
Delay-and-Sum Beamforming for Spherical Waves

**ECE 6279: Spatial Array Processing
Spring 2011
Lecture 7**

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Where We Are in J&D

- **Lecture material drawn from:**
 - Sec. 4.1.3
 - Sec. 4.2.1 (“Point Focusing” part on p. 123)

- **Please read section 4.1.1 on your own**



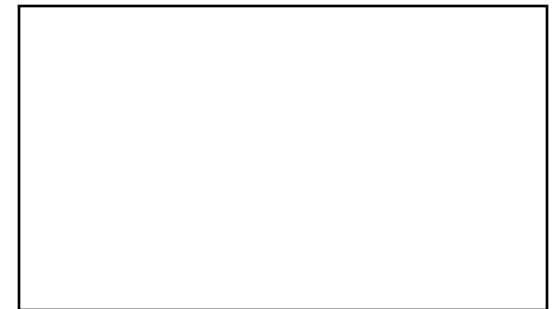
Delay-and-Sum Beamforming

- **Array of M sensors at positions $\vec{x}_0 \dots \vec{x}_{M-1}$**
- **For convenience, put the phase center at the origin**

$$\frac{1}{M} \sum_{m=0}^{M-1} \vec{x}_m = \vec{0}$$

- **Delay-and-sum beamforming**

$$z(t) \equiv \sum_{m=0}^{M-1} w_m y_m(t - \Delta_m)$$



Spherical Waves at the Sensors

Source location
↓

$$f(\vec{x}, t) = \frac{s(t - |\vec{x} - \vec{x}^0|/c)}{|\vec{x} - \vec{x}^0|}$$

$$y_m(t) = \frac{s(t - |\vec{x}_m - \vec{x}^0|/c)}{|\vec{x}_m - \vec{x}^0|} = \frac{s(t - r_m^0/c)}{r_m^0}$$

where $r_m^0 = |\vec{x}_m - \vec{x}^0|$ ← **Distance between source and sensor m**



Delay-and-Sum for Spherical Waves

$$y_m(t) = \frac{s(t - r_m^0 / c)}{r_m^0}$$

$$z(t) \equiv \sum_{m=0}^{M-1} w_m y_m(t - \Delta_m)$$

$$= \sum_{m=0}^{M-1} w_m \frac{s(t - \Delta_m - r_m^0 / c)}{r_m^0}$$



Matched Delays (1)

$$z(t) = \sum_{m=0}^{M-1} w_m \frac{s(t - \Delta_m - r_m^0 / c)}{r_m^0}$$

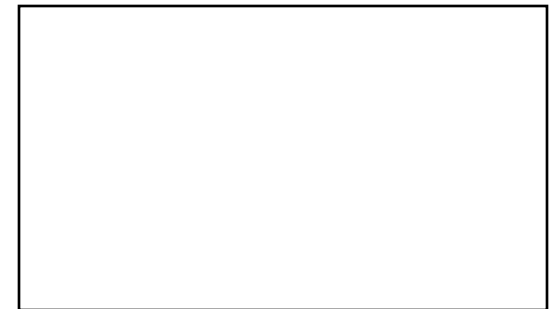
- If we pick $\Delta_m = (r^0 - r_m^0) / c$

then we get

Distance between source and origin

$$z(t) = \sum_{m=0}^{M-1} w_m \frac{s(t - r^0 / c)}{r_m^0}$$

$$= \frac{s(t - r^0 / c)}{r^0} \left[\sum_{m=0}^{M-1} w_m \frac{r^0}{r_m^0} \right]$$



Matched Delays (2)

$$z(t) = \underbrace{\frac{s(t - r^0 / c)}{r^0}}_{\text{Original signal, delayed and attenuated}} \underbrace{\left[\sum_{m=0}^{M-1} w_m \frac{r^0}{r_m} \right]}_{\text{Approaches sum of weights as } r^0, r_m \rightarrow \infty}$$



Mismatched Delays

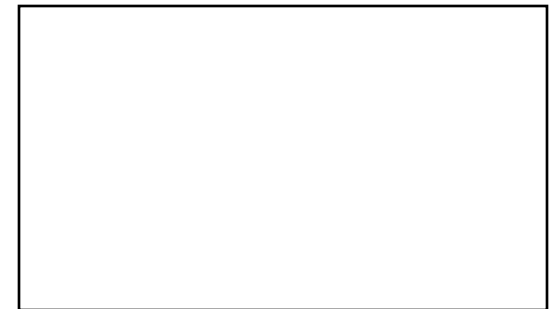
$$z(t) = \sum_{m=0}^{M-1} w_m \frac{s(t - \Delta_m - r_m^0 / c)}{r_m^0}$$

Distance between
assumed source
and origin

Distance
between
assumed
source and
sensor m

- In general, if we pick $\Delta_m = (r - r_m) / c$ then we get

$$z(t) = \sum_{m=0}^{M-1} \frac{w_m}{r_m^0} s\left(t - \frac{r - (r_m - r_m^0)}{c}\right)$$

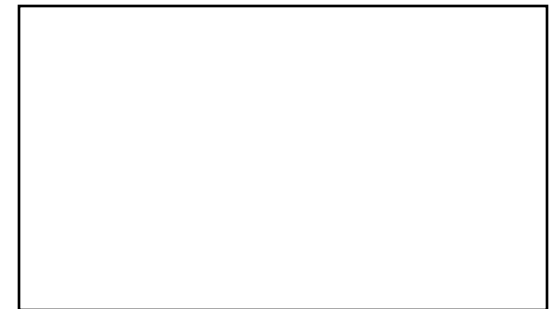


Beamforming Strategy

$$\begin{aligned} z(t) &\equiv \sum_{m=0}^{M-1} w_m y_m(t - \Delta_m) \\ &= \sum_{m=0}^{M-1} w_m y_m\left(t - \frac{r - r_m}{c}\right) \end{aligned}$$

$\Delta_m = (r - r_m) / c$

- Find the \vec{x} which gives the most energy in $z(t)$



Time Signal Seen at the Origin

$$s_0(t) = \frac{s(t - r^0 / c)}{r^0}$$

$$S_0(\omega) = \frac{1}{r^0} S(\omega) \exp(-j\omega r^0 / c)$$

$$r^0 S_0(\omega) \exp(j\omega r^0 / c) = S(\omega)$$

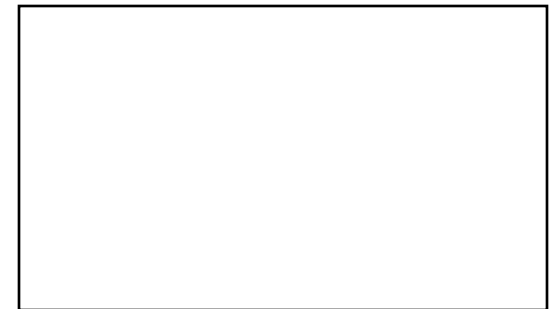


Output in Fourier Domain (1)

$$z(t) = \sum_{m=0}^{M-1} \frac{w_m}{r_m^0} S\left(t - \frac{r - (r_m - r_m^0)}{c}\right)$$

Take FT of both sides

$$Z(\omega) = \sum_{m=0}^{M-1} \frac{w_m}{r_m^0} S(\omega) \times \exp\left[-j\omega \frac{r - (r_m - r_m^0)}{c}\right]$$



Output in Fourier Domain (2)

$$Z(\omega) = \sum_{m=0}^{M-1} \frac{w_m}{r_m^0} S(\omega) \exp\left[-j\omega \frac{r - (r_m - r_m^0)}{c}\right]$$

$$S(\omega) = r^0 S_0(\omega) \exp(j\omega r^0 / c)$$

$$Z(\omega) = \sum_{m=0}^{M-1} w_m \frac{r^0}{r_m^0} S_0(\omega) \times$$

$$\exp\left[j\omega \left\{ \frac{(r^0 - r) - (r_m^0 - r_m)}{c} \right\}\right]$$

$$k = \omega / c$$



Array-Pattern-Like Thing

$$Z(\omega) = S_0(\omega) \widehat{W}(k, \vec{x}, \vec{x}^0)$$

Where $\widehat{W}(k, \vec{x}, \vec{x}^0) \equiv$

$$\sum_{m=0}^{M-1} w_m \frac{r^0}{r_m} \exp[jk\{(r^0 - r) - (r_m^0 - r_m)\}]$$

$$k = \omega / c$$

$\widehat{W}(k, \vec{x}, \vec{x}^0)$ plays a role analogous to $W(\vec{\omega}^0 \vec{\alpha} - \vec{k}^0)$ in previous lecture on beamforming for plane waves



When Perfectly Focused

- **If** $\vec{x} = \vec{x}^0$, i.e., $r^0 = r$ and $r_m^0 = r_m$

$$\widehat{W}(k, \vec{x}^0, \vec{x}^0) \equiv$$

$$\sum_{m=0}^{M-1} w_m \frac{r^0}{r_m^0} \exp[jk\{(r^0 - r^0) - (r_m^0 - r_m^0)\}]$$

$$= \sum_{m=0}^{M-1} w_m \frac{r^0}{r_m^0} \longrightarrow z(t) = s_0(t) \left[\sum_{m=0}^{M-1} w_m \frac{r^0}{r_m^0} \right]$$

We copy the signal exactly - no filtering!

Now show movie

