


Model Order Estimation

**ECE 6279: Spatial Array Processing
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Lecture 32**

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Where We Are in J&D

- **Section 7.3.5**

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Recall the Stochastic Gaussian Model

$\underline{\mathbf{y}} \sim CN(0, \mathbf{K}_y)$


- **Likelihood density:** $p(\mathbf{y}) =$

$$\frac{1}{[\pi^M \det \mathbf{K}_y(\xi)]^L} \exp \left[- \sum_{l=0}^{L-1} \mathbf{y}^H(l) \mathbf{K}_y^{-1}(\xi) \mathbf{y}(l) \right]$$

- **Loglikelihood:**

$$-L \ln \det \mathbf{K}_y(\xi) - L \operatorname{tr} \left[\hat{\mathbf{K}}_y \mathbf{K}_y^{-1}(\xi) \right]$$

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Highly Structured Covariance Model

- **For ML direction finding, we used a rather specific model:**

$$\mathbf{K}_y(\xi) = \mathbf{D}(\Theta) \mathbf{K}_s \mathbf{D}^H(\Theta) + \mathbf{K}_n$$

where ξ represents all the parameters (angles, signal powers, correlation coefficients)

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Less Structured Covariance Model

- For model order estimation, we will do something more in the spirit of MUSIC, EV method, ESPRIT, etc.

- Suppose \mathbf{K}_y has eigenvalues

$$\lambda_1, \lambda_2, \dots, \lambda_{N_s}, \sigma^2, \dots, \sigma^2$$

- Not putting any particular structure on the eigenvectors (unlike ML model for direction finding)

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ML Estimates of Eigenvalues

- ML estimate of sigma+noise space eigenvalues are just the corresponding eigenvalues of $\hat{\mathbf{K}}_y$

$$\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_{N_s}$$

- ML estimate of noise variance is

$$\hat{\sigma}^2 = \frac{1}{M - N_s} \sum_{i=N_s+1}^M \hat{\lambda}_i$$

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The Trace Term is “Constant”

$$-L \text{tr}[\hat{\mathbf{K}}_y \mathbf{K}_y^{-1}(\xi)]$$

$$= -L \left[\sum_{i=1}^{N_s} \hat{\lambda}_i \frac{1}{\hat{\lambda}_i} + \sum_{i=N_s+1}^M \hat{\lambda}_i \frac{1}{\frac{1}{M - N_s} \sum_{k=N_s+1}^M \hat{\lambda}_k} \right]$$

$$= -L[N_s + M - N_s] = -LM$$

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The Determinant Term (1)

$$-L \ln \det \mathbf{K}_y(\xi)$$

$$= -L \ln \left[\left(\prod_{i=1}^{N_s} \hat{\lambda}_i \right) \left(\frac{1}{M - N_s} \sum_{i=N_s+1}^M \hat{\lambda}_i \right)^{M - N_s} \right]$$

$$\stackrel{c}{=} -L \ln \left[\left(\prod_{i=1}^M \frac{1}{\hat{\lambda}_i} \right) \left(\prod_{i=1}^{N_s} \hat{\lambda}_i \right) \left(\frac{1}{M - N_s} \sum_{i=N_s+1}^M \hat{\lambda}_i \right)^{M - N_s} \right]$$

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The Determinant Term (2)

$$-L \ln \left[\left(\prod_{i=1}^M \frac{1}{\hat{\lambda}_i} \right) \left(\prod_{i=1}^{N_s} \hat{\lambda}_i \right) \left(\frac{1}{M - N_s} \sum_{i=N_s+1}^M \hat{\lambda}_i \right)^{M - N_s} \right]$$

$$= -L \ln \left[\left(\prod_{i=N_s+1}^M \frac{1}{\hat{\lambda}_i} \right) \left(\frac{1}{M - N_s} \sum_{i=N_s+1}^M \hat{\lambda}_i \right)^{M - N_s} \right]$$

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The Determinant Term (3)

$$L \ln \left[\frac{\prod_{i=N_s+1}^M \hat{\lambda}_i}{\left(\frac{1}{M - N_s} \sum_{i=N_s+1}^M \hat{\lambda}_i \right)^{M - N_s}} \right]$$

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The Determinant Term (4)

$$L \ln \left[\frac{\left(\prod_{i=N_s+1}^M \hat{\lambda}_i \right)^{\frac{1}{M - N_s}}}{\frac{1}{M - N_s} \sum_{i=N_s+1}^M \hat{\lambda}_i} \right]^{M - N_s}$$

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The Determinant Term (5)

- Goal: Find N_s that maximizes

$$\ell = L(M - N_s) \ln \left[\frac{\left(\prod_{i=N_s+1}^M \hat{\lambda}_i \right)^{\frac{1}{M - N_s}}}{\frac{1}{M - N_s} \sum_{i=N_s+1}^M \hat{\lambda}_i} \right]$$

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Uh Oh!

- **Problem: ML is greedy in model order estimation tasks!**

$$\hat{N}_s = M$$

- **Solution: subtract a penalty term and maximize the penalized likelihood**

$$AIC : \ell - \# \quad BIC : \ell - \frac{\#}{2} \ln L$$

- **Where # is the number of “free parameters”**

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Counting the Free Parameters (1)

- N_s complex eigenvectors: $2MN_s$
- N_s real eigenvalues
- One real noise variance

$$2MN_s + N_s + 1$$

- **But eigenvectors have unit norm, so subtract N_s**

$$2MN_s + 1$$

$$\begin{bmatrix} X & X & X \\ X & X & X \\ X & X & X \\ X & X & X \\ X & X & X \end{bmatrix}$$

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Counting the Free Parameters (2)

- **From last slide:**

$$2MN_s + 1$$

- **Since eigenvectors are orthogonal, subtract**

$$2 \sum_{i=1}^{N_s-1} i = N_s(N_s - 1)$$

yielding

$$\begin{aligned} \# &= 2MN_s + 1 - N_s(N_s - 1) \\ &= (2M - N_s + 1)N_s + 1 \end{aligned}$$

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$$\begin{bmatrix} X & X & X \\ X & X & X \\ X & X & X \\ X & X & \\ X & & \end{bmatrix}$$