Principal Stresses and Maximum Shear Stress

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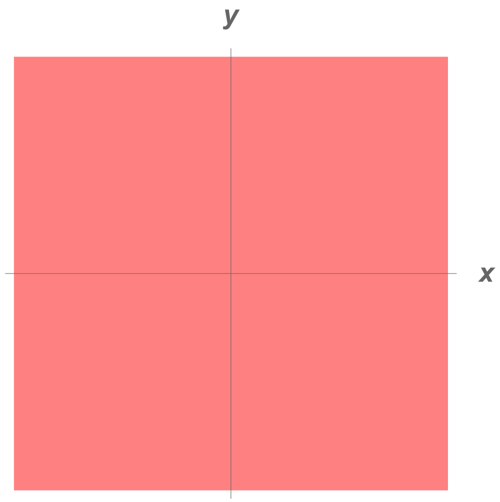
Variation of 2D stress components with rotation of axis

- Expression for stress transformation

$$\sigma_{x'x'} = \sigma_{xx}\cos^2\theta + 2\sigma_{xy}\sin\theta\cos\theta + \sigma_{yy}\sin^2\theta$$
$$\sigma_{x'y'} = -\sigma_{xx}\sin\theta\cos\theta + \sigma_{xy}\left(\cos^2\theta - \sin^2\theta\right) + \sigma_{yy}\sin\theta\cos\theta$$
$$\sigma_{y'y'} = \sigma_{xx}\sin^2\theta - 2\sigma_{xy}\sin\theta\cos\theta + \sigma_{yy}\cos^2\theta$$

$$\begin{bmatrix}\sigma_{x'x'} & \sigma_{x'y'}\\ \sigma_{x'y'} & \sigma_{y'y'}\end{bmatrix} = [Q]\begin{bmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{xy} & \sigma_{yy}\end{bmatrix}[Q]^T$$

- Lets plot for a specified state of stress



Plot of stress components as a function of rotation of axis

Maximum and Minimum normal stress

- Find the theta (direction) for max/min normal stress

- Find shear stress when normal stress is max/min

Maximum Shear Stress

- Find the theta (direction) for max/min shear stress

- Is the theta related to the theta for max/min normal stress?

- Is the max/min shear stress value related to the max/min normal stress?

Equivalence of stress states

Eigenvalue Analysis for Principal stresses

- Max/min normal stress or principal stresses are associated with zero shear stress
- Equation for traction on a surface perpendicular to unit vector $\{n\}$

$$\{T^{(n)}\} = [\sigma]\{n\} \qquad \{n\}^T\{n\} = 1$$

- Equation for zero shear stress (traction is in normal direction)

$$\{T^{(n)}\} = \sigma_p\{n\}$$

- Combining the above equations lead to an eigenvalue problem

$$[\sigma]\{n\} = \sigma_p\{n\} \qquad \{n\}^T\{n\} = 1$$

- Eigenvalues gives the principal stresses, eigenvectors the corresponding directions

Principal stresses from 3D state of stress

- Eigenvalue Analysis:

$$[\sigma]\{n\} = \sigma_p\{n\}$$

- We will obtain three eigenvalues
  - These are the normal stresses on surfaces where there is no shear stress
  - They are locally maximum, minimum, or saddle point normal stresses
  - These stresses are called the principal stress
- The three principal stresses are denoted by: σ_1, σ_2, σ_3
  - Typically $\sigma_1 > \sigma_2 > \sigma_3$
- Maximum shear stress for 3D state of stress:

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2}$$

Principal stresses from 2D state of stress

- Solving the 2D eigenvalue problem gives us two eigenvalues or two principal stresses: σ_1, σ_2
  - $\sigma_1 > \sigma_2$
- Maximum (in-plane) shear stress: $\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2}$

- Maximum shear stress (in any plane) for 2D state of stress
  - The third principal stress is zero so…
  - If $\sigma_1 > 0$ and $\sigma_2 < 0$, the three principal stresses are (σ_1, 0, σ_2): $\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2}$
  - If $\sigma_1 > 0$ and $\sigma_2 > 0$, the three principal stresses are (σ_1, σ_2, 0): $\tau_{\max} = \frac{\sigma_1 - 0}{2} = \frac{\sigma_1}{2}$
  - If $\sigma_1 < 0$ and $\sigma_2 < 0$, the three principal stresses are (0, σ_1, σ_2): $\tau_{\max} = \frac{0 - \sigma_2}{2} = -\frac{\sigma_2}{2}$