

Introduction to Cramér-Rao Bounds

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Basic Univariate Cramér-Rao Bound

- For any unbiased estimator $\hat{\xi}$
 $MSE = \text{var}_{\xi}[\hat{\xi}(y)] \geq 1 / F(\xi)$
where the Fisher Information is

$$F(\xi) = E_{\xi} \left\{ \left[\frac{d}{d\tilde{\xi}} \ln p(\underline{y}; \tilde{\xi}) \right]^2 \bigg|_{\tilde{\xi}=\xi} \right\}$$

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Two Ways to Compute the F.I.

- Under some common conditions,

$$F(\xi) = E_{\xi} \left\{ \left[\frac{d}{d\tilde{\xi}} \ln p(\underline{y}; \tilde{\xi}) \right]^2 \bigg|_{\tilde{\xi}=\xi} \right\}$$
$$= -E_{\xi} \left\{ \frac{d^2}{d\tilde{\xi}^2} \ln p(\underline{y}; \tilde{\xi}) \bigg|_{\tilde{\xi}=\xi} \right\}$$

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Gaussian Example, 1st Derivative (1)

- Consider F.I. for one data point for

$$\underline{y} \sim N(f(\xi), \sigma^2)$$

$$\frac{d}{d\xi} \ln p(y; \xi) =$$
$$\frac{d}{d\xi} \ln \left(\frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{[y - f(\xi)]^2}{2\sigma^2} \right\} \right)$$

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Gaussian Example, 1st Derivative (2)

$$\begin{aligned} \frac{d}{d\xi} \ln \left(\frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{[y - f(\xi)]^2}{2\sigma^2} \right\} \right) \\ = \frac{d}{d\xi} \left\{ -\frac{[y - f(\xi)]^2}{2\sigma^2} \right\} \\ = \frac{[y - f(\xi)] df(\xi)}{\sigma^2 d\xi} \end{aligned}$$

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Gaussian Example, 2st Derivative (1)

$$\begin{aligned} \frac{d^2}{d\xi^2} \ln p(y; \xi) \\ = \frac{d}{d\xi} \left\{ \frac{[y - f(\xi)] df(\xi)}{\sigma^2} \right\} \\ = \frac{1}{\sigma^2} \left\{ [y - f(\xi)] \frac{d^2 f(\xi)}{d\xi^2} - \left[\frac{df(\xi)}{d\xi} \right]^2 \right\} \end{aligned}$$

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Using Derivative-Squared Version (1)

$$\begin{aligned} F(\xi) &= E_{\xi} \left\{ \left[\frac{\partial}{\partial \xi} \ln p(y; \xi) \right]^2 \right\} \\ &= E_{\xi} \left\{ \left[\frac{[y - f(\xi)] df(\xi)}{\sigma^2 d\xi} \right]^2 \right\} \end{aligned}$$

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Using Derivative-Squared Version (2)

$$\begin{aligned} F(\xi) &= \left[\frac{df(\xi)}{d\xi} \right]^2 \frac{E_{\xi} \{ [y - f(\xi)]^2 \}}{(\sigma^2)^2} \\ &= \left[\frac{df(\xi)}{d\xi} \right]^2 / \sigma^2 \end{aligned}$$

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Using Double-Derivative Version (1)

$$F(\xi) = -E_{\xi} \left\{ \frac{\partial^2}{\partial \xi^2} \ln p(\underline{y}; \xi) \right\}$$

$$= -\frac{1}{\sigma^2} E_{\xi} \left\{ [\underline{y} - f(\xi)] \frac{d^2 f(\xi)}{d\xi^2} - \left[\frac{df(\xi)}{d\xi} \right]^2 \right\}$$

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Using Double-Derivative Version (2)

$$-\frac{1}{\sigma^2} \left\{ E_{\xi} [\underline{y} - f(\xi)] \frac{d^2 f(\xi)}{d\xi^2} - \left[\frac{df(\xi)}{d\xi} \right]^2 \right\}$$

$$= \left[\frac{df(\xi)}{d\xi} \right]^2 / \sigma^2$$

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Independent, Identically Dist. Data

- If you have L i.i.d. data points, the F.I. is just L times the F.I. for one data point:

$$F(\xi) = LF_1(\xi)$$

- For our ex., $F(\xi) = L \left[\frac{df(\xi)}{d\xi} \right]^2 / \sigma^2$

$$\text{var}_{\xi} [\hat{\xi}(\underline{y})] \geq \sigma^2 / \left\{ L \left[\frac{df(\xi)}{d\xi} \right]^2 \right\}$$

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Basic Multivariate Cramér-Rao Bound

- For any unbiased estimator $\hat{\xi}$

$$\text{cov}_{\xi} [\hat{\xi}(\underline{y})] \geq \mathbf{F}^{-1}(\hat{\xi})$$

($\mathbf{A} \geq \mathbf{B}$ means $\mathbf{A} - \mathbf{B}$ is nonneg. def.)

where the entries of
of the F.I. matrix are

$$\mathbf{F}_{rc} = E_{\xi} \left\{ \left[\frac{\partial}{\partial \xi_r} \ln p(\underline{y}; \xi) \right] \left[\frac{\partial}{\partial \xi_c} \ln p(\underline{y}; \xi) \right] \right\}$$

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Nonnegative Definiteness

- If \mathbf{A} is nonnegative definite, then
 - $\mathbf{x}^T \mathbf{A} \mathbf{x} \geq 0$ for any real vector \mathbf{x}
 - Eigenvalues of \mathbf{A} are nonnegative
- Useful consequences: if $\mathbf{A} \geq \mathbf{B}$,
 - Diagonals dominated: $\mathbf{A}_{ii} \geq \mathbf{B}_{ii}$
 - Does not mean $\mathbf{A}_{rc} \geq \mathbf{B}_{rc}$ in general!
 - Total sum property: $\sum_{r,c} \mathbf{A}_{rc} \geq \sum_{r,c} \mathbf{B}_{rc}$

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Two Ways to Compute the F.I.M.

- Under some common conditions:

$$\mathbf{F}_{rc} = E_{\xi} \left\{ \left[\frac{\partial}{\partial \tilde{\xi}_r} \ln p(\underline{y}; \tilde{\xi}) \right] \left[\frac{\partial}{\partial \tilde{\xi}_c} \ln p(\underline{y}; \tilde{\xi}) \right] \right\}$$
$$= -E_{\xi} \left\{ \frac{\partial^2}{\partial \tilde{\xi}_r \partial \tilde{\xi}_c} \ln p(\underline{y}; \tilde{\xi}) \right\}$$

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Efficiency

- An unbiased estimator that achieves the CR bound with equality is called **efficient**
- If an efficient estimator exists, the maximum-likelihood estimator is it!
- Even if an estimator isn't efficient, it may be **asymptotically efficient**
- ML estimators are asymptotically efficient

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CR Bounds for Biased Estimators

- Both univariate and multivariate versions for biased estimators exist...
- ...but they require taking derivatives of the bias...
- ...which requires you have an analytic form for the bias...
- ...which you almost never have...
- ...so it's rarely used; people usually just compute the CR bound for the unbiased estimator and then handwave

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