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# ***Robust Constrained Optimization***

**ECE 6279: Spatial Array Processing  
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Lecture 23**

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# Where We Are in J&D

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- **Section 7.4.3**
- **Note: these slides will use  $R$  instead of  $R_y$**



# Adaptive Techniques So Far

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- **Recall MVDR formulation:**

$$\min_{\mathbf{w}} \mathbf{w}^H \mathbf{R} \mathbf{w} \quad \text{s.t.} \quad \mathbf{e}^H (\vec{k}) \mathbf{w} = 1$$

- **Then derived EV, MUSIC, etc.**



# Some Potential Problems

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- **Problem: If we don't know  $e(\vec{k})$  exactly, these techniques may break down**
- **Sensitivity to model mismatches increases with SNR!**
- **Overall sensitive to quality of estimate of  $\mathbf{R}$**



# Problems with Few Snapshots

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- **All methods use empirical correlation matrix:**

$$\hat{\mathbf{R}} = \frac{1}{L} \sum_{l=0}^{L-1} \mathbf{y}(l) \mathbf{y}^H(l)$$

- **All methods need a “full rank”  $\hat{\mathbf{R}}$**
- **Hence, need  $L \geq M$**   
No. of snapshots  $\nearrow$   $L \geq M$   $\nwarrow$  No. of sensors



# Try Building Errors into Formulation

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$$\min_{\mathbf{w}} \mathbf{w}^H \mathbf{R} \mathbf{w} \quad \text{s.t.} \quad (\mathbf{e} + \boldsymbol{\delta})^H \mathbf{w} = 1$$

$$\text{and} \quad \|\boldsymbol{\delta}\|^2 = \boldsymbol{\delta}^H \boldsymbol{\delta} \leq \varepsilon^2$$

- **Note will also need to “find”  $\boldsymbol{\delta}$**
- **Use Lagrange multipliers**

$$\begin{aligned} L = & \mathbf{w}^H \mathbf{R} \mathbf{w} \\ & + \lambda_1^* [(\mathbf{e} + \boldsymbol{\delta})^H \mathbf{w} - 1] + \lambda_1 [(\mathbf{e} + \boldsymbol{\delta})^T \mathbf{w}^* - 1] \\ & + \lambda_2 [\boldsymbol{\delta}^H \boldsymbol{\delta} - \varepsilon^2] \end{aligned}$$



# Taking Gradients

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$$\begin{aligned} L = & \mathbf{w}^H \mathbf{R} \mathbf{w} \\ & + \lambda_1^* [(\mathbf{e} + \boldsymbol{\delta})^H \mathbf{w} - 1] + \lambda_1 [\mathbf{w}^H (\mathbf{e} + \boldsymbol{\delta}) - 1] \\ & + \lambda_2 [\boldsymbol{\delta}^H \boldsymbol{\delta} - \varepsilon^2] \end{aligned}$$

$$\nabla_{\mathbf{w}^*} L = \mathbf{R} \mathbf{w} + \lambda_1 (\mathbf{e} + \boldsymbol{\delta}) = 0$$

$$\nabla_{\boldsymbol{\delta}^*} L = \lambda_1^* \mathbf{w} + \lambda_2 \boldsymbol{\delta} = 0$$



# Constraint on Errors is Operative

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$$\lambda_1^* \mathbf{w} + \lambda_2 \boldsymbol{\delta} = 0$$

- **From Appendix C, either**

$$\lambda_2 = 0 \text{ or } \|\boldsymbol{\delta}\|^2 = \varepsilon^2$$

- **Taking  $\lambda_2 = 0 \longrightarrow \mathbf{w} = 0$**

- **So instead take  $\lambda_2 \neq 0$**

$$\longrightarrow \boldsymbol{\delta} = -\frac{\lambda_1^*}{\lambda_2} \mathbf{w}$$





# Eliminating the Explicit Errors

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$$\delta = -\frac{\lambda_1^*}{\lambda_2} \mathbf{w}$$

$$\mathbf{R}\mathbf{w} = -\lambda_1(\mathbf{e} + \delta)$$

$$\mathbf{R}\mathbf{w} = -\lambda_1 \left( \mathbf{e} - \frac{\lambda_1^*}{\lambda_2} \mathbf{w} \right)$$

$$\mathbf{R}\mathbf{w} = -\lambda_1 \mathbf{e} + \frac{|\lambda_1|^2}{\lambda_2} \mathbf{w}$$



# Boring Manipulations

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$$\mathbf{R}\mathbf{w} = -\lambda_1\mathbf{e} + \frac{|\lambda_1|^2}{\lambda_2}\mathbf{w}$$

$$\left(\mathbf{R} - \frac{|\lambda_1|^2}{\lambda_2}\mathbf{I}\right)\mathbf{w} = -\lambda_1\mathbf{e}$$

$$\mathbf{w} = -\lambda_1\left(\mathbf{R} - \frac{|\lambda_1|^2}{\lambda_2}\mathbf{I}\right)^{-1}\mathbf{e}$$



# Yes, the Inverse is Well Defined

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$$\mathbf{w}_\diamond = -\lambda_1 \left( \mathbf{R} - \frac{|\lambda_1|^2}{\lambda_2} \mathbf{I} \right)^{-1} \mathbf{e}(\vec{k})$$

- **Can show  $\lambda_1 < 0$ ,  $\lambda_2 < 0$ , so**

**$\mathbf{R} - \frac{|\lambda_1|^2}{\lambda_2} \mathbf{I}$  is invertible**



# Running into Trouble

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$$\mathbf{w}_{\diamond} = -\lambda_1 \left( \mathbf{R} - \frac{|\lambda_1|^2}{\lambda_2} \mathbf{I} \right)^{-1} \mathbf{e}(\vec{k})$$

- **Computing  $\lambda_1, \lambda_2$  is difficult**
- **Several possible solutions for  $\lambda_1, \lambda_2$ ; must check which minimizes  $\mathbf{w}^H \mathbf{R} \mathbf{w}$**
- **$\lambda_1, \lambda_2$  change with  $\vec{k}$**



# Punting

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$$\mathbf{w}_\diamond = -\lambda_1 \left( \mathbf{R} - \frac{|\lambda_1|^2}{\lambda_2} \mathbf{I} \right)^{-1} \mathbf{e}(\vec{k})$$

- **Let's try something ad-hoc:**

$$\mathbf{w}_\diamond = (\mathbf{R} + \alpha \mathbf{I})^{-1} \mathbf{e}(\vec{k}), \alpha > 0$$

- **“diagonal loading,” “ridge regression,” “Tikhonov regularization”**



# Interpretation of Added Term

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$$\mathbf{w}_{\diamond} = (\mathbf{R} + \alpha \mathbf{I})^{-1} \mathbf{e}(\vec{k}), \alpha > 0$$

- **Adding additional “white noise” to  $\mathbf{R}$  used in computing weights**
- **Remember sensitivity to errors increases with SNR**
- **Artificially decrease apparent SNR to decrease sensitivity**



# Alternative Problem Formulation

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$$\min_{\mathbf{w}} \mathbf{w}^H \mathbf{R} \mathbf{w} \quad \text{s.t.} \quad \mathbf{e}^H \mathbf{w} = 1$$
$$\text{and} \quad \|\mathbf{w}\|^2 = \mathbf{w}^H \mathbf{w} \leq \beta$$

- **Use Lagrange multipliers**

$$L = \mathbf{w}^H \mathbf{R} \mathbf{w}$$
$$+ \lambda_1^* [\mathbf{e}^H \mathbf{w} - 1] + \lambda_1 [\mathbf{e}^T \mathbf{w}^* - 1]$$
$$+ \lambda_2 [\mathbf{w}^H \mathbf{w} - \beta]$$



# Taking Gradients

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$$\begin{aligned} L = & \mathbf{w}^H \mathbf{R} \mathbf{w} \\ & + \lambda_1^* [\mathbf{e}^H \mathbf{w} - 1] + \lambda_1 [\mathbf{w}^H \mathbf{e} - 1] \\ & + \lambda_2 [\mathbf{w}^H \mathbf{w} - \beta] \end{aligned}$$

$$\nabla_{\mathbf{w}^*} L = \mathbf{R} \mathbf{w} + \lambda_1 \mathbf{e} + \lambda_2 \mathbf{w} = 0$$

- **If  $\lambda_2 = 0$ , get original MVDR  
(means  $\|\mathbf{w}\|^2 \leq \beta$  is inoperative)**





# Results for Alternate Formulation

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- **If  $\lambda_2 \neq 0$ , we have**

$$\mathbf{R}\mathbf{w} + \lambda_1\mathbf{e} + \lambda_2\mathbf{w} = \mathbf{0}$$

$$(\mathbf{R} + \lambda_2\mathbf{I})\mathbf{w} = -\lambda_1\mathbf{e}$$

$$\mathbf{w} = -\lambda_1(\mathbf{R} + \lambda_2\mathbf{I})^{-1}\mathbf{e}$$

- **Same sort of structure as before!**

