Transformations of Cramér-Rao Bounds

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Lecture 31

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Transformation of ML Estimates

- **Suppose we find a maximum likelihood estimate**

$$\hat{\gamma}_{ML} = \arg\max_{\gamma} \ln p(y;\gamma)$$

- **But what we really want is to find a ML estimate in terms of $\phi = g(\gamma)$ (assume *g* is continuous & invertible)**
- **It turns out ML estimation "commutes:"** $\hat{\phi}_{ML} = g(\hat{\gamma}_{ML})$

Properties May Not Commute

$$\hat{\phi} = g(\hat{\gamma})$$

- **If $\hat{\gamma}$ is unbiased/efficient, that doesn't necessarily imply that $\hat{\phi}$ is unbiased/efficient**
- **If using ML estimators, $\hat{\phi}$ will be at least be *asymptotically* unbiased/ efficient**

Transformations of Fisher Information

$$F_\phi(\phi) = \left[\frac{dg(\gamma)}{d\gamma}\right]^{-2} F_\gamma(\gamma)\Bigg|_{\gamma=g^{-1}(\phi)}$$

$$= \left[\frac{dg^{-1}(\phi)}{d\phi}\right]^{2} F_\gamma(\gamma)\Bigg|_{\gamma=g^{-1}(\phi)}$$

Transformations of CR Bounds

$$CRB_\phi(\phi) = \left[\frac{dg(\gamma)}{d\gamma}\right]^2 CRB_\gamma(\gamma)\Bigg|_{\gamma = g^{-1}(\phi)}$$

$$= \left[\frac{dg^{-1}(\phi)}{d\phi}\right]^{-2} CRB_\gamma(\gamma)\Bigg|_{\gamma = g^{-1}(\phi)}$$

Our Example Transformation

$$\gamma = g^{-1}(\phi) = \frac{2\pi}{\lambda} d \sin(\phi)$$

$$\phi = g(\gamma) = \sin^{-1}\left(\gamma \frac{\lambda}{2\pi d}\right)$$

The First Way (1)

$$\frac{dg(\gamma)}{d\gamma} = \frac{d}{d\gamma}\left\{\sin^{-1}\left(\gamma\frac{\lambda}{2\pi d}\right)\right\}$$

$$= \frac{\lambda}{2\pi d}\frac{1}{\sqrt{1-\left[\gamma\frac{\lambda}{2\pi d}\right]^2}}$$

The First Way (2)

$$\left.\frac{dg(\gamma)}{d\gamma}\right|_{\gamma=f^{-1}(\phi)} = \frac{\lambda}{2\pi d}\frac{1}{\left.\sqrt{1-\left[\gamma\frac{\lambda}{2\pi d}\right]^2}\right|_{\gamma=\frac{2\pi}{\lambda}d\sin(\phi)}}$$

$$= \frac{\lambda}{2\pi d}\frac{1}{\sqrt{1-\sin^2(\phi)}} = \frac{\lambda}{2\pi d}\frac{1}{\cos(\phi)}$$

The Second Way

$$\frac{dg^{-1}(\phi)}{d\phi} = \frac{d}{d\phi}\left\{\frac{2\pi}{\lambda} d\sin(\phi)\right\}$$

$$= \frac{2\pi}{\lambda} d\cos(\phi)$$

Hopefully Get the Same Answer

$$\left[\frac{dg(\gamma)}{d\gamma}\right]^2\Bigg|_{\gamma=f^{-1}(\phi)} = \left(\frac{\lambda}{2\pi d}\right)^2 \frac{1}{\cos^2(\phi)}$$

$$\left[\frac{dg^{-1}(\phi)}{d\phi}\right]^{-2} = \left[\frac{2\pi}{\lambda} d\cos(\phi)\right]^{-2} = \left(\frac{\lambda}{2\pi d}\right)^2 \frac{1}{\cos^2(\phi)}$$

Putting it All Together

From last lecture: $$\mathrm{var}_{\gamma}[\hat{\gamma}(\underline{y})] \gtrsim \frac{6}{M^3 L(SNR)}$$

$$\mathrm{var}_{\phi}[\hat{\phi}(\underline{y})] \gtrsim \frac{6}{M^3 L(SNR)} \left(\frac{\lambda}{2\pi d} \right)^2 \frac{1}{\cos^2(\phi)}$$

$$\mathrm{var}_{0}[\hat{\phi}(\underline{y})] \gtrsim \frac{6}{M^3 L(SNR)} \qquad \mathrm{var}_{\pi/2}[\hat{\phi}(\underline{y})] \gtrsim \infty$$

One Random Tidbit for This Example

From (5.9) of Stoica & Nehorai:

$$\frac{\mathrm{var}_\gamma[\hat{\gamma}_{ML}(\underline{y})]}{CRB(\gamma)} = 1 + \frac{1}{M(SNR)}$$

$$SNR = \frac{1}{L}\sum_{l=0}^{L-1} \left|s(l)\right|^2 \Big/ \sigma^2$$

$\hat{\gamma}_{ML}$ **is inefficient for finite *M*, even if** $L \to \infty$

Defining a Gradient

$$\nabla_{\xi}\{g(\boldsymbol{\xi})\} = \begin{bmatrix} \dfrac{\partial g_1(\boldsymbol{\xi})}{\partial \xi_1} & \cdots & \dfrac{\partial g_N(\boldsymbol{\xi})}{\partial \xi_1} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_1(\boldsymbol{\xi})}{\partial \xi_N} & \cdots & \dfrac{\partial g_N(\boldsymbol{\xi})}{\partial \xi_N} \end{bmatrix}$$

Multivariate Transformations

- **Suppose** $\alpha = g(\xi)$, **continuous and invertible**
- **Nice result from p. 230 of Scharf:**

$$F_\alpha(\alpha) = [\nabla_\xi \{g(\xi)\}]^{-1} F_\xi(\xi) [\nabla_\xi^T \{g(\xi)\}]^{-1} \Big|_{\xi = g^{-1}(\alpha)}$$

$$\mathrm{cov}_\alpha\{\hat{\alpha}(\underline{y})\} \geq \nabla_\xi^T \{g(\xi)\} F_\xi^{-1}(\xi) [\nabla_\xi \{g(\xi)\}] \Big|_{\xi = g^{-1}(\alpha)}$$

- **Another version:**

$$F_\alpha(\alpha) = \nabla_\alpha \{g^{-1}(\alpha)\} F_\xi(\xi) \nabla_\alpha^T \{g^{-1}(\alpha)\} \Big|_{\xi = g^{-1}(\alpha)}$$

$$\mathrm{cov}_\alpha\{\hat{\alpha}(\underline{y})\} \geq [\nabla_\alpha^T \{g^{-1}(\alpha)\}]^{-1} F_\xi^{-1}(\xi) [\nabla_\alpha \{g^{-1}(\alpha)\}]^{-1} \Big|_{\xi = g^{-1}(\alpha)}$$