Root MUSIC

ECE 6279: Spatial Array Processing
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Lecture 17

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Leaving J&D for a Little Bit

- Van Trees Vol. IV, pp. 1159-1160
- Journal paper (linked on class website)
- Lecture notes from Doug Williams
- Notation in every source I've found seems maximally confusing relative to J&D notation, so I'll make up some notation for this lecture

Original MUSIC Method

- From last lecture (sometimes called "spectral MUSIC")

$$P^{MUSIC}(\vec{k}) = \left[\mathbf{e}^H(\vec{k})\mathbf{R}_{MUSIC}^{-1}\mathbf{e}(\vec{k})\right]^{-1}$$

$$\mathbf{R}_{MUSIC}^{-1} = \sum_{i=N_s+1}^{M} \mathbf{v}_i \mathbf{v}_i^H$$

Looking at it Upside Down

- **To find peaks in**

$$P^{MUSIC}(\vec{k}) = \left[\mathbf{e}^H(\vec{k}) \left(\sum_{i=N_s+1}^{M} \mathbf{v}_i \mathbf{v}_i^H \right) \mathbf{e}(\vec{k}) \right]^{-1}$$

can find valleys in

$$\mathbf{e}^H(\vec{k}) \left(\sum_{i=N_s+1}^{M} \mathbf{v}_i \mathbf{v}_i^H \right) \mathbf{e}(\vec{k})$$

Linear Array Assumption

- Spectral MUSIC can use arbitrarily shaped arrays
- Basic “Root MUSIC” requires an equally spaced linear array
- Poking around IEEExplore will uncover many Root MUSIC-style algorithms for other specialized array shapes

Vandermonde Structure (Phi)

$$\exp\left\{-jk\sin(\phi)\frac{M-1}{2}d\right\}\begin{bmatrix} 1 \\ \exp\{jkd\sin(\phi)\} \\ \vdots \\ \exp\{j\underbrace{kd\sin(\phi)}_{\gamma}(M-1)\} \end{bmatrix}$$

"electrical angle"

$$= const \begin{bmatrix} (e^{j\gamma})^0 \\ (e^{j\gamma})^1 \\ \vdots \\ (e^{j\gamma})^{M-1} \end{bmatrix} = const \begin{bmatrix} z^0 \\ z^1 \\ \vdots \\ z^{M-1} \end{bmatrix}$$

Vandermonde structure

Vandermonde Structure (Theta)

$$\exp\left\{-jk\cos(\theta)\frac{M-1}{2}d\right\}\begin{bmatrix}1\\ \exp\{jkd\cos(\theta)\}\\ \vdots \\ \exp\{j\underbrace{kd\cos(\theta)}_{\gamma}(M-1)\}\end{bmatrix}$$

“electrical angle”

$$= const\begin{bmatrix}(e^{j\gamma})^0\\ (e^{j\gamma})^1\\ \vdots \\ (e^{j\gamma})^{M-1}\end{bmatrix} = const\begin{bmatrix}z^0\\ z^1\\ \vdots \\ z^{M-1}\end{bmatrix}$$

Vandermonde structure

Writing in Terms of Electrical Angle

- Seeking valleys of

$$\mathbf{e}^H(\gamma)\left(\sum_{i=N_s+1}^{M}\mathbf{v}_i\mathbf{v}_i^H\right)\mathbf{e}(\gamma)$$

$$=\sum_{i=N_s+1}^{M}\mathbf{e}^H(\gamma)\mathbf{v}_i\mathbf{v}_i^H\mathbf{e}(\gamma)$$

Polynomial Construction

$$z = e^{j\gamma}$$

$$P_i(z) = \sum_{k=1}^{M} (\mathbf{v}_i)_k z^{-(k-1)} = (const)\mathbf{e}^H(\gamma)\mathbf{v}_i$$

$$P_i^*\left(\frac{1}{z^*}\right) = \left(\sum_{k=1}^{M}(\mathbf{v}_i)_k (z^*)^{k-1}\right)^* = \sum_{k=1}^{M} (\mathbf{v}_i^*)_k z^{k-1}$$

$$= (const^*)\mathbf{v}_i^H \mathbf{e}(\gamma)$$

The Root MUSIC Polynomial

$$D(z) = \sum_{i=N_s+1}^{M} P_i(z) P_i^* \left(\frac{1}{z^*} \right)$$

$$P^{MUSIC}(\gamma) = \left. \frac{1}{D(z)} \right|_{z=\exp(j\gamma)}$$

- To find γ for which $P^{MUSIC}(\gamma)$ is big, find γ for which $D(e^{j\gamma})$ is small

In an Ideal World

- **Ideally, if a particular γ corresponds to a source, then**

$$\mathbf{e}^H(\gamma)\mathbf{v}_i = 0 \text{ for } i = N_s + 1, \ldots, M$$

$$\text{hence } D(e^{j\gamma}) = 0$$

- **Ideally, could find the zeros of $D(z)$ on the unit circle and extract the electrical angles of the sources**

In a Less Ideal World

- In practice, eigenvectors are computed from an empirical covariance matrix
- Zeros won't be exactly on unit circle
- Hack: find the N_s zeros z_i inside and closest to the unit circle, then take

$$\gamma_i = \arg\{z_i\} \text{ for } i = 1,\ldots,N_s$$

Not Done Yet!

$$\gamma_i = \frac{2\pi}{\lambda} d \sin(\phi_i) \qquad \gamma_i = \frac{2\pi}{\lambda} d \cos(\theta_i)$$

$$\frac{\lambda}{2\pi d}\gamma_i = \sin(\phi_i) \qquad \frac{\lambda}{2\pi d}\gamma_i = \cos(\theta_i)$$

$$\phi_i = \sin^{-1}\left(\frac{\lambda}{2\pi d}\gamma_i\right) \qquad \theta_i = \cos^{-1}\left(\frac{\lambda}{2\pi d}\gamma_i\right)$$