Introduction to Estimation Theory Part II

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Lecture 25

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Variance of ML Est. of Gaussian Mean

Jumping right in where we left off:

$$\mathrm{var}_\xi[\hat{\mu}_{ML}(\underline{y})] = \mathrm{var}_\xi\left[\frac{1}{L}\sum_{l=0}^{L-1}\underline{y}(l)\right]$$

$$= \frac{1}{L^2}\sum_{l=0}^{L-1}\mathrm{var}_\xi[\underline{y}(l)] = \frac{1}{L^2}L\sigma^2 = \frac{\sigma^2}{L}$$

Recall: $b_\xi(\hat{\mu}_{ML}) = 0$

Estimators of Gaussian Variance

$$\hat{\sigma}^2_{ML}(y) = \frac{1}{L}\sum_{l=0}^{L-1}[y(l) - \hat{\mu}_{ML}(y)]^2$$

$$b_\xi(\hat{\sigma}^2_{ML}) = -\sigma^2 / L$$

$$\sigma^2_{UB}(y) = \frac{1}{L-1}\sum_{l=0}^{L-1}[y(l) - \hat{\mu}_{ML}(y)]^2$$

$$b_\xi(\hat{\sigma}^2_{UB}) = 0$$

Vars. of Estimators of Gaussian Vars.

$$\mathrm{var}_{\xi}[\hat{\sigma}^2_{ML}(\underline{y})] = 2\frac{(\sigma^2)^2}{L}\left(\frac{L-1}{L}\right)$$

$$\mathrm{var}_{\xi}[\hat{\sigma}^2_{UB}(\underline{y})] = 2\frac{(\sigma^2)^2}{L}\left(\frac{L}{L-1}\right)$$

(from estimation theory notes by Al Hero, U. of Michigan)

Notion of Mean Squared Error

We will explore the idea of bias/variance tradeoffs in the MSE:

$$E_\xi[(\hat{\xi} - \xi)^2] = \mathrm{var}_\xi(\hat{\xi}) + b_\xi^2(\hat{\xi})$$

Bias and Covariance Matrices

$$\mathbf{b}_\xi(\hat{\boldsymbol{\xi}}) = E_\xi[\hat{\boldsymbol{\xi}}(\underline{y})] - \boldsymbol{\xi}$$

$$= E_\xi\left\{\begin{bmatrix} \hat{\mu}(\underline{y}) \\ \hat{\sigma}^2(\underline{y}) \end{bmatrix}\right\} - \begin{bmatrix} \mu \\ \sigma^2 \end{bmatrix}$$

$$\mathrm{cov}_\xi[\hat{\boldsymbol{\xi}}(\underline{y})] = E_\xi[\hat{\boldsymbol{\xi}}\hat{\boldsymbol{\xi}}^{\mathrm{T}}] - E_\xi[\hat{\boldsymbol{\xi}}]E_\xi[\hat{\boldsymbol{\xi}}^{\mathrm{T}}]$$

$$= \begin{bmatrix} E[\hat{\mu}^2] & E[\hat{\mu}\hat{\sigma}^2] \\ E[\hat{\mu}\hat{\sigma}^2] & E[(\hat{\sigma}^2)^2] \end{bmatrix} - \begin{bmatrix} E^2[\hat{\mu}] & E[\hat{\mu}]E[\hat{\sigma}^2] \\ E[\hat{\mu}]E[\hat{\sigma}^2] & E^2[\hat{\sigma}^2] \end{bmatrix}$$

Decomposition of MSE Matrix

$$E_\xi\{[\hat{\xi}(\underline{y})-\xi][\hat{\xi}(\underline{y})-\xi]^T]\}$$

$$=E_\xi[\hat{\xi}(\underline{y})\hat{\xi}^T(\underline{y})-\hat{\xi}(\underline{y})\xi^T-\xi\hat{\xi}^T(\underline{y})+\xi\xi^T]$$

$$=E[\hat{\xi}\hat{\xi}^T]-E[\hat{\xi}]\xi^T-\xi E[\hat{\xi}^T]+\xi\xi^T$$
$$+E[\hat{\xi}]E[\hat{\xi}^T]-E[\hat{\xi}]E[\hat{\xi}^T]$$

$$=E[\hat{\xi}\hat{\xi}^T]-E[\hat{\xi}]E[\hat{\xi}^T]$$
$$+E[\hat{\xi}]E[\hat{\xi}^T]-E[\hat{\xi}]\xi^T-\xi E[\hat{\xi}^T]+\xi\xi^T$$

Decomposition of MSE Matrix

$$E_{\xi}\{[\hat{\xi}(\underline{y})-\xi][\hat{\xi}(\underline{y})-\xi]^T]\}$$

$$= E[\hat{\xi}\hat{\xi}^T]-E[\hat{\xi}]E[\hat{\xi}^T]$$

$$+\underbrace{E[\hat{\xi}]E[\hat{\xi}^T]-E[\hat{\xi}]\xi^T-\xi E[\hat{\xi}^T]+\xi\xi^T}_{(E[\hat{\xi}]-\xi)(E[\hat{\xi}]-\xi)^T}$$

$$=\mathrm{cov}_{\xi}(\hat{\xi})+\mathbf{b}_{\xi}(\hat{\xi})\mathbf{b}_{\xi}^T(\hat{\xi})$$

In scalar case: $E_{\xi}[(\hat{\xi}-\xi)^2]=\mathrm{var}_{\xi}(\hat{\xi})+b_{\xi}^2(\hat{\xi})$

Composite MSE

$$MSE = \mathrm{tr}\left\{ E_{\xi}\{[\hat{\xi}(\underline{y}) - \xi][\hat{\xi}(\underline{y}) - \xi]^T]\}\right\}$$

$$= \mathrm{tr}\{\mathrm{cov}_{\xi}(\hat{\xi}) + \mathbf{b}_{\xi}(\hat{\xi})\mathbf{b}_{\xi}^T(\hat{\xi})\}$$

$$= \mathrm{tr}\{\mathrm{cov}_{\xi}(\hat{\xi})\} + \mathrm{tr}\{\mathbf{b}_{\xi}(\hat{\xi})\mathbf{b}_{\xi}^T(\hat{\xi})\}$$