ESPRIT, Part II: Total Least Squares

ECE 6279: Spatial Array Processing
Spring 2011
Lecture 22

Prof. Aaron D. Lanterman

School of Electrical & Computer Engineering
Georgia Institute of Technology
AL: 404-385-2548
<lanterma@ece.gatech.edu>

References

- ESPRIT journal papers (posted on website)

- On total least squares: see Section 12.3 in Matrix Computations by Golub & Van Loan

Ideal ESPRIT Algorithm

- Do eigendecomposition of $\mathbf{R}_{\mathbf{y}}$

$$\mathbf{V}_{s+n} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_{N_s} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{E}_0 \\ \mathbf{E}_1 \end{bmatrix}$$

- Solve $\mathbf{E}_1 = \mathbf{E}_0 \Psi$ for Ψ

$$\phi_i = \sin^{-1}\left(\arg(\lambda_i^{(\Psi)}) \frac{\lambda}{2\pi\Delta} \right)$$

Total Least Squares for Practical ESPIRT

- **Goal: "Solve" $\hat{\mathbf{E}}_1 = \hat{\mathbf{E}}_0 \Psi$ when both sides have have errors**
- **Formulation:**

$$(\hat{\mathbf{E}}_1 + \Delta\mathbf{E}_1) = (\hat{\mathbf{E}}_0 + \Delta\mathbf{E}_0)\Psi$$

- **Find $\hat{\mathbf{E}}_0$, $\hat{\mathbf{E}}_1$, and Ψ that minimizes**

$$\left\| [\Delta\mathbf{E}_0 \; \Delta\mathbf{E}_1] \right\|_F^2 \equiv \mathrm{tr}([\Delta\mathbf{E}_0 \; \Delta\mathbf{E}_1][\Delta\mathbf{E}_0 \; \Delta\mathbf{E}_1]^H)$$

The Null Subspace - Ideally

- Ideally $\mathbf{E}_1 = \mathbf{E}_0\Psi$, so $\mathbf{E}_0$ and $\mathbf{E}_1$ share the same subspace

$$\overbrace{\left[\mathbf{E}_0 \; \mathbf{E}_1\right]}^{2N_s} \text{ has rank } N_s$$

$$\exists \mathbf{F} = \left.\begin{bmatrix} \mathbf{F}_0 \\ \mathbf{F}_1 \end{bmatrix}\right\} 2N_s \quad \begin{aligned} 0 &= \left[\mathbf{E}_0 \; \mathbf{E}_1\right]\mathbf{F} \\ &= \mathbf{E}_0\mathbf{F}_0 + \mathbf{E}_1\mathbf{F}_1 \end{aligned}$$

spans null subspace of $\left[\mathbf{E}_0 \; \mathbf{E}_1\right]$

Exploiting the Null Subspace

- Ideally $\mathbf{E}_1 = \mathbf{E}_0 \Psi$, so $\mathbf{E}_0$ and $\mathbf{E}_1$ share the same subspace

$$0 = \mathbf{E}_0 \mathbf{F}_0 + \mathbf{E}_1 \mathbf{F}_1$$

$$\mathbf{E}_1 \mathbf{F}_1 = -\mathbf{E}_0 \mathbf{F}_0$$

$$\mathbf{E}_1 = \mathbf{E}_0 (-\mathbf{F}_0 \mathbf{F}_1^{-1})$$

$$\Rightarrow \Psi = -\mathbf{F}_0 \mathbf{F}_1^{-1}$$

The Null Subspace – in Reality

- **In reality, won't have** $\left[\hat{\mathbf{E}}_0 \; \hat{\mathbf{E}}_1\right]\mathbf{F} = 0$
- **It "is easily shown" we should replace it with**

$$\hat{\mathbf{E}}_{01}\mathbf{F} = \left[\Delta\mathbf{E}_0 \; \Delta\mathbf{E}_1\right]$$

where $\hat{\mathbf{E}}_{01} = \left[\hat{\mathbf{E}}_0 \; \hat{\mathbf{E}}_1\right]$

- **Seek** $\mathbf{F}$ **that minimizes** $\left\|\hat{\mathbf{E}}_{01}\mathbf{F}\right\|_F$ **s.t.** $\mathbf{F}^H\mathbf{F} = \mathbf{I}$

Magic

- "applying standard Lagrange techniques leads to a solution"
- Compute eigenvectors of $\hat{\mathbf{E}}_{01}^H \hat{\mathbf{E}}_{01}$

$$\begin{bmatrix} \mathbf{F}_0 \\ \mathbf{F}_1 \end{bmatrix} = \mathbf{F} = \underbrace{\begin{bmatrix} \tilde{\mathbf{v}}_{N_s+1} & \cdots & \tilde{\mathbf{v}}_{2N_s} \end{bmatrix}}_{N_s \text{ smallest eigenvectors of } \hat{\mathbf{E}}_{01}^H \hat{\mathbf{E}}_{01}}$$

- In general cases, total least squares uses a SVD

Practical ESPRIT Algorithm

- Do eigendecomposition of $\hat{\mathbf{R}}_{\mathbf{y}}$

$$\hat{\mathbf{V}}_{s+n} = \left[\hat{\mathbf{v}}_1 \ \cdots \ \hat{\mathbf{v}}_{N_s} \right] \equiv \begin{bmatrix} \hat{\mathbf{E}}_0 \\ \hat{\mathbf{E}}_1 \end{bmatrix}$$

- Let $\mathbf{F} = N_s$ smallest eigvecs of

$$\left[\hat{\mathbf{E}}_0 \ \hat{\mathbf{E}}_1 \right]^H \left[\hat{\mathbf{E}}_0 \ \hat{\mathbf{E}}_1 \right]$$

$$\Psi = -\mathbf{F}_0 \mathbf{F}_1^{-1} \qquad \phi_i = \sin^{-1}\left(\arg(\lambda_i^{(\Psi)}) \frac{\lambda}{2\pi\Delta} \right)$$