Model Order Estimation

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Lecture 32

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Where We Are in J&D

- **Section 7.3.5**

Recall the Stochastic Gaussian Model

$$\underline{\mathbf{y}} \sim CN(0, \mathbf{K}_{\mathbf{y}})$$

- **Likelihood density:** $p(\mathbf{y}) =$

$$\frac{1}{[\pi^M \det \mathbf{K}_{\mathbf{y}}(\xi)]^L} \exp\left[-\sum_{l=0}^{L-1} \mathbf{y}^H(l) \mathbf{K}_{\mathbf{y}}^{-1}(\xi) \mathbf{y}(l)\right]$$

- **Loglikelihood:**

$$-L \ln \det \mathbf{K}_{\mathbf{y}}(\xi) - L \operatorname{tr}\left[\hat{\mathbf{K}}_{\mathbf{y}} \mathbf{K}_{\mathbf{y}}^{-1}(\xi)\right]$$

Highly Structured Covariance Model

- For ML direction finding, we used a rather specific model:

$$\mathbf{K}_{\mathbf{y}}(\xi) = \mathbf{D}(\Theta)\mathbf{K}_{\mathbf{s}}\mathbf{D}^{H}(\Theta) + \mathbf{K}_{\mathbf{n}}$$

where ξ represents all the parameters (angles, signal powers, correlation coefficients)

Less Structured Covariance Model

- For model order estimation, we will do something more in the spirit of MUSIC, EV method, ESPRIT, etc.
- Suppose $\mathbf{K}_{\mathbf{y}}$ has eigenvalues

$$\lambda_1, \lambda_2, \ldots, \lambda_{N_s}, \sigma^2, \ldots, \sigma^2$$

- Not putting any particular structure on the eigenvectors (unlike ML model for direction finding)

ML Estimates of Eigenvalues

- **ML estimate of sigma+noise space eigenvalues are just the corresponding eigenvalues of $\hat{\mathbf{K}}_{\mathbf{y}}$**

$$\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_{N_s}$$

- **ML estimate of noise variance is**

$$\hat{\sigma}^2 = \frac{1}{M - N_s} \sum_{i=N_s+1}^{M} \hat{\lambda}_i$$

The Trace Term is "Constant"

$$-L\,\mathrm{tr}\left[\hat{\mathbf{K}}_{\mathbf{y}}\mathbf{K}_{\mathbf{y}}^{-1}(\xi)\right]$$

$$=-L\left[\sum_{i=1}^{N_s}\hat{\lambda}_i\frac{1}{\hat{\lambda}_i}+\sum_{i=N_s+1}^{M}\hat{\lambda}_i\frac{1}{\frac{1}{M-N_s}\sum_{k=N_s+1}^{M}\hat{\lambda}_k}\right]$$

$$=-L[N_s+M-N_s]=-LM$$

The Determinant Term (1)

$$-L \ln \det \mathbf{K}_{\mathbf{y}}(\xi)$$

$$= -L \ln \left[\left(\prod_{i=1}^{N_s} \hat{\lambda}_i \right) \left(\frac{1}{M - N_s} \sum_{i=N_s+1}^{M} \hat{\lambda}_i \right)^{M-N_s} \right]$$

$$\stackrel{c}{=} -L \ln \left[\left(\prod_{i=1}^{M} \frac{1}{\hat{\lambda}_i} \right) \left(\prod_{i=1}^{N_s} \hat{\lambda}_i \right) \left(\frac{1}{M - N_s} \sum_{i=N_s+1}^{M} \hat{\lambda}_i \right)^{M-N_s} \right]$$

The Determinant Term (2)

$$-L\ln\left[\left(\prod_{i=1}^{M}\frac{1}{\hat{\lambda}_i}\right)\left(\prod_{i=1}^{N_s}\hat{\lambda}_i\right)\left(\frac{1}{M-N_s}\sum_{i=N_s+1}^{M}\hat{\lambda}_i\right)^{M-N_s}\right]$$

$$=-L\ln\left[\left(\prod_{i=N_s+1}^{M}\frac{1}{\hat{\lambda}_i}\right)\left(\frac{1}{M-N_s}\sum_{i=N_s+1}^{M}\hat{\lambda}_i\right)^{M-N_s}\right]$$

The Determinant Term (3)

$$L \ln \left[\frac{\prod_{i=N_s+1}^{M} \hat{\lambda}_i}{\left(\frac{1}{M-N_s} \sum_{i=N_s+1}^{M} \hat{\lambda}_i \right)^{M-N_s}} \right]$$

The Determinant Term (4)

$$L \ln \left[\frac{\left(\prod_{i=N_s+1}^{M} \hat{\lambda}_i \right)^{\frac{1}{M-N_s}}}{\frac{1}{M-N_s} \sum_{i=N_s+1}^{M} \hat{\lambda}_i} \right]^{M-N_s}$$

The Determinant Term (5)

- **Goal: Find N_s that maximizes**

$$\ell = L(M - N_s)\ln\left[\frac{\left(\prod_{i=N_s+1}^{M}\hat{\lambda}_i\right)^{\frac{1}{M-N_s}}}{\frac{1}{M-N_s}\sum_{i=N_s+1}^{M}\hat{\lambda}_i}\right]$$

Uh Oh!

- **Problem: ML is greedy in model order estimation tasks!**

$$\hat{N}_s = M$$

- **Solution: subtract a penalty term and maximize the penalized likelihood**

$$AIC : \ell - \#$$

$$BIC : \ell - \frac{\#}{2} \ln L$$

- **Where # is the number of “free parameters”**

Counting the Free Parameters (1)

- N_s complex eigenvectors: $2MN_s$
- N_s real eigenvalues
- One real noise variance

$$2MN_s + N_s + 1$$

- But eigenvectors have unit norm, so subtract N_s

$$2MN_s + 1$$

$$\begin{bmatrix} X & X & X \\ X & X & X \\ X & X & X \\ X & X & X \\ X & X & X \end{bmatrix}$$

Counting the Free Parameters (2)

- **From last slide:**

$$2MN_s + 1$$

- **Since eigenvectors are orthogonal, subtract**

$$2\sum_{i=1}^{N_s - 1} i = N_s(N_s - 1)$$

yielding

$$\# = 2MN_s + 1 - N_s(N_s - 1)$$
$$= (2M - N_s + 1)N_s + 1$$

$$\begin{bmatrix} X & X & X \\ X & X & X \\ X & X & X \\ X & X & \\ X & & \end{bmatrix}$$