Delay-and-Sum Beamforming for Spherical Waves

ECE 6279: Spatial Array Processing
Spring 2011
Lecture 7

Prof. Aaron D. Lanterman

School of Electrical & Computer Engineering
Georgia Institute of Technology
AL: 404-385-2548
<lanterma@ece.gatech.edu>

Where We Are in J&D

- **Lecture material drawn from:**
  - Sec. 4.1.3
  - Sec. 4.2.1 (“Point Focusing” part on p. 123)

- **Please read section 4.1.1 on your own**

Delay-and-Sum Beamforming

- **Array of M sensors at positions $\vec{x}_0 \ldots \vec{x}_{M-1}$**
- **For convenience, put the phase center at the origin**

$$\frac{1}{M}\sum_{m=0}^{M-1}\vec{x}_m = \vec{0}$$

- **Delay-and-sum beamforming**

$$z(t) \equiv \sum_{m=0}^{M-1} w_m y_m(t-\Delta_m)$$

Spherical Waves at the Sensors

Source location

$$f(\vec{x},t) = \frac{s(t - |\vec{x} - \vec{x}^0|/c)}{|\vec{x} - \vec{x}^0|}$$

$$y_m(t) = \frac{s(t - |\vec{x}_m - \vec{x}^0|/c)}{|\vec{x}_m - \vec{x}^0|} = \frac{s(t - r_m^0/c)}{r_m^0}$$

where $r_m^0 = |\vec{x}_m - \vec{x}^0| \longleftarrow$ **Distance between source and sensor *m***

Delay-and-Sum for Spherical Waves

$$y_m(t) = \frac{s(t - r_m^0 / c)}{r_m^0}$$

$$z(t) \equiv \sum_{m=0}^{M-1} w_m y_m(t - \Delta_m)$$

$$= \sum_{m=0}^{M-1} w_m \frac{s(t - \Delta_m - r_m^0 / c)}{r_m^0}$$

Matched Delays (1)

$$z(t) = \sum_{m=0}^{M-1} w_m \frac{s(t - \Delta_m - r_m^0 / c)}{r_m^0}$$

- **If we pick** $\Delta_m = (r^0 - r_m^0)/c$ **then we get**

Distance between source and origin

$$z(t) = \sum_{m=0}^{M-1} w_m \frac{s(t - r^0 / c)}{r_m^0}$$

$$= \frac{s(t - r^0 / c)}{r^0} \left[\sum_{m=0}^{M-1} w_m \frac{r^0}{r_m^0} \right]$$

Matched Delays (2)

$$z(t) = \underbrace{\frac{s(t - r^0/c)}{r^0}}_{\text{Original signal, delayed and attenuated}} \underbrace{\left[\sum_{m=0}^{M-1} w_m \frac{r^0}{r_m^0} \right]}_{\text{Approaches sum of weights as } r^0, r_m^0 \to \infty}$$

Original signal, delayed and attenuated

Approaches sum of weights as $r^0, r_m^0 \to \infty$

Mismatched Delays

$$z(t) = \sum_{m=0}^{M-1} w_m \frac{s(t - \Delta_m - r_m^0 / c)}{r_m^0}$$

Distance between **assumed** source and origin

Distance between **assumed** source and sensor *m*

- **In general, if we pick** $\Delta_m = (r - r_m)/c$ **then we get**

$$z(t) = \sum_{m=0}^{M-1} \frac{w_m}{r_m^0} s\left(t - \frac{r - (r_m - r_m^0)}{c} \right)$$

Beamforming Strategy

$$\Delta_m = (r - r_m)/c$$

$$z(t) \equiv \sum_{m=0}^{M-1} w_m y_m (t - \Delta_m)$$

$$= \sum_{m=0}^{M-1} w_m y_m \left(t - \frac{r - r_m}{c} \right)$$

- **Find the $\vec{x}$ which gives the most energy in $z(t)$**

Time Signal Seen at the Origin

$$s_0(t) = \frac{s(t - r^0 / c)}{r^0}$$

$$S_0(\omega) = \frac{1}{r^0} S(\omega) \exp(-j\omega r^0 / c)$$

$$r^0 S_0(\omega) \exp(j\omega r^0 / c) = S(\omega)$$

Output in Fourier Domain (1)

$$z(t) = \sum_{m=0}^{M-1} \frac{w_m}{r_m^0} s\left(t - \frac{r - (r_m - r_m^0)}{c} \right)$$

Take FT of both sides

$$Z(\omega) = \sum_{m=0}^{M-1} \frac{w_m}{r_m^0} S(\omega) \times$$

$$\exp\left[-j\omega \frac{r - (r_m - r_m^0)}{c} \right]$$

Output in Fourier Domain (2)

$$Z(\omega) = \sum_{m=0}^{M-1} \frac{w_m}{r_m^0} S(\omega) \exp\left[-j\omega \frac{r - (r_m - r_m^0)}{c} \right]$$

$$S(\omega) = r^0 S_0(\omega) \exp(j\omega r^0 / c)$$

$$Z(\omega) = \sum_{m=0}^{M-1} w_m \frac{r^0}{r_m^0} S_0(\omega) \times$$

$$\exp\left[j\omega \left\{ \frac{(r^0 - r) - (r_m^0 - r_m)}{c} \right\} \right]$$

$$\boxed{k = \omega / c}$$

Array-Pattern-Like Thing

$$Z(\omega) = S_0(\omega)\widehat{W}(k,\vec{x},\vec{x}^0)$$

Where $\widehat{W}(k,\vec{x},\vec{x}^0) \equiv$

$$\sum_{m=0}^{M-1} w_m \frac{r^0}{r_m^0} \exp[jk\{(r^0 - r) - (r_m^0 - r_m)\}]$$

$k = \omega / c$

$\widehat{W}(k,\vec{x},\vec{x}^0)$ **plays a role analogous to** $W(\vec{\omega}^0\vec{\alpha} - \vec{k}^0)$ **in previous lecture on beamforming for plane waves**

When Perfectly Focused

- **If** $\vec{x} = \vec{x}^0$, **i.e.,** $r^0 = r$ **and** $r_m^0 = r_m$

$$\widehat{W}(k, \vec{x}^0, \vec{x}^0) \equiv$$

$$\sum_{m=0}^{M-1} w_m \frac{r^0}{r_m^0} \exp[jk\{(r^0 - r^0) - (r_m^0 - r_m^0)\}]$$

$$= \sum_{m=0}^{M-1} w_m \frac{r^0}{r_m^0} \longrightarrow z(t) = s_0(t)\left[\sum_{m=0}^{M-1} w_m \frac{r^0}{r_m^0}\right]$$

We copy the signal exactly - no filtering!

Now show movie