigvecs of $\begin{bmatrix} \hat{\mathbf{E}}_0 & \hat{\mathbf{E}}_1 \end{bmatrix}^H \begin{bmatrix} \hat{\mathbf{E}}_0 & \hat{\mathbf{E}}_1 \end{bmatrix}$

$$\Psi = -\mathbf{F}_0 \mathbf{F}_1^{-1} \quad \phi_i = \sin^{-1} \left(\arg(\lambda_i^{(\Psi)}) \frac{\lambda}{2\pi\Delta} \right)$$

