
Introduction to Cramér-Rao Bounds

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Basic Univariate Cramér-Rao Bound

- For any unbiased estimator $\hat{\xi}$

$$MSE = \text{var}_{\xi} [\hat{\xi}(\underline{y})] \geq 1/F(\xi)$$

where the Fisher Information is

$$F(\xi) = E_{\xi} \left\{ \left[\frac{d}{d\tilde{\xi}} \ln p(\underline{y}; \tilde{\xi}) \right]^2 \Big|_{\tilde{\xi} = \xi} \right\}$$



Two Ways to Compute the F.I.

- Under some common conditions,

$$F(\xi) = E_{\xi} \left\{ \left[\frac{d}{d\tilde{\xi}} \ln p(\underline{y}; \tilde{\xi}) \right]^2 \Big|_{\tilde{\xi} = \xi} \right\}$$
$$= -E_{\xi} \left\{ \frac{d^2}{d\tilde{\xi}^2} \ln p(\underline{y}; \tilde{\xi}) \Big|_{\tilde{\xi} = \xi} \right\}$$



Gaussian Example, 1st Derivative (1)

- Consider F.I. for one data point for

$$\underline{y} \sim N(f(\xi), \sigma^2)$$

$$\frac{d}{d\xi} \ln p(y; \xi) =$$

$$\frac{d}{d\xi} \ln \left(\frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{[y - f(\xi)]^2}{2\sigma^2} \right\} \right)$$



Gaussian Example, 1st Derivative (2)

$$\begin{aligned} & \frac{d}{d\xi} \ln \left(\frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{[y - f(\xi)]^2}{2\sigma^2} \right\} \right) \\ &= \frac{d}{d\xi} \left\{ \frac{[y - f(\xi)]^2}{2\sigma^2} \right\} \\ &= \frac{[y - f(\xi)] df(\xi)}{\sigma^2 d\xi} \end{aligned}$$



Gaussian Example, 2st Derivative (1)

$$\begin{aligned} & \frac{d^2}{d\xi^2} \ln p(y; \xi) \\ &= \frac{d}{d\xi} \left\{ \frac{[y - f(\xi)] df(\xi)}{\sigma^2 d\xi} \right\} \\ &= \frac{1}{\sigma^2} \left\{ [y - f(\xi)] \frac{d^2 f(\xi)}{d\xi^2} - \left[\frac{df(\xi)}{d\xi} \right]^2 \right\} \end{aligned}$$



Using Derivative-Squared Version (1)

$$F(\xi) = E_{\xi} \left\{ \left[\frac{\partial}{\partial \xi} \ln p(\underline{y}; \xi) \right]^2 \right\}$$
$$= E_{\xi} \left\{ \left[\frac{[\underline{y} - f(\xi)]}{\sigma^2} \frac{df(\xi)}{d\xi} \right]^2 \right\}$$



Using Derivative-Squared Version (2)

$$F(\xi) = \left[\frac{df(\xi)}{d\xi} \right]^2 \frac{E_{\xi} \{ [y - f(\xi)]^2 \}}{(\sigma^2)^2}$$
$$= \left[\frac{df(\xi)}{d\xi} \right]^2 / \sigma^2$$



Using Double-Derivative Version (1)

$$F(\xi) = -E_{\xi} \left\{ \frac{\partial^2}{\partial \xi^2} \ln p(\underline{y}; \xi) \right\}$$

$$= -\frac{1}{\sigma^2} E_{\xi} \left\{ [\underline{y} - f(\xi)] \frac{d^2 f(\xi)}{d\xi^2} - \left[\frac{df(\xi)}{d\xi} \right]^2 \right\}$$



Using Double-Derivative Version (2)

$$-\frac{1}{\sigma^2} \left\{ E_{\xi} [y - f(\xi)] \frac{d^2 f(\xi)}{d\xi^2} - \left[\frac{df(\xi)}{d\xi} \right]^2 \right\}$$
$$= \left[\frac{df(\xi)}{d\xi} \right]^2 / \sigma^2$$



Independent, Identically Dist. Data

- If you have L i.i.d. data points, the F.I. is just L times the F.I. for one data point:

$$F(\xi) = LF_1(\xi)$$

- For our ex., $F(\xi) = L \left[\frac{df(\xi)}{d\xi} \right]^2 / \sigma^2$

$$\text{var}_{\xi} [\hat{\xi}(\underline{y})] \geq \sigma^2 / \left\{ L \left[\frac{df(\xi)}{d\xi} \right]^2 \right\}$$



Basic Multivariate Cramér-Rao Bound

- For any unbiased estimator $\hat{\xi}$

$$\text{COV}_{\xi} [\hat{\xi}(\underline{y})] \geq \mathbf{F}^{-1}(\hat{\xi})$$

($A \geq B$ means $A - B$ is nonneg. def.)

where the entries of
of the F.I. matrix are

$$\mathbf{F}_{rc} = E_{\xi} \left\{ \left[\frac{\partial}{\partial \xi_r} \ln p(\underline{y}; \xi) \right] \left[\frac{\partial}{\partial \xi_c} \ln p(\underline{y}; \xi) \right] \right\}$$



Nonnegative Definiteness

- **If A is nonnegative definite, then**
 - $\mathbf{x}^T \mathbf{A} \mathbf{x} \geq 0$ for any real vector \mathbf{x}
 - Eigenvalues of A are nonnegative
- **Useful consequences: if $A \geq B$,**
 - Diagonals dominated: $A_{ii} \geq B_{ii}$
 - Does not mean $A_{rc} \geq B_{rc}$ in general!
 - Total sum property:
$$\sum_{r,c} A_{rc} \geq \sum_{r,c} B_{rc}$$



Two Ways to Compute the F.I.M.

- Under some common conditions:

$$\mathbf{F}_{rc} = E_{\xi} \left\{ \left[\frac{\partial}{\partial \tilde{\xi}_r} \ln p(\underline{y}; \tilde{\xi}) \right] \left[\frac{\partial}{\partial \tilde{\xi}_c} \ln p(\underline{y}; \tilde{\xi}) \right] \right\}$$
$$= -E_{\xi} \left\{ \frac{\partial^2}{\partial \xi_r \partial \xi_c} \ln p(\underline{y}; \xi) \right\}$$



Efficiency

- An unbiased estimator that achieves the CR bound with equality is called **efficient**
- If an efficient estimator exists, the maximum-likelihood estimator is it!
- Even if an estimator isn't efficient, it may be **asymptotically efficient**
- ML estimators are asymptotically efficient



CR Bounds for Biased Estimators

- **Both univariate and multivariate versions for biased estimators exist...**
- **...but they require taking derivatives of the bias...**
- **...which requires you have an analytic form for the bias...**
- **...which you almost never have...**
- **...so it's rarely used; people usually just compute the CR bound for the unbiased estimator and then handwave**

