

## Transformations of Cramér-Rao Bounds

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Lecture 31

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## Transformation of ML Estimates

- Suppose we find a maximum likelihood estimate

$$\hat{\gamma}_{ML} = \arg \max_{\gamma} \ln p(y; \gamma)$$

- But what we really want is to find a ML estimate in terms of  $\phi = g(\gamma)$  (assume  $g$  is continuous & invertible)

- It turns out ML estimation

“commutes:”  $\hat{\phi}_{ML} = g(\hat{\gamma}_{ML})$

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## Properties May Not Commute

$$\hat{\phi} = g(\hat{\gamma})$$

- If  $\hat{\gamma}$  is unbiased/efficient, that doesn't necessarily imply that  $\hat{\phi}$  is unbiased/efficient
- If using ML estimators,  $\hat{\phi}$  will be at least be *asymptotically* unbiased/efficient

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## Transformations of Fisher Information

$$\begin{aligned} F_{\phi}(\phi) &= \left[ \frac{dg(\gamma)}{d\gamma} \right]^{-2} F_{\gamma}(\gamma) \Big|_{\gamma=g^{-1}(\phi)} \\ &= \left[ \frac{dg^{-1}(\phi)}{d\phi} \right]^2 F_{\gamma}(\gamma) \Big|_{\gamma=g^{-1}(\phi)} \end{aligned}$$

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### Transformations of CR Bounds

$$\begin{aligned}
 CRB_{\phi}(\phi) &= \left[ \frac{dg(\gamma)}{d\gamma} \right]^2 CRB_{\gamma}(\gamma) \Big|_{\gamma=g^{-1}(\phi)} \\
 &= \left[ \frac{dg^{-1}(\phi)}{d\phi} \right]^{-2} CRB_{\gamma}(\gamma) \Big|_{\gamma=g^{-1}(\phi)}
 \end{aligned}$$

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### Our Example Transformation

$$\gamma = g^{-1}(\phi) = \frac{2\pi}{\lambda} d \sin(\phi)$$

$$\phi = g(\gamma) = \sin^{-1}\left(\gamma \frac{\lambda}{2\pi d}\right)$$

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### The First Way (1)

$$\begin{aligned}
 \frac{dg(\gamma)}{d\gamma} &= \frac{d}{d\gamma} \left\{ \sin^{-1}\left(\gamma \frac{\lambda}{2\pi d}\right) \right\} \\
 &= \frac{\lambda}{2\pi d} \frac{1}{\sqrt{1 - \left[\gamma \frac{\lambda}{2\pi d}\right]^2}}
 \end{aligned}$$

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### The First Way (2)

$$\begin{aligned}
 \frac{dg(\gamma)}{d\gamma} \Big|_{\gamma=f^{-1}(\phi)} &= \frac{\lambda}{2\pi d} \frac{1}{\sqrt{1 - \left[\gamma \frac{\lambda}{2\pi d}\right]^2}} \Big|_{\gamma=\frac{2\pi}{\lambda} d \sin(\phi)} \\
 &= \frac{\lambda}{2\pi d} \frac{1}{\sqrt{1 - \sin^2(\phi)}} = \frac{\lambda}{2\pi d} \frac{1}{\cos(\phi)}
 \end{aligned}$$

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### The Second Way

$$\begin{aligned} \frac{dg^{-1}(\phi)}{d\phi} &= \frac{d}{d\phi} \left\{ \frac{2\pi}{\lambda} d \sin(\phi) \right\} \\ &= \frac{2\pi}{\lambda} d \cos(\phi) \end{aligned}$$

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### Hopefully Get the Same Answer

$$\begin{aligned} \left[ \frac{dg(\gamma)}{d\gamma} \right]^2 \Big|_{\gamma=f^{-1}(\phi)} &= \left( \frac{\lambda}{2\pi d} \right)^2 \frac{1}{\cos^2(\phi)} \\ \left[ \frac{dg^{-1}(\phi)}{d\phi} \right]^{-2} &= \left[ \frac{2\pi}{\lambda} d \cos(\phi) \right]^{-2} = \left( \frac{\lambda}{2\pi d} \right)^2 \frac{1}{\cos^2(\phi)} \end{aligned}$$

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### Putting it All Together

From last lecture:  $\text{var}_\gamma[\hat{\gamma}(\underline{y})] \gtrsim \frac{6}{M^3 L(SNR)}$

$$\text{var}_\phi[\hat{\phi}(\underline{y})] \gtrsim \frac{6}{M^3 L(SNR)} \left( \frac{\lambda}{2\pi d} \right)^2 \frac{1}{\cos^2(\phi)}$$

$$\text{var}_0[\hat{\phi}(\underline{y})] \gtrsim \frac{6}{M^3 L(SNR)} \quad \text{var}_{\pi/2}[\hat{\phi}(\underline{y})] \gtrsim \infty$$

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### One Random Tidbit for This Example

From (5.9) of Stoica & Nehorai:

$$\frac{\text{var}_\gamma[\hat{\gamma}_{ML}(\underline{y})]}{CRB(\gamma)} = 1 + \frac{1}{M(SNR)}$$

$$SNR = \frac{1}{L} \sum_{l=0}^{L-1} |s(l)|^2 / \sigma^2$$

$\hat{\gamma}_{ML}$  is inefficient for finite  $M$ ,  
even if  $L \rightarrow \infty$

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## Defining a Gradient

$$\nabla_{\xi} \{g(\xi)\} = \begin{bmatrix} \frac{\partial g_1(\xi)}{\partial \xi_1} & \dots & \frac{\partial g_N(\xi)}{\partial \xi_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_1(\xi)}{\partial \xi_N} & \dots & \frac{\partial g_N(\xi)}{\partial \xi_N} \end{bmatrix}$$

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## Multivariate Transformations

- Suppose  $\alpha = g(\xi)$ , continuous and invertible

- Nice result from p. 230 of Scharf:

$$F_{\alpha}(\alpha) = [\nabla_{\xi} \{g(\xi)\}]^{-1} F_{\xi}(\xi) [\nabla_{\xi} \{g(\xi)\}]^{-1} \Big|_{\xi=g^{-1}(\alpha)}$$

$$\text{cov}_{\alpha} \{\hat{\alpha}(\underline{y})\} \geq \nabla_{\xi}^T \{g(\xi)\} F_{\xi}^{-1}(\xi) [\nabla_{\xi} \{g(\xi)\}] \Big|_{\xi=g^{-1}(\alpha)}$$

- Another version:

$$F_{\alpha}(\alpha) = \nabla_{\alpha} \{g^{-1}(\alpha)\} F_{\xi}(\xi) \nabla_{\alpha}^T \{g^{-1}(\alpha)\} \Big|_{\xi=g^{-1}(\alpha)}$$

$$\text{cov}_{\alpha} \{\hat{\alpha}(\underline{y})\} \geq [\nabla_{\alpha}^T \{g^{-1}(\alpha)\}]^{-1} F_{\xi}^{-1}(\xi) [\nabla_{\alpha} \{g^{-1}(\alpha)\}]^{-1} \Big|_{\xi=g^{-1}(\alpha)}$$

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