ACADEMIC PRESS

Journal of Algebra 255 (2002) 247–257

JOURNAL OF Algebra

www.academicpress.com

On the Littlewood–Richardson polynomials

Harm Derksen [a] and Jerzy Weyman [b,*,1]

[a] *Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA*
[b] *Department of Mathematics, Northeastern University, Boston, MA 02115, USA*

Received 20 October 2000

Communicated by Craig Huneke

Abstract

We prove the equivalence of several descriptions of generators of rings of semi-invariants of quivers, due to Domokos and Zubkov, Schofield and van den Bergh, and our earlier work. We also show that the dimensions of semi-invariants of weights $n\sigma$ depend polynomially on n.

Keywords: Semi-invariants; Quivers; Littlewood–Richardson coefficients

0. Introduction

Let Q be a quiver without oriented cycles. Let β be a dimension vector for Q. We denote by $\mathrm{SI}(Q,\beta)$ the ring of semi invariants of the β-dimensional representations of Q over a fixed algebraically closed field K. In [DW] we proved that the set

$$\Sigma(Q,\beta)=\left\{\sigma \;\middle|\; \mathrm{SI}(Q,\beta)_\sigma\neq 0\right\}$$

of weights for which the weight space is non-zero is defined in the space of all weights by one homogeneous linear equation and by a finite set of homogeneous linear inequalities. In particular the set $\Sigma(Q,\beta)$ is saturated, i.e., if $n\sigma\in\Sigma(Q,\beta)$

* Corresponding author.
1 Supported by NSF, grant DMS 9700884.

0021-8693/02/$ – see front matter
PII: S0021-8693(02)00125-4

then also $\sigma \in \Sigma(Q,\beta)$. The proof was based on the general result about semi- invariants. In the paper [S2] Schofield defined for each indecomposable representation W of Q a semi-invariant c_W. We showed that the semi-invariants of this type span each weight space in $\mathrm{SI}(Q,\beta)$.

In this note we continue the investigation of the rings of semi-invariants $\mathrm{SI}(Q,\beta)$. We prove that our results from [DW] imply the LeBruyn–Procesi–Donkin theorem on the rings of polynomial invariants of quivers. More generally, we derive from results of [DW] the recent result of Domokos and Zubkov [DZ] and Schofield and van den Bergh [SV] which says that for an arbitrary quiver the semi invariants are spanned by determinants of linear combinations of paths.

We also prove some results on the structure of the rings

$$\mathrm{SI}(Q,\beta,\sigma)=\bigoplus_{n\geqslant 0}\mathrm{SI}(Q,\beta)_{n\sigma}.$$

These rings play an important role in investigating various quotients of the space $\mathrm{Rep}(Q,\beta)$ of representations of Q of dimension β given by geometric invariant theory (comp. [K]).

We prove that if n is big enough then there exists a homogeneous system of parameters in $\mathrm{SI}(Q,\beta,\sigma)$ in degree n. We deduce that for a given weight σ the function $n\mapsto \dim \mathrm{SI}(Q,\beta)_{n\sigma}$ is polynomial in n. We also give consequences for the Hilbert function of the ring $\mathrm{SI}(Q,\beta,\sigma)$. Applying these results to the quiver $Q=T_{n,n,n}$, as in [DW], we deduce the following result on Littlewood–Richardson coefficients.

Let λ, μ, and ν be three highest weights for the special linear group. Denote by $c_\nu^{\lambda,\mu}$ the multiplicity of V_ν in $V_\lambda\otimes V_\mu$, where V_λ denotes the irreducible representation of $\mathrm{SL}(n)$ of highest weight λ (Schur functor). Then for three highest weights λ, μ, and ν the function

$$n\mapsto c_{n\nu}^{n\lambda,n\mu}$$

is a polynomial in n.

The paper is organized as follows. In Section 1 we recall the results from [DW]. In Section 2 we give the proofs of the Domokos–Schofield–van den Bergh–Zubkov and the LeBruyn–Procesi–Donkin theorems.

In Section 3 we discuss the new results on systems of parameters in the rings of semi invariants.

1. The results from [DW]

A quiver Q is a pair $Q=(Q_0,Q_1)$ consisting of the set of vertices Q_0 and the set of arrows Q_1. Each edge a has its head ha and tail ta, both in Q_0:

$$ta\xrightarrow{a} ha.$$

We fix an algebraically closed field K. A representation V of Q is a family of finite dimensional vector spaces $\{V(x) \mid x \in Q_0\}$ and of linear maps $V(a): V(ta) \to V(ha)$. The dimension vector of a representation V is the function $\underline{d}(V): Q_0 \to \mathbf{Z}$ defined by $\underline{d}(V)(x) := \dim V(x)$. The dimension vectors lie in the space Γ of integer-valued functions on Q_0. A morphism $\phi: V \to V'$ of two representations is the collection of linear maps $\phi(x): V(x) \to V'(x)$ such that for each $a \in Q_1$ we have $V'(a)\phi(ta) = \phi(ha)V(a)$. We denote the linear space of morphisms from V to V' by $\mathrm{Hom}_Q(V, V')$.

A path p in Q is a sequence of arrows $p = a_1, \ldots, a_n$ such that $ha_i = ta_{i+1}$ $(1 \leqslant i \leqslant n-1)$. We define $tp = ta_1$, $hp = ha_n$. For each $x \in Q_0$ we have trivial path $e(x)$ from x to x. We denote by $[x, y]$ the vector space on the basis of paths from x to y.

An oriented cycle is a path $p = a_1 \ldots a_n$ such that $ta_1 = ha_n$. In this section we assume that Q has no oriented cycles. This implies that the spaces $[x, y]$ are finite dimensional.

The category of representations of Q is hereditary, i.e., a subobject of a projective object is projective. This means that every representation has projective dimension $\leqslant 1$.

The spaces $\mathrm{Hom}_Q(V, W)$ and $\mathrm{Ext}_Q(V, W)$ can be calculated as the kernel and cokernel of the following linear map:

$$d_W^V : \bigoplus_{x \in Q_0} \mathrm{Hom}\bigl(V(x), W(x)\bigr) \to \bigoplus_{a \in Q_1} \mathrm{Hom}\bigl(V(ta), W(ha)\bigr),$$

where the map d_W^V restricted to $\mathrm{Hom}(V(x), W(x))$ is equal to

$$\sum_{a;\ ta=x} \mathrm{Hom}\bigl(\mathrm{id}_{V(x)}, W(a)\bigr) - \sum_{a;\ ha=x} \mathrm{Hom}\bigl(V(a), \mathrm{id}_{W(x)}\bigr).$$

Let α, β be two elements of Γ. We define the Euler inner product

$$\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha).$$

It follows that $\langle \underline{d}(V), \underline{d}(W) \rangle = \dim_K \mathrm{Hom}_Q(V, W) - \dim_K \mathrm{Ext}_Q(V, W)$.

For a dimension vector β we denote by $\mathrm{Rep}(Q, \beta)$ the vector space of representations of Q of dimension vector β. The groups $\mathrm{GL}(Q, \beta) := \prod_{x \in Q_0} \mathrm{GL}(\beta(x))$ and its subgroup $\mathrm{SL}(Q, \beta) = \prod_{x \in Q_0} \mathrm{SL}(\beta(x))$ acts on $\mathrm{Rep}(Q, \beta)$ in an obvious way. We are interested in the rings of semi invariants

$$\mathrm{SI}(Q, \beta) = K\bigl[\mathrm{Rep}(Q, \beta)\bigr]^{\mathrm{SL}(Q, \beta)}.$$

The ring $\mathrm{SI}(Q, \beta)$ has a weight space decomposition

$$\mathrm{SI}(Q, \beta) = \bigoplus_{\sigma} \mathrm{SI}(Q, \beta)_\sigma,$$

where σ runs through the characters of $\mathrm{GL}(Q,\beta)$ and

$$\mathrm{SI}(Q,\beta)_\sigma = \bigl\{ f \in K\bigl[\mathrm{Rep}(Q,\beta)\bigr] \bigm| g(f) = \sigma(g) f \ \forall g \in \mathrm{GL}(Q,\beta) \bigr\}.$$

Since every character of $\mathrm{GL}(Q,\beta)$ is a product of determinants the group of characters of $\mathrm{GL}(Q,\beta)$ is naturally identified with the space $\Gamma^* = \mathrm{Hom}_{\mathbf{Z}}(\Gamma, \mathbf{Z})$.

Let us choose the dimension vectors $\alpha = \underline{d}(V)$, $\beta = \underline{d}(W)$ of V, W in such way that $\langle \alpha, \beta \rangle = 0$. Then the matrix of d^V_W is a square matrix. Following [S2] we can therefore define the semi invariant

$$c(V, W) := \det d^V_W$$

of the action of $\mathrm{GL}(Q,\alpha) \times \mathrm{GL}(Q,\beta)$ on $\mathrm{Rep}(Q,\alpha) \times \mathrm{Rep}(Q,\beta)$. Notice that the semi-invariant c vanishes on the point (V, W) if and only if $\mathrm{Hom}_Q(V, W) \neq 0$ which is equivalent to $\mathrm{Ext}_Q(V, W) \neq 0$.

For a fixed V the restriction of c to $\{V\} \times \mathrm{Rep}(Q,\beta)$ defines a semi-invariant c^V in $\mathrm{SI}(Q, \underline{d}(W))$. Schofield proves [S2, Lemma 1.4] that the weight of c^V equals $\langle \alpha, - \rangle$. Similarly, for a fixed W the restriction of c to $\mathrm{Rep}(Q,\alpha) \times \{w\}$ defines a semi-invariant c_W in $\mathrm{SI}(Q, \underline{d}(V))$ of weight $-\langle -, \underline{d}(W) \rangle$ [S2, Lemma 1.4].

Notice that semi-invariants c^V have the following multiplicative property. If

$$0 \to V' \to V \to V'' \to 0$$

is an exact sequence, $\alpha' = \dim V'$, $\alpha'' = \dim V''$ with $\langle \alpha', \beta \rangle = \langle \alpha'', \beta \rangle = 0$, then $c^V = c^{V'} c^{V''}$.

The main result of [DW] is that the semi invariants of type c^V (respectively c_W) span all the weight spaces in the rings $\mathrm{SI}(Q,\beta)$.

Theorem 1. *Let Q be a quiver without oriented cycles and let β be a dimension vector. Then the weight space $\mathrm{SI}(Q,\beta)_\sigma$ is non-zero only when $\sigma = \langle \alpha, - \rangle$ for some dimension vector α with $\langle \alpha, \beta \rangle = 0$, and in such case the weight space $\mathrm{SI}(Q,\beta)_{\langle \alpha, - \rangle}$ is spanned over K by the semi-invariants c^V where V is a module of dimension α.*

Of course the analogous result is true for the semi-invariants c_W. For two-dimension vectors β, β' we will write $\beta' \hookrightarrow \beta$ if a general representation of dimension β contains a subrepresentation of dimension β'. Theorem 1 has the following remarkable consequence.

Theorem 2. *Let Q be a quiver without oriented cycles and let β be a dimension vector. The semigroup $\Sigma(Q,\beta)$ is the set of all weights σ such that $\sigma(\beta) = 0$ and $\sigma(\beta') \leqslant 0$ for all β' such that $\beta' \hookrightarrow \beta$. In particular the set $\Sigma(Q,\beta)$ is saturated.*

2. Semi-invariants and invariants for arbitrary quivers

In this section we apply the double quiver construction due to Schofield [S1, p. 56] to extend the results from [DW] to general quivers. In particular we prove that the description of semi invariants given by Domokos and Zubkov [DZ] and Schofield-van den Bergh [SV] can be derived in this way. In particular the same construction gives an easy proof of the LeBruyn–Procesi–Donkin theorem on polynomial invariants.

Let $Q=(Q_0,Q_1)$ be an arbitrary quiver. We construct the quiver $\widehat{Q}=(\widehat{Q}_0,\widehat{Q}_1)$ as follows. We set

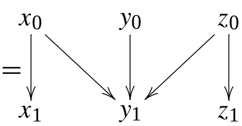
$$\widehat{Q}_0=Q_0\times\{0,1\}.$$

We denote the vertices corresponding to $x\in Q_0$ by x_0 and x_1. We set

$$\widehat{Q}_1=\{c_x:x_0\to x_1\}_{x\in Q_0}\cup\bigl\{\hat{a}:t(a)_0\to h(a)_1\bigr\}_{a\in Q_1}.$$

Example.

$$Q=x\longrightarrow y\longleftarrow z$$

$\widehat{Q}=$ 

Let Q be as above and let β be a dimension vector. We define the dimension $\hat{\beta}$ by setting $\hat{\beta}(x_0)=\hat{\beta}(x_1)=\beta(x)$. Consider the open set $U(\beta)$ in $\operatorname{Rep}(\widehat{Q},\hat{\beta})$ consisting of the representations V such that $V(c_x)$ is an isomorphism for all $x\in Q_0$. Then there is an isomorphism of algebraic varieties

$$\phi(\beta):U(\beta)\to\operatorname{Rep}(Q,\beta)\times\prod_{x\in Q_0}\operatorname{Hom}_K\bigl(V(x_0),V(x_1)\bigr)^{\times},$$

where $\operatorname{Hom}_K(V(x_0),V(x_1))^{\times}$ is the subset of non-singular linear maps in $\operatorname{Hom}_K(V(x_0),V(x_1))$. The map $\phi(\beta)$ is given by

$$\phi(\beta):\bigl(W(\hat{a}),W(c_x)\bigr)_{a\in Q_1,\ x\in Q_0}\mapsto\bigl(W(\hat{a})W(c_{ta})^{-1},W(c_x)\bigr)_{a\in Q_1,\ x\in Q_0}.$$

The map $\phi(\beta)$ is $\mathrm{GL}(Q,\beta)\times\prod_{x\in Q_0}\mathrm{GL}(W(x_0))$ equivariant, where $\mathrm{GL}(Q,\beta)$ acts via $\prod_{x\in Q_0}\mathrm{GL}(\widehat{Q},x_1)$ on $\operatorname{Rep}(\widehat{Q},\hat{\beta})$.

Interpreting this statement in terms of the rings $\mathrm{SI}(\widehat{Q},\hat{\beta})$ we get the following proposition.

Proposition 1. *There is an isomorphism of rings*

$$\phi:\mathrm{SI}(Q,\beta)\bigl[t_x,t_x^{-1}\bigr]_{x\in Q_0}\to\mathrm{SI}\bigl(\widehat{Q},\hat{\beta}\bigr)\bigl[\det(c_x)^{-1}\bigr]_{x\in Q_0},$$

where $\{t_x\}_{x\in Q_0}$ are independent variables. For $f \in \mathrm{SI}(Q,\beta)$ we define $\phi(f)$ by substituting the entries of $W(\hat{a})W((c_{ta}))^{-1}$ for the entries of $W(a)$, and $\phi(t_x) = \det(c_x)$.

Let us choose the dimension vector $\hat{\alpha}$ such that $\langle\hat{\alpha},\hat{\beta}\rangle = 0$, and let $\hat{\sigma} = \langle\hat{\alpha}, -\rangle$. For a representation $\widehat{V}$ of $\widehat{Q}$ we interpret the semi-invariant $c^{\widehat{V}}$ in terms of the quiver Q. The relation between $\hat{\sigma}$ and $\hat{\alpha}$ is as follows:

$$\begin{aligned}\forall x \in Q_0 \quad & \hat{\sigma}(x_1) = \hat{\alpha}(x_1),\\ & \hat{\sigma}(x_0) = \hat{\alpha}(x_0) - \hat{\alpha}(x_1) - \sum_{a\in Q_1,\ ta=x} \hat{\alpha}(ha_1),\\ & \hat{\alpha}(x_0) = \hat{\sigma}(x_0) + \hat{\sigma}(x_1) + \sum_{a\in Q_1,\ tp=x} \hat{\sigma}(hp_1).\end{aligned}$$

The condition $\langle\hat{\alpha},\hat{\beta}\rangle = 0$ means

$$\sum_{x\in Q_0}\Bigg(\hat{\alpha}(x_1) - \sum_{a\in Q_1,\ ha=x} \hat{\alpha}(ta_0)\Bigg)\beta(x) = 0.$$

Let $\widehat{V}$ be a representation of $\widehat{Q}$ of dimension $\hat{\alpha}$. By definition the semi-invariant $c^{\widehat{V}}$ is a determinant of a linear map

$$d^{\widehat{V}}_{\widehat{W}} : \bigoplus_{x\in\widehat{Q}_0} \mathrm{Hom}_K\big(\widehat{V}(x), \widehat{W}(x)\big) \to \bigoplus_{a\in\widehat{Q}_1} \mathrm{Hom}_K\big(\widehat{V}(ta), \widehat{W}(ha)\big)$$

defined as follows:

$$\big\{f(x)\big\}_{x\in\widehat{Q}_0} \mapsto \big\{f(ha)\widehat{V}(a) - \widehat{W}(a)f(ta)\big\}_{a\in\widehat{Q}_1}.$$

Let us decompose the domain of $d^{\widehat{V}}_{\widehat{W}}$ into the direct sum of

$$X(0) = \bigoplus_{x\in Q_0} \mathrm{Hom}_K\big(\widehat{V}(x_0), \widehat{W}(x_0)\big) \quad \text{and}$$

$$X(1) = \bigoplus_{x\in Q_0} \mathrm{Hom}_K\big(\widehat{V}(x_1), \widehat{W}(x_1)\big).$$

The codomain of $d^{\widehat{V}}_{\widehat{W}}$ decomposes into a direct sum of

$$Y(0) = \bigoplus_{x\in Q_0} \mathrm{Hom}_K\big(\widehat{V}(x_0), \widehat{W}(x_1)\big) \quad \text{and}$$

$$Y(1) = \bigoplus_{a\in Q_1} \mathrm{Hom}_K\big(\widehat{V}(ta_0), \widehat{W}(ha_1)\big).$$

We can write the matrix of $d_{\widehat{W}}^{\widehat{V}}$ in the block form

$$d_{\widehat{W}}^{\widehat{V}} = \begin{pmatrix} B_{0,0} & B_{0,1} \\ B_{1,0} & B_{1,1} \end{pmatrix}$$

corresponding to these decompositions.

It is clear that if $\widehat{W}$ is chosen from the set $U(\beta)$ then the block $B_{0,0}$ is an invertible square matrix. Its determinant is a monomial in $\det \widehat{W}(c_x)_{x\in Q_0}$. By row operations we can bring the matrix $d_{\widehat{W}}^{\widehat{V}}$ to the form

$$\begin{pmatrix} B_{0,0} & B_{0,1} \\ 0 & B_{1,1} - B_{1,0}B_{0,0}^{-1}B_{0,1} \end{pmatrix}.$$

Thus the determinant of $c^{\widehat{V}}$ is a product of the determinant of $B_{0,0}$ and the determinant of $B_{1,1} - B_{1,0}B_{0,0}^{-1}B_{0,1}$.

This last matrix can be written in block form as follows:

$$\{f_{x_1}\}_{x\in Q_0} \mapsto \sum_{b\in\widehat{Q}_1,\ hb=x_1} f_{x_1}\widehat{V}(b) - \sum_{a\in\widehat{Q}_1,\ ta=x_0} \widehat{W}(a)\widehat{W}(c_x)^{-1} f_{x_1}\widehat{V}(c_x).$$

Notice that for a fixed $\widehat{V}$ all the linear maps involved are linear combinations of identity maps and the maps of the form $\widehat{W}(a)\widehat{W}(c_{ta})^{-1}$ so as a semi-invariant determinant is a semi invariant of the type $\phi(f),\ f \in \mathrm{SI}(Q,\beta)$. Therefore, by Theorem 1 they span $\mathrm{SI}(Q,\beta)$ as a linear space over K. However, each of them is written in the following form:

$$g : \bigoplus_{i=1}^{n} W(x_i) \to \bigoplus_{j=1}^{m} W(y_j),$$

where $x_1,\dots,x_n, y_1,\dots,y_m \in Q_0$, $\sum_i \beta(x_i) = \sum_j \beta(y_j)$, and

$$g = \begin{pmatrix} g_{1,1} & \cdots & g_{1,n} \\ \vdots & \ddots & \vdots \\ g_{m,1} & \cdots & g_{m,n} \end{pmatrix},$$

where each $g_{j,i}$ is a linear combination of paths from x_i to y_j in Q, with the identity allowed when $x_i = y_j$. This means we have proved the next theorem.

Theorem 3 (Domokos, Schofield, van den Bergh, Zubkov). *Let Q be an arbitrary quiver. The determinants of the form $(*)$ span $\mathrm{SI}(Q,\beta)$ as a linear space over K.*

Let $Q = (Q_0, Q_1)$ be an arbitrary quiver. Let β be a dimension vector. The ring of polynomial invariants is just $K[\mathrm{Rep}(Q,\beta)]^{\mathrm{GL}(Q,\beta)} = \mathrm{SI}(Q,\beta)_0$ where 0 denotes the zero weight. Of course when Q has no oriented cycles this ring contains just the constants. The well-known result on polynomial invariants states that the ring $K[\mathrm{Rep}(Q,\beta)]^{\mathrm{GL}(Q,\beta)}$ is generated by the coefficients of characteristic polynomials of oriented cycles in Q. This result was proved by LeBruyn–Procesi [L-P] in characteristic 0 and by Donkin [D] in arbitrary characteristic. To be more precise, denote by $\mathrm{Loop}(Q)$ the set of paths p in Q such that $h(p) = t(p)$. For given dimension vector α and $p \in \mathrm{Loop}(Q)$ with $x = t(p) = h(p)$, we denote by $P_1(\alpha, p), \dots, P_{\alpha(x)}(\alpha, p)$ the coefficients of characteristic polynomial of the endo-morphism $V(p)$ of the space $V(x)$. These are clearly the elements of $K[\mathrm{Rep}(Q,\alpha)]^{\mathrm{GL}(Q,\alpha)}$.

Theorem 4 (LeBruyn–Procesi, Donkin). *The ring $K[\mathrm{Rep}(Q,\alpha)]^{\mathrm{GL}(Q,\alpha)}$ is generated by the invariants $P_1(\alpha, p), \dots, P_{\alpha(x)}(\alpha, p)$ for all $p \in \mathrm{Loop}(Q)$.*

Proof. Let f be a GL-invariant for Q, i.e., the semi invariant of weight 0. It is clear that $\phi(f) = g/h$ where h is a monomial $\prod_{x\in Q_0} \det(c_x)^{u_x}$ and g is a semi invariant of weight $\sum_{x\in Q_0} u_x \sigma(\det(c_x))$.

To prove the theorem we have to analyze the semi-invariant g. We perform the analysis used to prove Theorem 3. The semi-invariant g occurs in the weight $\hat{\sigma} = \sum_{x\in Q_0} u_x \sigma(c_x)$, i.e., $\hat{\sigma}(x_0) = u_x$, $\hat{\sigma}(x_1) = -u_x$. Expressing $\hat{\sigma}$ in the form $\langle \hat{\alpha}, -\rangle$ where $\langle\,,\rangle$ is the Euler form for $\widehat{Q}$, we get $\hat{\alpha}(x_0) = u_x$, $\hat{\alpha}(x_1) = \sum_{b\in Q_1,\ hb=x} u_{tb}$.

Considering the block $B_{1,1} - B_{1,0}B_{0,0}^{-1}B_{0,1}$ used in the proof of Theorem 3 and making the above identification, we see that the determinant of $B_{1,1} - B_{1,0}B_{0,0}^{-1}B_{0,1}$ can be written in the block form

$$\psi : \bigoplus_{i=1}^{m} W(x_i) \to \bigoplus_{i=1}^{m} W(x_i),$$

where the block $\psi_{i,j}$ is a linear combination of paths in Q, with the identity path (corresponding to $c_x^{-1}c_x$) included for $x_i = x_j$. This means that the semi-invariant $c^{\widehat{V}}$ can be written as a monomial in $\det(c_x)$ times the determinant of ψ. But working on the appropriate Zariski open set in the space $\mathrm{Rep}(\widehat{Q}, \alpha)$, we can use Gauss elimination to make the matrix ψ block uppertriangular. This means the determinant of ψ is a product of determinants of $\psi_{i,i}$. The block $\psi_{i,i}$ is a linear combination of loops in Q, with identity included, so its determinant is obviously a polynomial in coefficients of characteristic polynomials of loops in Q. This brings the semi-invariant g in a form from which the theorem follows. $\square$

3. The systems of parameters in the rings of semi-invariants

Let us fix the quiver Q without oriented cycles, the dimension vector β and the weight σ. In this section we investigate the ring

$$\mathrm{SI}(Q,\beta,\sigma) = \bigoplus_{n\geqslant 0} \mathrm{SI}(Q,\beta)_{n\sigma}.$$

Identifying σ with the character of $\mathrm{GL}(Q,\beta)$ we can identify the ring $\mathrm{SI}(Q,\beta,\sigma)$ with $K[\mathrm{Rep}(Q,\beta)]^H$ where $H=\mathrm{Ker}(\sigma)\subset\mathrm{GL}(Q,\beta)$.

Since H is reductive, the rings $\mathrm{SI}(Q,\beta,\sigma)$ are Cohen–Macaulay by Hochster–Roberts theorem.

Moreover, if $\mathrm{char}\,K=0$ then by Boutot theorem [Bou] the rings $\mathrm{SI}(Q,\beta,\sigma)$ have rational singularities.

The main result of this section is the following theorem.

Theorem 5. *For given Q,β,σ there exists such $N>0$ that for $m\geqslant N$ the ring $\mathrm{SI}(Q,\beta,\sigma)$ has a system of parameters consisting of homogeneous elements of degree m.*

Proof. Let $G=\mathrm{GL}(Q,\beta)$. The weight σ is a character of G. Let H be the kernel of σ. Let $W\in\mathrm{Rep}(Q,\beta)$ be a semistable element with respect to H-action. Let $Z=\overline{G\cdot W}$. The ring $K[Z]^H$ is a graded ring with respect to the grading induced by σ. Let us assume that the H-orbit of W is closed (otherwise we can take the closed H-orbit $H\cdot W'$ in the closure of $H\cdot W$, and $Z'=G\cdot W'$, we will have $K[Z]^H=K[Z'|^H)$.

We want to show that for to $m\gg 0$ there exists an H-invariant f of degree m such that $f(W)\neq 0$. Suppose that $f(W)\neq 0$ implies that the degree of f is divisible by d. We want to show that $d=1$. Let ζ be a primitive root of 1 of degree d. Let $g\in G$ be such that $\sigma(g)=\zeta$. There is no invariant which distinguishes closed orbits $gH\cdot W$ and $H\cdot W$, therefore $gH\cdot W$ and $H\cdot W$ must be the same orbit. This means there exists an element $g'\in G$ such that $g'\cdot W=W$ and $\sigma(g')=\zeta$. The stabilizer B of W in $\mathrm{Rep}(Q,\beta)$ is connected, because it is the open subset of invertible elements of $\mathrm{Hom}_Q(W,W)$. In particular σ maps B onto K^*. This means there exists $h\in B$ such that $\sigma(h)$ is not a root of 1. If $f\in K[Z]^H$, then f in invariant with respect to H and h. But the Zariski closure of the subgroup generated by H and h is G, so f is G-invariant and therefore constant. This gives a contradiction with the fact that W is semistable. This contradiction shows that in fact $d=1$ which proves the theorem. □

Corollary 1. *Let $H(Q,\beta,\sigma)(t)$ be the Poincaré series of the ring $\mathrm{SL}(Q,\beta,\sigma)$. Then*

$$H(Q,\beta,\sigma)(t)=\frac{P(t)}{(1-t)^d},$$

where $P(t)$ is a polynomial with rational coefficients and d is the Krull dimension of $\mathrm{SI}(Q,\beta,\sigma)$.

Proof. Let p,q be two primes bigger than N. The ring $\mathrm{SI}(Q,\beta,\sigma)$ has systems of parameters in degrees p and q, so

$$H(Q,\beta,\sigma)(t)=\frac{P_1(t)}{(1-t^p)^s}=\frac{P_2(t)}{(1-t^q)^s},$$

where $P_1(t), P_2(t)$ are polynomials with rational (non-negative) coefficients. Writing $H(Q,\beta,\sigma)(t)$ as a rational function with numerator and denominator relatively prime we get the statement of the corollary. □

Lemma 1. *The polynomial $P(t)$ in Corollary* 1 *has degree s where s is smaller than d.*

Proof. The invariant ring $A = \mathrm{SI}(Q,\beta,\sigma)$ has rational singularities by Boutot Theorem [Bou]. In particular A is Cohen–Macaulay and it has a canonical module K_A. We therefore have

$$H(Q,\beta,\sigma)(t) = (-1)^d t^a H_{K_A}(Q,\beta,\sigma)\bigl(t^{-1}\bigr),$$

$d = \dim A$. The degree a can be easily seen to be $s - d$.

This means that Lemma 1 follows instantly from the next lemma. We are grateful to Kei-Ichi Watanabe for providing the proof of Lemma 2. □

Lemma 2. *Let A be a graded finitely generated d-dimensional algebra over a field K of characteristic* 0 *with rational singularities. Let K_A be a canonical module over A. Assume that the Poincaré functions $H(A,t)$, $H_{K_A}(A,t)$ satisfy*

$$H(A,t) = (-1)^d t^a H_{K_A}\bigl(A, t^{-1}\bigr).$$

Then we have $a < 0$.

Proof. Using local duality we can reduce the question about the Poincaré function of the canonical module K_A to the question about the grading on top local cohomology module $H^d_m(A)$ where $m = \bigoplus_{n>0} A_n$ is the irrelevant maximal ideal of A. By the results of [Sm,H] we know that the rational singularities property is equivalent to the reductions of A modulo p being F-rational for primes $p \gg 0$ (see these papers for the definition of F-rationality). This means we can assume that A is F-rational in characteristic p. Then the socle z of the highest local cohomology module $H^d_m(A)$ is in degree a. Smith showed in [Sm] that the images of z under successive Frobenius action ($F^e(z)$'s) generate the whole module $H^d_m(A)$. Since $F^e(z)$ has degree qa ($q = p^e$) and $H^d_m(A) \cong A^*(-a)$, there are elements of negative degree in $H^d_m(A)$, and we conclude that the degree a is negative. □

Corollary 2. *The function $n \mapsto \dim \mathrm{SI}(Q,\beta)_{n\sigma}$ is polynomial in n.*

Proof. Obvious from Corollary 1 and Lemma 1. □

Corollary 3. *Let λ, μ, ν be three partitions. The function $n \mapsto c^{n\nu}_{n\lambda,n\mu}$ (where $c^{\nu}_{\lambda,\mu}$ is a Littlewood–Richardson coefficient) is polynomial in n.*

Proof. Apply the above corollary to the quiver $T_{n,n,n}$, as in Section 3 of [DW]. □

Remark. We understand that A. Knutson [Kn] has another proof of Corollary 3.

References

[Bou] J.-F. Boutot, Singularités rationelles et quotients par les groups reductifs, Invent. Math. 88 (1987) 65–68.

[D] S. Donkin, Polynomial invariants of representations of quivers, Comment. Math. Helv. 69 (1994) 137–141

[DW] H. Derksen, J. Weyman, Semi-invariants of quivers and saturation for Littlewood–Richardson coefficients, preprint, 1999.

[DZ] M. Domokos, A. Zubkov, Semi-invariants of quivers as determinants, preprint, 1999.

[H] N. Hara, Amer. Math. J. 120 (1998) 981–996.

[K] A.D. King, Moduli of representations of finite dimensional algebras, Quart. J. Math. Oxford (2) 45 (1994) 515–530.

[Kn] A. Knutson, personal communication, 1999.

[L-P] L. LeBruyn, C. Procesi, Semisimple representations of quivers, Trans. Amer. Math. Soc. 317 (1990) 585–598.

[S1] A. Schofield, General Representations of quivers, Proc. London Math. Soc. (3) 65 (1992) 46–64.

[S2] A. Schofield, Semi-invariants of quivers, J. London Math. Soc. 43 (1991) 383–395.

[SV] A. Schofield, M. van den Bergh, Semi-invariants of quivers for arbitrary dimension vectors, Indag. Math. (N.S.) 12 (2001) 125–138.

[Sm] K. Smith, Amer. Math. J. 119 (1997) 159–180.