Time Averaging

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Lecture 12

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Where We Are (and Aren't) in J&D

- **Inspired by Section 4.9.1**

Average Beamformer Output

- **Average power of beamformer output over time**

$$\hat{z} = \frac{1}{L}\sum_{l=0}^{L-1}\left|z(l)\right|^2 = \frac{1}{L}\sum_{l=0}^{L-1}\left|\mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{y}(l)\right|^2$$

$$= \frac{1}{L}\sum_{l=0}^{L-1}\mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{y}(l)\mathbf{y}^H(l)\mathbf{W}^H\mathbf{e}(\vec{k})$$

$$= \mathbf{e}^H(\vec{k})\mathbf{W}\left[\frac{1}{L}\sum_{l=0}^{L-1}\mathbf{y}(l)\mathbf{y}^H(l)\right]\mathbf{W}^H\mathbf{e}(\vec{k})$$

Under the Stochastic Model

$$\hat{z} = \mathbf{e}^H(\vec{k})\mathbf{W}\hat{\mathbf{R}}_{\mathbf{y}}\mathbf{W}^H\mathbf{e}(\vec{k})$$

where $$\hat{\mathrm{R}}_{\mathbf{y}} = \frac{1}{L}\sum_{l=0}^{L-1}\mathbf{y}(l)\mathbf{y}^H(l)$$

- **Under the stochastic signal model, $\hat{\mathrm{R}}_{\mathrm{y}}$ is the unconstrained maximum-likelihood estimate of the correlation matrix of a Gaussian random vector $\underline{\mathrm{y}}$**

Improvement from Time Averaging

$$\hat{\mathbf{R}}_{\mathrm{y}} \rightarrow \mathbf{R}_{\mathrm{y}} = E[\underline{\mathbf{y}}\,\underline{\mathbf{y}}^{T}] \text{ as } L \rightarrow \infty$$

- **If snapshots are independent, variance of estimate improves as O(1/*L*), standard deviation improves as O(1/sqrt(*L*))**
- **Can't make *L* too big if the sources are moving!**

Outer Product of Steering Vectors

- **For linear array**

$$\gamma = k\sin(\phi)\cos(\theta)d$$

$$= \frac{2\pi}{\lambda}\sin(\phi)\cos(\theta)d$$

$$\begin{bmatrix} 1 \\ e^{j\gamma} \\ \vdots \\ e^{j(M-1)\gamma} \end{bmatrix} \begin{bmatrix} 1 & e^{-j\gamma} & \cdots & e^{-j(M-1)\gamma} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & e^{-j\gamma} & \cdots & e^{-j(M-1)\gamma} \\ e^{j\gamma} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & e^{-j\gamma} \\ e^{j(M-1)\gamma} & \cdots & e^{j\gamma} & 1 \end{bmatrix}$$

Linear Array, Single Plane Wave

- **Assuming no noise**

$$\mathbf{y}(l) = s(l)\exp\left(-j\frac{M-1}{2}\gamma\right)\exp\begin{bmatrix} 0 \\ j\gamma \\ \vdots \\ j(M-1)\gamma \end{bmatrix}$$

$$\mathbf{y}(l)\mathbf{y}^H(l) =$$

$$|s(l)|^2 \exp\begin{bmatrix} 0 & -j\gamma & \cdots & -j(M-1)\gamma \\ j\gamma & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -j\gamma \\ j(M-1)\gamma & \cdots & j\gamma & 0 \end{bmatrix}$$

Toeplitz Structure of Est. Correlation

$$\hat{\mathbf{R}}_{\mathbf{y}} = \frac{1}{L}\sum_{l=0}^{L-1} \mathbf{y}(l)\mathbf{y}^H(l)$$

- $\hat{\mathbf{R}}_{\mathbf{y}}$ is Toeplitz since a sum of Toeplitz matrices is Toeplitz
- Remember we are assuming no noise

Convergence to True Correlation

$$\hat{\mathbf{R}}_{\mathbf{y}} \rightarrow E\left[|\underline{s}|^2\right]\exp\begin{bmatrix} 0 & -j\gamma & \cdots & -j(M-1)\gamma \\ j\gamma & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -j\gamma \\ j(M-1)\gamma & \cdots & j\gamma & 0 \end{bmatrix}$$

as $L \rightarrow \infty$

Linear Array, Two Plane Waves

- Again assuming no noise

$$\mathbf{y}(l) = s_1(l)\exp\left(-j\frac{M-1}{2}\gamma_1\right)\exp\begin{bmatrix} 0 \\ j\gamma_1 \\ \vdots \\ j(M-1)\gamma_1 \end{bmatrix}$$

$$+s_2(l)\exp\left(-j\frac{M-1}{2}\gamma_2\right)\exp\begin{bmatrix} 0 \\ j\gamma_2 \\ \vdots \\ j(M-1)\gamma_2 \end{bmatrix}$$

Two Signals

$$\mathbf{y}(l)\mathbf{y}^H(l) = |s_1(l)|^2 \begin{bmatrix} 1 & e^{-j\gamma_1} & \cdots & e^{-j(M-1)\gamma_1} \\ e^{j\gamma_1} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & e^{-j\gamma_1} \\ e^{j(M-1)\gamma_1} & \cdots & e^{j\gamma_1} & 1 \end{bmatrix}$$

$$+|s_2(l)|^2 \left[\text{same but with } \gamma_2\right]$$

$$+\text{crossterms!!!}$$

Crossterms (1)

$$s_1(l)e^{-j\frac{M-1}{2}\gamma_1}\begin{bmatrix}1\\e^{j\gamma_1}\\\vdots\\e^{j(M-1)\gamma_1}\end{bmatrix}s_2^*(l)e^{j\frac{M-1}{2}\gamma_2}\begin{bmatrix}1\\e^{-j\gamma_2}\\\vdots\\e^{-j(M-1)\gamma_2}\end{bmatrix}^T$$

$$+s_2(l)e^{-j\frac{M-1}{2}\gamma_2}\begin{bmatrix}1\\e^{j\gamma_2}\\\vdots\\e^{j(M-1)\gamma_2}\end{bmatrix}s_1^*(l)e^{j\frac{M-1}{2}\gamma_1}\begin{bmatrix}1\\e^{-j\gamma_1}\\\vdots\\e^{-j(M-1)\gamma_1}\end{bmatrix}^T$$

Crossterms (2)

$$s_1(l)s_2^*(l)e^{-j\frac{(M-1)}{2}(\gamma_1-\gamma_2)}\times$$

$$\exp\begin{bmatrix} 0 & -j\gamma_2 & \cdots & -j(M-1)\gamma_2 \\ j\gamma_1 & j(\gamma_1-\gamma_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & j[(M-2)\gamma_1-(M-2)\gamma_2] \\ j(M-1)\gamma_1 & \cdots & j[(M-1)\gamma_1-(M-2)\gamma_2] & j(M-1)(\gamma_1-\gamma_2) \end{bmatrix}$$

$$+s_2(l)s_1^*(l)e^{-j\frac{(M-1)}{2}(\gamma_2-\gamma_1)}\times$$

$$\exp\begin{bmatrix} 0 & -j\gamma_1 & \cdots & -j(M-1)\gamma_1 \\ j\gamma_2 & j(\gamma_2-\gamma_1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & j[(M-2)\gamma_2-(M-2)\gamma_1] \\ j(M-1)\gamma_2 & \cdots & j[(M-1)\gamma_2-(M-2)\gamma_1] & j(M-1)(\gamma_2-\gamma_1) \end{bmatrix}$$

Crossterm Matrix Elements

- ***i*th diagonal element of crossterm matrix given by**

$$2\,\mathrm{Re}\left\{ s_1(l) s_2^*(l) \exp\left(-j \frac{M-1}{2} [\gamma_1 - \gamma_2] \right) \exp\left(j[\gamma_1 - \gamma_2] i \right) \right\}$$

$$= 2 \left| s_1(l) s_2^*(l) \right| \cos\left([\gamma_1 - \gamma_2] \left[i - \frac{M-1}{2} \right] + \angle\{ s_1(l) s_2^*(l) \} \right)$$

Intersignal Coherence

$$\frac{1}{L}\sum_{l=0}^{L-1} s_1(l)s_2^*(l) \rightarrow \underbrace{E[\underline{s}_1\underline{s}_2^*]}_{\text{Intersignal coherence}} \quad \textbf{as } L \rightarrow \infty$$

- Incoherent signals have an intersignal coherence of zero
- Coherence suggests multipath effects