"Stochastic Signal" Gaussian Model

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Lecture 26

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A Stochastic Gaussian Model

- ***N*** **sources in additive noise**

$$\underline{\mathbf{y}}(l) = \sum_{n=1}^{N_s} \mathbf{e}(\boldsymbol{\theta}_n)\underline{s}_n(l) + \underline{\mathbf{n}}(l)$$

$$\underline{\mathbf{n}} \sim CN(0, \mathbf{K}_{\mathbf{n}})$$

$$\underline{\mathbf{s}} = \begin{bmatrix} \underline{s}_1 \\ \vdots \\ \underline{s}_{N_s} \end{bmatrix} \sim CN(0, \mathbf{K}_{\mathbf{s}})$$

Uncorrelated Sources

- Sources may be uncorrelated (usual model we've used in class)…

$$\mathbf{K}_\mathbf{s} = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{N_s}^2 \end{bmatrix}$$

Correlated Sources

- Or they may be correlated (for instance, due to multipath)

$$\mathbf{K_s} = \begin{bmatrix} \sigma_1^2 & c_{12}\sigma_1\sigma_2 & \cdots & c_{1N_s}\sigma_1\sigma_{N_s} \\ c_{12}^*\sigma_1\sigma_2 & \sigma_2^2 & & c_{2N}\sigma_2\sigma_{N_s} \\ \vdots & & \ddots & \vdots \\ c_{1N_s}^*\sigma_1\sigma_{N_s} & c_{2N_s}^*\sigma_2\sigma_{N_s} & \cdots & \sigma_{N_s}^2 \end{bmatrix}$$

Fun Fact from Probability

- **Independent $\Rightarrow$ uncorrelated**
- **Uncorrelated $\not\Rightarrow$ independent**
  (in general)
- **Uncorrelated+Gaussian $\Rightarrow$ independent**

Distribution of the Data

- *N* sources in additive noise

$$\underline{\mathbf{y}} \sim CN(0, \mathbf{K}_{\mathbf{y}})$$

$$\mathbf{K}_{\mathbf{y}} = \mathbf{D}(\boldsymbol{\Theta})\mathbf{K}_{\mathbf{s}}\mathbf{D}^{H}(\boldsymbol{\Theta}) + \mathbf{K}_{\mathbf{n}}$$

$$\mathbf{D}(\boldsymbol{\Theta}) = \left[\mathbf{e}(\boldsymbol{\theta}_1) \;\cdots\; \mathbf{e}(\boldsymbol{\theta}_{N_s})\right]$$

$$\boldsymbol{\Theta} = \left[\boldsymbol{\theta}_1 \;\cdots\; \boldsymbol{\theta}_{N_s}\right]$$

Data Covariance (1)

$$\underline{\mathbf{y}}(l) = \mathbf{D}(\Theta)\underline{\mathbf{s}}(l) + \underline{\mathbf{n}}(l)$$

$$\mathbf{K}_{\mathbf{y}} = E[\underline{\mathbf{y}}\underline{\mathbf{y}}^H]$$

$$= E[\{\mathbf{D}(\Theta)\underline{\mathbf{s}} + \underline{\mathbf{n}}\}\{\mathbf{D}(\Theta)\underline{\mathbf{s}} + \underline{\mathbf{n}}\}^H]$$

$$= E[\mathbf{D}(\Theta)\underline{\mathbf{s}}\underline{\mathbf{s}}^H\mathbf{D}^H(\Theta)] + E[\underline{\mathbf{n}}\underline{\mathbf{n}}^H]$$

$$+ E[\mathbf{D}(\Theta)\underline{\mathbf{s}}\underline{\mathbf{n}}^H] + E[\underline{\mathbf{n}}\underline{\mathbf{s}}^H\mathbf{D}^H(\Theta)]$$

$$= E[\mathbf{D}(\Theta)\underline{\mathbf{s}}\underline{\mathbf{s}}^H\mathbf{D}^H(\Theta)] + E[\underline{\mathbf{n}}\underline{\mathbf{n}}^H]$$

Data Covariance (2)

$$\mathbf{K}_{\mathbf{y}} = E[\mathbf{D}(\Theta)\underline{\mathbf{s}}\underline{\mathbf{s}}^H\mathbf{D}^H(\Theta)] + E[\underline{\mathbf{n}}\underline{\mathbf{n}}^H]$$

$$= \mathbf{D}(\Theta)E[\underline{\mathbf{s}}\underline{\mathbf{s}}^H]\mathbf{D}^H(\Theta) + E[\underline{\mathbf{n}}\underline{\mathbf{n}}^H]$$

$$= \mathbf{D}(\Theta)\mathbf{K}_{\mathbf{s}}\mathbf{D}^H(\Theta) + \mathbf{K}_{\mathbf{n}}$$

Probability Density of the Data

$$\underline{\mathbf{y}} \sim CN(0, \mathbf{K}_{\mathbf{y}})$$

- **Circular (Goodman) zero-mean Gaussian density for *L* independent snapshots:** $p(\mathbf{y}) =$

$$\frac{1}{(\pi^{M} \det \mathbf{K}_{\mathbf{y}})^{L}} \exp\left[-\sum_{l=0}^{L-1} \mathbf{y}^{H}(l) \mathbf{K}_{\mathbf{y}}^{-1} \mathbf{y}(l)\right]$$

Loglikelihood Calculation

- **Loglikelihood** $\ln p(\mathbf{y}) =$

$$-LM\ln\pi - L\ln\det\mathbf{K}_{\mathbf{y}}$$

$$-\sum_{l=0}^{L-1}\mathbf{y}^{H}(l)\mathbf{K}_{\mathbf{y}}^{-1}\mathbf{y}(l)$$

$$L\,\mathrm{tr}\left[\frac{1}{L}\sum_{l=0}^{L-1}\mathbf{y}^{H}(l)\mathbf{K}_{\mathbf{y}}^{-1}\mathbf{y}(l)\right]$$

Trace Rearrangement Trick

$$\mathrm{tr}\left[\frac{1}{L}\sum_{l=0}^{L-1}\mathbf{y}^H(l)\mathbf{K}_{\mathbf{y}}^{-1}\mathbf{y}(l)\right]$$

$$=\frac{1}{L}\sum_{l=0}^{L-1}\mathrm{tr}\left[\mathbf{y}^H(l)\mathbf{K}_{\mathbf{y}}^{-1}\mathbf{y}(l)\right]$$

$$=\mathrm{tr}\left[\underbrace{\frac{1}{L}\sum_{l=0}^{L-1}\mathbf{y}(l)\mathbf{y}^H(l)}_{\equiv\hat{\mathbf{K}}_{\mathbf{y}}}\mathbf{K}_{\mathbf{y}}^{-1}\right]=\mathrm{tr}\left[\hat{\mathbf{K}}_{\mathbf{y}}\mathbf{K}_{\mathbf{y}}^{-1}\right]$$

Maximum-Likelihood Estimation

- **Goal: maximize**

$$-L \ln \det \mathbf{K}_{\mathbf{y}}(\xi) - L \, \mathrm{tr}\left[\hat{\mathbf{K}}_{\mathbf{y}} \mathbf{K}_{\mathbf{y}}^{-1}(\xi)\right]$$

where

$$\mathbf{K}_{\mathbf{y}}(\xi) = \mathbf{D}(\Theta)\mathbf{K}_{\mathbf{s}}\mathbf{D}^{H}(\Theta) + \mathbf{K}_{\mathbf{n}}$$

over ξ, which represents all the parameters (angles, signal powers, correlation coefficients)

Important Caveat

- The maximization must be over the feasible set, i.e. $\mathbf{K}_{\mathbf{y}}(\xi)$ must be a legitimate covariance matrix (nonnegative definite)

- To make sense at all, we need

$$\dim(\xi) \le M^2$$

How Can We Maximize?

- **No closed form solution in general**
- **Still an open research problem!**
- **One approach: try an expectation-maximization algorithm (sometimes ECE7251 covers EM algorithms)**

M.I. Miller and D.R. Fuhrmann, "Maximum-Likelihood Narrow-Band Direction Finding and the EM Algorithm," *IEEE Trans. ASSP*, 38 (9), Sept. 1990, pp. 1560-1577.

Adding More Detail

- **Could extend model with:**
  - Unknown noise covariance, for instance

$$\mathbf{K_n} = \mathbf{I}\underbrace{\sigma_n^2}_{\text{Make a parameter}}$$

  - Uncertainties in sensor positions

$$x = \breve{x} + \underbrace{\delta x}_{\text{Make a parameter}}$$

Alternative: Least Squares Hack

- **Find feasible parameter ξ that minimizes**

$$\left\| \hat{\mathbf{K}}_{\mathbf{y}} - \mathbf{K}_{\mathbf{y}}(\xi) \right\|^2$$

- **Nonlinear Least Squares problem**
  - Large literature on numerical techniques
  - Can use MATLAB Optimization Toolbox
- **Discussed in Sec. 7.1.2 of J&D**