
Root MUSIC

**ECE 6279: Spatial Array Processing
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Lecture 17**

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Leaving J&D for a Little Bit

- **Van Trees Vol. IV, pp. 1159-1160**
- **Journal paper (linked on class website)**
- **Lecture notes from Doug Williams**
- **Notation in every source I've found seems maximally confusing relative to J&D notation, so I'll make up some notation for this lecture**

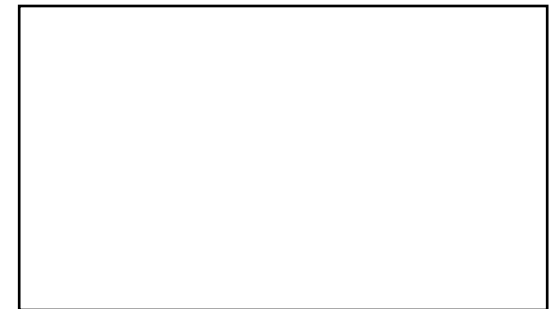


Original MUSIC Method

- From last lecture (sometimes called “spectral MUSIC”)

$$P^{MUSIC}(\vec{k}) = \left[\mathbf{e}^H(\vec{k}) \mathbf{R}_{MUSIC}^{-1} \mathbf{e}(\vec{k}) \right]^{-1}$$

$$\mathbf{R}_{MUSIC}^{-1} = \sum_{i=N_s+1}^M \mathbf{v}_i \mathbf{v}_i^H$$



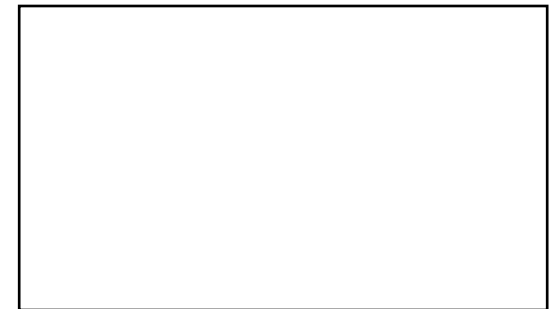
Looking at it Upside Down

- To find peaks in

$$P^{MUSIC}(\vec{k}) = \left[\mathbf{e}^H(\vec{k}) \left(\sum_{i=N_s+1}^M \mathbf{v}_i \mathbf{v}_i^H \right) \mathbf{e}(\vec{k}) \right]^{-1}$$

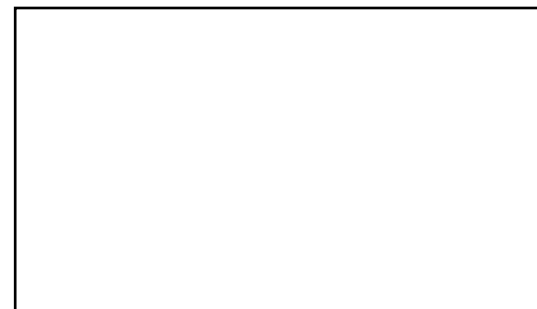
can find valleys in

$$\mathbf{e}^H(\vec{k}) \left(\sum_{i=N_s+1}^M \mathbf{v}_i \mathbf{v}_i^H \right) \mathbf{e}(\vec{k})$$



Linear Array Assumption

- **Spectral MUSIC can use arbitrarily shaped arrays**
- **Basic “Root MUSIC” requires an equally spaced linear array**
- **Poking around IEEExplore will uncover many Root MUSIC-style algorithms for other specialized array shapes**



Vandermonde Structure (Phi)

$$\exp\left\{-jk \sin(\phi) \frac{M-1}{2} d\right\} \begin{bmatrix} 1 \\ \exp\{jkd \sin(\phi)\} \\ \vdots \\ \exp\{jkd \sin(\phi)(M-1)\} \end{bmatrix} \\
 = \text{const} \begin{bmatrix} (e^{j\gamma})^0 \\ (e^{j\gamma})^1 \\ \vdots \\ (e^{j\gamma})^{M-1} \end{bmatrix} = \text{const} \begin{bmatrix} z^0 \\ z^1 \\ \vdots \\ z^{M-1} \end{bmatrix}$$

$\underbrace{jkd \sin(\phi)}_{\gamma}$
← “electrical angle”
← Vandermonde structure



Vandermonde Structure (Theta)

$$\exp\left\{-jk \cos(\theta) \frac{M-1}{2} d\right\} \begin{bmatrix} 1 \\ \exp\{jkd \cos(\theta)\} \\ \vdots \\ \exp\{jkd \cos(\theta)(M-1)\} \end{bmatrix}$$

$$= \text{const} \begin{bmatrix} (e^{j\gamma})^0 \\ (e^{j\gamma})^1 \\ \vdots \\ (e^{j\gamma})^{M-1} \end{bmatrix} = \text{const} \begin{bmatrix} z^0 \\ z^1 \\ \vdots \\ z^{M-1} \end{bmatrix}$$

$\underbrace{\hspace{10em}}_{\gamma}$

“electrical angle”

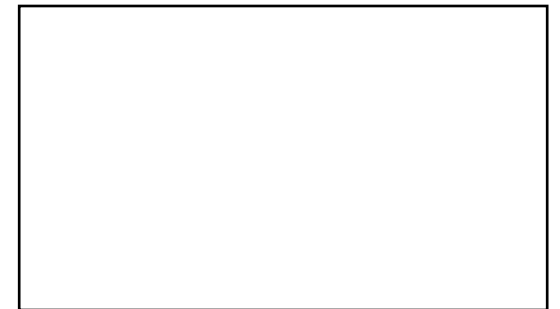
Vandermonde structure



Writing in Terms of Electrical Angle

- Seeking valleys of

$$\mathbf{e}^H(\gamma) \left(\sum_{i=N_s+1}^M \mathbf{v}_i \mathbf{v}_i^H \right) \mathbf{e}(\gamma)$$
$$= \sum_{i=N_s+1}^M \mathbf{e}^H(\gamma) \mathbf{v}_i \mathbf{v}_i^H \mathbf{e}(\gamma)$$



Polynomial Construction

$$P_i(z) = \sum_{k=1}^M (\mathbf{v}_i)_k z^{-(k-1)} = (\text{const}) \mathbf{e}^H(\gamma) \mathbf{v}_i$$

$z = e^{j\gamma}$

$$P_i^* \left(\frac{1}{z^*} \right) = \left(\sum_{k=1}^M (\mathbf{v}_i)_k (z^*)^{k-1} \right)^* = \sum_{k=1}^M (\mathbf{v}_i^*)_k z^{k-1}$$

$$= (\text{const}^*) \mathbf{v}_i^H \mathbf{e}(\gamma)$$



The Root MUSIC Polynomial

$$D(z) = \sum_{i=N_s+1}^M P_i(z) P_i^* \left(\frac{1}{z^*} \right)$$

$$P^{MUSIC}(\gamma) = \frac{1}{D(z)} \Big|_{z=\exp(j\gamma)}$$

- To find γ for which $P^{MUSIC}(\gamma)$ is big, find γ for which $D(e^{j\gamma})$ is small



In an Ideal World

- **Ideally, if a particular γ corresponds to a source, then**

$$\mathbf{e}^H(\gamma)\mathbf{v}_i = 0 \text{ for } i = N_s + 1, \dots, M$$

$$\text{hence } D(e^{j\gamma}) = 0$$

- **Ideally, could find the zeros of $D(z)$ on the unit circle and extract the electrical angles of the sources**



In a Less Ideal World

- In practice, eigenvectors are computed from an empirical covariance matrix
- Zeros won't be exactly on unit circle
- Hack: find the N_s zeros z_i inside and closest to the unit circle, then take

$$\gamma_i = \arg\{z_i\} \text{ for } i = 1, \dots, N_s$$



Not Done Yet!

$$\gamma_i = \frac{2\pi}{\lambda} d \sin(\phi_i)$$

$$\gamma_i = \frac{2\pi}{\lambda} d \cos(\theta_i)$$

$$\frac{\lambda}{2\pi d} \gamma_i = \sin(\phi_i)$$

$$\frac{\lambda}{2\pi d} \gamma_i = \cos(\theta_i)$$

$$\phi_i = \sin^{-1} \left(\frac{\lambda}{2\pi d} \gamma_i \right)$$

$$\theta_i = \cos^{-1} \left(\frac{\lambda}{2\pi d} \gamma_i \right)$$

