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Semi-invariants for quivers with relations

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Dedicated to Professor Claudio Procesi on the occasion of his 60th birthday

0. Introduction

For a given quiver Q and a dimension vector β, let $\mathrm{SI}(Q,\beta)$ be the ring of semi-invariants for β-dimensional representations. Schofield defined in [S] for any α-dimensional representation V with $\langle\alpha,\beta\rangle = 0$ a semi-invariant $c^V \in \mathrm{SI}(Q,\beta)$. We recall that we proved in [DW] that the ring of semi-invariants $\mathrm{SI}(Q,\beta)$ is generated by such Schofield semi-invariants.

In this note we generalize the results of [DW] to quivers with relations, in the case when the base field has characteristic zero. We make this assumption throughout the paper.

Let Q be a quiver and let I be an ideal generated by admissible relations. The variety $\mathrm{Rep}(Q/I,\beta)$ of representations of Q satisfying the relations from I does not have to be irreducible. We are interested in the generators of rings of semi-invariants on the irreducible components of $\mathrm{Rep}(Q/I,\beta)$.

It is clear that the ring of semi-invariants on such components will be generated by the restrictions of the semi-invariants c^V where V is a representation of Q. However, such description is not very useful. We might have a case when the quiver Q with relations I is of a finite or tame type, but Q is wild. In such case one would still like to describe the semi-invariants of Q/I explicitly, but the description of indecomposable representations of Q is impossible. Therefore it

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is desirable to construct the generators of semi-invariants for Q/I that are related to the representations of Q/I.

In this note we solve this problem by exhibiting such sets of generators. We prove that if a component of $\operatorname{Rep}(Q/I,\beta)$ is faithful (i.e., a general module in this component does not satisfy additional relations) then the ring of semi-invariants is generated by the semi-invariants of type c^V for representations V of Q/I of projective dimension 1. The general case reduces easily to the case of a faithful component.

We also give an example showing that the saturation theorem from [DW] does not generalize to quivers with relations.

1. Basic definitions

Throughout this paper $Q=(Q_0,Q_1)$ will be a quiver without oriented cycles, where Q_0 is the (finite) set of vertices and Q_1 is the (finite) set of arrows. If $a\in Q_1$ is an arrow, then ta and ha denote its tail and its head, respectively.

A path is a sequence of arrows

$$p=a_1a_2\cdots a_s$$

with $ta_i=ha_{i+1}$ for all i. We define $tp=ta_s$ and $hp=ha_1$. For each vertex $x\in Q_0$ we also define the trivial path e_x of length 0, satisfying $te_x=he_x=x$. An oriented cycle is a nontrivial path satisfying $hp=tp$.

Let K be an algebraically closed field. The path algebra KQ is the K-vector space spanned by all paths. If p and q are paths, then their product $p\cdot q$ is the concatenation of the paths if $tp=hq$, and is defined 0 otherwise. The category $\operatorname{Rep}_K(Q)$ of representations of the quiver Q is the category of finite dimensional KQ-modules. If V is a representation of Q (i.e., a finite dimensional KQ-module) then we define $V(x)=e_xV$ for all $x\in Q_0$ and $V(p):V(tp)\to V(hp)$ is the restriction of multiplication with p to $V(tp)=e_{tp}V$ for every path p.

The path algebra is graded:

$$KQ=\bigoplus_{x,y\in Q_0} e_xKQe_y.$$

Let $r\in KQ$ be a relation, i.e.,

$$r=\sum_{i=1}^{s} c_ip_i$$

with p_i a path and $c_i\in K$ for all i. We say that the relation r is *admissible* if r is homogeneous with respect to the grading, i.e., there exist $tr,hr\in Q_0$ such that $tp_i=tr$ and $hp_i=hr$ for all i. Let us assume that I is an admissible ideal, i.e., a two sided ideal generated by admissible relations. We will call Q/I

a quiver with relations. The category $\mathrm{Rep}_K(Q/I)$ of representations of Q/I is the category of finite dimensional KQ/I-modules. We may assume that I is generated by admissible relations of length $\geqslant 2$, because otherwise the algebra KQ/I is a factor of a path algebra of a smaller quiver.

A dimension vector for Q is an element $\alpha \in \mathbb{N}^{Q_0}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of non-negative integers. We say that a representation V is α-dimensional if $\dim V(x) = \alpha(x)$ for all $x \in Q_0$.

For a dimension vector α we define the representation space by

$$\mathrm{Rep}_K(Q, \alpha) = \bigoplus_{a \in Q_1} \mathrm{Hom}\big(K^{\alpha(ta)}, K^{\alpha(ha)}\big).$$

Note that every element

$$V = \{V(a) \mid a \in Q_1\} \in \mathrm{Rep}_K(Q, \alpha)$$

can be viewed as a representation of Q. Groups $\mathrm{GL}(Q, \alpha) := \prod_{x \in Q_0} \mathrm{GL}(\alpha(x))$ and $\mathrm{SL}(Q, \alpha) := \prod_{x \in Q_0} \mathrm{SL}(\alpha(x))$ act on $\mathrm{Rep}_K(Q, \alpha)$ in a natural way. Two representations $V, W \in \mathrm{Rep}_K(Q, \alpha)$ are isomorphic if they lie in the same $\mathrm{GL}(Q, \alpha)$-orbit.

We also define

$$\mathrm{Rep}_K(Q/I, \alpha) \subseteq \mathrm{Rep}_K(Q, \alpha)$$

as the Zariski-closed subset defined by

$$\mathrm{Rep}_K(Q/I, \alpha) = \big\{V \in \mathrm{Rep}_K(Q, \alpha) \;\big|\; V(r) = 0 \text{ for all } r \in I \text{ homogeneous}\big\}.$$

The space $\mathrm{Rep}_K(Q/I, \alpha)$ does not have to be irreducible. We denote its irreducible components by $\mathrm{Rep}_K(Q/I, \alpha)_i$ $(i = 1, 2, \ldots, N(Q/I; \alpha))$. We are interested in the rings of semi-invariants

$$\mathrm{SI}\big(\mathrm{Rep}(Q/I, \alpha)_i\big) := K\big[\mathrm{Rep}(Q/I, \alpha)_i\big]^{\mathrm{SL}(Q,\alpha)}.$$

We recall that the Euler form for Q is a bilinear form on the space $\Gamma := \mathbb{Z}^{Q_0}$ defined by

$$\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha).$$

Every representation $V \in \mathrm{Rep}_K(Q)$ has a canonical resolution

$$0 \to \bigoplus_{a \in Q_1} V(ta) \otimes P_{ha} \to \bigoplus_{x \in Q_0} V(x) \otimes P_x \to V \to 0 \tag{1}$$

where $P_x = KQe_x$ is the indecomposable projective module associated to the vertex $x \in Q_0$. More precisely, denoting by $[x, y] := e_y KQ e_x$ the K-span of all paths from x to y, we have $P_x(y) = [x, y]$ with the linear map $P_x(a)$ acting by the left composition with a. We can also characterize P_x by the property $\mathrm{Hom}_R(P_x, W) = W(x)$ for all $W \in \mathrm{Rep}_K(Q)$.

Now (1) implies that the category $\mathrm{Rep}_K(Q)$ is hereditary (in particular, $\mathrm{Ext}^i_R(V,W)=0$ for all $i \geqslant 2$) and for $V \in \mathrm{Rep}_K(Q,\alpha)$, $W \in \mathrm{Rep}_K(Q,\beta)$ we have that the Euler characteristic is equal to

$$\chi(V,W) := \dim_K \mathrm{Hom}_R(V,W) - \dim_K \mathrm{Ext}_R(V,W) = \langle\alpha,\beta\rangle.$$

For a quiver Q with relations I we notice that the indecomposable projective modules again correspond to vertices from Q_0 and the module corresponding to $x \in Q_0$ is just $P'_x := P_x/IP_x$. They are characterized by the property that for each $W' \in \mathrm{Rep}_K(Q/I)$ we have $\mathrm{Hom}_{R/I}(P'_x, W') = W'(x)$.

Let $V' \in \mathrm{Rep}_K(Q/I,\alpha)$. We construct the module $\overline{P}_0 = \bigoplus_{x\in Q_0} V'(x)\otimes P'_x$. Then for each $V' \in \mathrm{Rep}_K(Q/I,\alpha)$ the kernel

$$0 \to V'_{(1)} \to \overline{P}_0 \to V' \to 0$$

has the same dimension vector. We define $\overline{P}_1$ by using the construction of $\overline{P}_0$ for $V'_{(1)}$. Continuing like that we construct the family of projective resolutions of modules from $\mathrm{Rep}_K(Q/I,\alpha)$ with fixed terms. The category of representations of Q satisfying the relations I is of finite global dimension ($\leqslant \operatorname{card} Q_0$) so the kernel $V'_{(m)}$ becomes projective for big m.

This construction allows to define the Euler form for the quiver Q with relations I. For two dimension vectors α and β we set

$$\langle\langle\alpha,\beta\rangle\rangle = \sum_{s\geqslant 0}(-1)^s \dim_K \mathrm{Ext}^s_{R/I}(V',W')$$

where V', W' are the modules from α, β respectively. The value of $\langle\langle\alpha,\beta\rangle\rangle$ does not depend on the choice of V', W' because we can use the family of resolutions of modules $V' \in \mathrm{Rep}_K(Q/I,\alpha)$ with fixed terms constructed above to calculate this Euler characteristic.

Finally we mention that the category $\mathrm{Rep}_K(Q/I)$ has projective covers. Thus we can talk about minimal sets of generators, minimal presentations and minimal projective resolutions.

2. The semi-invariants $\bar{c}^V$ for quivers with relations

Let $V' \in \mathrm{Rep}_K(Q/I)$. Let

$$\widetilde{P}_1 \to \widetilde{P}_0 \to V' \to 0$$

be the minimal presentation of V' in $\mathrm{Rep}_K(Q/I)$.

Definition. Let $W' \in \mathrm{Rep}_K(Q/I)$. We define the semi-invariant $\bar{c}^{V'}$ by setting $\bar{c}^{V'}(W')$ to be the determinant of the matrix

$$\mathrm{Hom}_{R/I}\big(\widetilde{P}_0, W'\big) \to \mathrm{Hom}_{R/I}\big(\widetilde{P}_1, W'\big)$$

whenever it is a square matrix.

The semi-invariant $\bar{c}^V$ is defined on the components $\mathrm{Rep}_K(Q/I,\beta)_j$ of the representation space $\mathrm{Rep}_K(Q/I,\beta)$ such that, for $W' \in \mathrm{Rep}_K(Q/I,\beta)_j$, $\dim \mathrm{Hom}_{R/I}(\widetilde{P}_0, W') = \dim \mathrm{Hom}_{R/I}(\widetilde{P}_1, W')$, while $\mathrm{Hom}_{R/I}(V', W') = 0$ for general $W' \in \mathrm{Rep}_K(Q/I,\beta)_j$.

Proposition 1. *Assume that* $\operatorname{char} K = 0$. *For each component* $\mathrm{Rep}_K(Q/I,\beta)_j$, *the semi-invariants* $\bar{c}^V$ span $\mathrm{SI}(Q/I, \mathrm{Rep}_K(Q/I,\beta)_j)$.

Proof. Let V be a representation of Q such that c^V is non-zero on component $\mathrm{Rep}_K(Q/I,\beta)_j$. Let $V' = V/IV$. Let

$$0 \to P_1 \to P_0 \to V \to 0$$

be a minimal resolution of V in $\mathrm{Rep}_K(Q)$. Tensoring with R/I over R we get the exact sequence

$$P_1' \to P_0' \to V' \to 0$$

where $P_s' = P_s/IP_s$ for $s = 0, 1$. The module P_0 is a projective cover of V, so P_0' is a projective cover of V'. Thus we have $\widetilde{P}_0 = P_0'$. Also the module $\widetilde{P}_1$ is a direct summand of P_1'.

Let W' be a general module from $\mathrm{Rep}_K(Q/I,\beta)_j$. We know that

$$\mathrm{Hom}_R(P_0, W') \to \mathrm{Hom}_R(P_1, W')$$

is a square matrix. It follows that

$$\mathrm{Hom}_{R/I}(P_0', W') \to \mathrm{Hom}_{R/I}(P_1', W')$$

is a square matrix. On the other hand we have an exact sequence

$$0 \to \mathrm{Hom}_{R/I}(V', W') \to \mathrm{Hom}_{R/I}(\widetilde{P}_0, W') \to \mathrm{Hom}_{R/I}(\widetilde{P}_1, W').$$

Since $\mathrm{Hom}_{R/I}(V', W') = \mathrm{Hom}_R(V, W')$ is zero for generic W', we know that

$$\dim \mathrm{Hom}_{R/I}\big(\widetilde{P}_1, W'\big) \geqslant \dim \mathrm{Hom}_{R/I}\big(\widetilde{P}_0, W'\big).$$

Since $\widetilde{P}_1$ is a direct summand of P_1', we must have the equality. This means that $\bar{c}^{V'}$ is the restriction of c^V to $\mathrm{Rep}(Q/I,\beta)_j$. Since in characteristic zero the general linear groups is linearly reductive, we know that the restrictions of semi-invariants c^V to $\mathrm{Rep}_K(Q/I,\beta)_j$ span $\mathrm{SI}(Q/I,\beta_j)$ which concludes the proof. □

The proof of the proposition can be also applied to arbitrary presentation of V' to obtain the following result.

Corollary 1. *The semi-invariant* $\bar{c}^{V'}$ *exists and is non-zero on the component with a general module* W' *if there exists a presentation*

$$P_1' \to P_0' \to V' \to 0$$

for which the matrix

$$\mathrm{Hom}_{R/I}(P_0', W') \to \mathrm{Hom}_{R/I}(P_1', W')$$

is invertible. This condition implies that $\mathrm{Hom}_{R/I}(V', W') = \mathrm{Ext}^1_{R/I}(V', W') = 0$.

The semi-invariants $\bar{c}^{V'}$ have the same multiplicative property as the semi-invariants c^V.

Proposition 2. *Consider the exact sequence*

$$0 \to V_1' \to V' \to V_2' \to 0$$

of representations of Q *satisfying the relations* I. *Assume that* $\bar{c}^{V_1'}$ *and* $\bar{c}^{V_2'}$ *are defined and non-zero on some component* $\mathrm{Rep}_K(Q/I, \beta)_j$. *Then* $\bar{c}^{V'}$ *is also defined and it equals* $\bar{c}^{V_1'}\bar{c}^{V_2'}$.

Proof. Let

$$P_1^{(i)} \to P_0^{(i)} \to V_i' \to 0$$

be minimal presentations of V_i' for $i = 1, 2$. This gives a partial projective resolution

$$P_1^{(1)} \oplus P_1^{(2)} \to P_0^{(1)} \oplus P_0^{(2)} \to V' \to 0$$

which might not be minimal. However it gives rise to an invertible matrix

$$\mathrm{Hom}_{R/I}\big(P_0^{(1)} \oplus P_0^{(2)}, W'\big) \to \mathrm{Hom}_{R/I}\big(P_1^{(1)} \oplus P_1^{(2)}, W'\big)$$

which shows by Corollary 1 that $\bar{c}^{V'}$ is non-zero and $\bar{c}^{V'} = \bar{c}^{V_1'}\bar{c}^{V_2'}$. □

Let us look at semi-invariants on some fixed component $\mathrm{Rep}_K(Q/I, \beta)_j$. It might happen that the general module W' in this component satisfies more relations than just relation from I. Then it is more convenient to treat this component as a component of some quiver Q with a bigger ideal of relations. To avoid this phenomenon we make the following definition.

Definition. Let Q be a quiver and let I be an admissible ideal in KQ. A component $\mathrm{Rep}_K(Q/I, \beta)_j$ is faithful over R/I if every element from R acting trivially on every module from $\mathrm{Rep}_K(Q/I, \beta)_j$ is in I.

Theorem 1. *Assume that* $\operatorname{char} K = 0$. *Let Q be a quiver without oriented cycles and let I be an admissible ideal of relations. Let* $\operatorname{Rep}_K(Q/I, \beta)_j$ *be a faithful component of* $\operatorname{Rep}_K(Q/I, \beta)$. *Then* $\operatorname{SI}(Q/I, \operatorname{Rep}_K(Q/I, \beta)_j)$ *is spanned by the semi-invariants $\bar{c}^{V'}$ for R/I-modules V' of projective dimension* 1 *and of dimension vectors α such that* $\langle\langle \alpha, \beta \rangle\rangle = 0$.

Proof. We know by Proposition 1 that the ring $\operatorname{SI}(Q/I, \operatorname{Rep}_K(Q/I, \beta)_j)$ is spanned by the semi-invariants of the type $\bar{c}^{V'}$. Let us assume that a representation V' such that the semi-invariant $\bar{c}^{V'}$ is non-zero on $\operatorname{Rep}_K(Q/I, \beta)_j$ has projective dimension $\geqslant 2$. Let us write a minimal resolution of V'

$$\cdots \to \widetilde{P}_2 \to \widetilde{P}_1 \to \widetilde{P}_0.$$

Let W' be a general representation in $\operatorname{Rep}_K(Q/I, \beta)_j$. Let us look at the induced complex

$$\operatorname{Hom}_{R/I}\big(\widetilde{P}_0, W'\big) \to \operatorname{Hom}_{R/I}\big(\widetilde{P}_1, W'\big) \to \operatorname{Hom}_{R/I}\big(\widetilde{P}_2, W'\big).$$

The left hand side map is generically an isomorphism, so the right hand side map is zero on $\operatorname{Rep}_K(Q/I, \beta)_j$. Let us look at this map in more detail. Let us write $\widetilde{P}_2 = \bigoplus_{s=1}^m \overline{P}(x_s)$ and $\widetilde{P}_1 = \bigoplus_{t=1}^p \overline{P}(y_t)$. Then the map in the resolution of V' is a matrix of (linear combinations of) paths $p_{s,t}$ in Q from y_t to x_s. The induced map is a map from $\bigoplus_{t=1}^p W'(y_t)$ to $\bigoplus_{s=1}^m W'(x_s)$ with matrix entries being $W'(p_{s,t})$. We conclude that all the maps $W'(p_{s,t})$ are zero. Since the component $\operatorname{Rep}_K(Q/I, \beta)_j$ is faithful, we see that all paths $p_{s,t}$ are in I. This means that the map $\widetilde{P}_2 \to \widetilde{P}_1$ is zero which gives a contradiction. The conclusion is that if V' has projective dimension > 1 then $\bar{c}^{V'}$ is zero when restricted to $\operatorname{Rep}_K(Q/I, \beta)_j$. This concludes the proof of the theorem. □

Example. Let Q be a quiver

$$x \xrightarrow{a} y \xrightarrow{b} z$$

with the ideal I generated by ba. Let us consider the dimension vector β with $\beta(x) = \beta(y) = \beta(z) = 1$. The variety $\operatorname{Rep}_K(Q/I, \beta)$ has two components. Let us look at the component $\operatorname{Rep}_K(Q/I, \beta)_1$ of modules W' for which $W'(b) = 0$. Let V' be a representation

$$K \to 0 \to 0.$$

The representation V' has projective dimension 2 and $\bar{c}^{V'}(W') = \det W'(a) \neq 0$. The reason is precisely that the component $\operatorname{Rep}_K(Q/I, \beta)_1$ is not faithful. In our approach it would be treated as a representation of Q with the relation $b = 0$.

3. An example

In this section we illustrate our result by analyzing an example of a quiver with relations. We consider the quiver Q

$$x \underset{a_2}{\overset{a_1}{\rightrightarrows}} y \underset{b_2}{\overset{b_1}{\rightrightarrows}} z$$

with the relations $b_1a_1 = 0$, $b_2a_2 = 0$. We consider the dimension vector β defined by $\beta(x) = m$, $\beta(y) = m + n$, $\beta(z) = n$. We assume that $\mathrm{GCD}(m, n) = 1$.

The Euler form for Q is given by the matrix

$$E = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The Euler form for the quiver Q with relations I is given by the matrix

$$E' = \begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The variety $\mathrm{Rep}_K(Q/I, \beta)$ is irreducible because it is a product of two varieties of complexes of dimension β which are irreducible by [DS]. The only component of this variety is faithful.

We analyze the ring of semi-invariants $\mathrm{SI}(Q/I, \beta)$ on $\mathrm{Rep}_K(Q/I, \beta)$.

For a group $\mathrm{GL}(n)$ we work with highest weights $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \in \mathbb{Z}$, $\lambda_1 \geqslant \cdots \geqslant \lambda_n$. For a highest weight λ we denote by $S_\lambda(K^n)$ the irreducible representation of $\mathrm{GL}(n)$ with highest weight λ. It is always a Schur functor tensored with a power of the determinant representation.

The coordinate ring of the variety $\mathrm{Rep}(Q/I, \beta)$ has the following decomposition to irreducible representations of the group $\mathrm{GL}(V(x)) \times \mathrm{GL}(V(y)) \times \mathrm{GL}(V(z))$ (see [DS]):

$$\begin{aligned} &K\bigl[\mathrm{Rep}(Q/I, \beta)\bigr] \\ &\quad = \Bigl(\bigoplus_{\lambda^{(1)},\mu^{(1)}} S_{\lambda^{(1)}} V(x) \otimes S_{\mu^{(1)},-\lambda^{(1)}} V(y) \otimes S_{\mu^{(1)}} V(z)^*\Bigr) \\ &\qquad \otimes \Bigl(\bigoplus_{\lambda^{(2)},\mu^{(2)}} S_{\lambda^{(2)}} V(x) \otimes S_{\mu^{(2)},-\lambda^{(2)}} V(y) \otimes S_{\mu^{(2)}} V(z)^*\Bigr). \end{aligned}$$

Here we sum over all partitions $\lambda^{(i)} = (\lambda_1^{(i)}, \ldots, \lambda_m^{(i)})$ and $\mu^{(i)} = (\mu_1^{(i)}, \ldots, \mu_n^{(i)})$ (for $i = 1, 2$) and the weight $(\mu^{(i)}, -\lambda^{(i)})$ is the weight

$$\bigl(\mu_1^{(i)}, \ldots, \mu_n^{(i)}, -\lambda_m^{(i)}, \ldots, -\lambda_1^{(i)}\bigr).$$

The summand corresponding to $\lambda^{(1)}, \lambda^{(2)}, \mu^{(1)}, \mu^{(2)}$ can contain at most one semi-invariant. This follows from Littlewood–Richardson rule. Moreover, Corollary 1.6 from [SW] implies that the semi-invariant occurs in such summand if and only if

$$\lambda_j^{(1)} - \lambda_{j+1}^{(1)} = \lambda_{m+1-j}^{(2)} - \lambda_{m-j}^{(2)}$$

for $j = 1, \ldots, m-1$,

$$\begin{aligned}\big(\mu^{(1)}, -\lambda^{(1)}\big)_j - \big(\mu^{(1)}, -\lambda^{(1)}\big)_{j+1} = {} & \big(\mu^{(2)}, -\lambda^{(2)}\big)_{m+n+1-j} \\ & - \big(\mu^{(2)}, -\lambda^{(2)}\big)_{m+n-j}\end{aligned}$$

for $j = 1, \ldots, m+n-1$,

$$\mu_j^{(1)} - \mu_{j+1}^{(1)} = \mu_{n+1-j}^{(2)} - \mu_{n-j}^{(2)}$$

for $j = 1, \ldots, n-1$.

Since $\mathrm{GCD}(m, n) = 1$, it follows easily that the semi-invariant occurs in our summand if and only if all the differences of consecutive numbers in all six weights are the same. If all those differences are zero, all partitions $\lambda^{(i)}, \mu^{(i)}$ are zero and the corresponding semi-invariant is a constant. Let us analyze the possibilities for the case when all differences are 1. In particular, we have

$$\lambda_m^{(i)} + \mu_n^{(i)} = 1$$

for $i = 1, 2$. This means that we have four possibilities

$$\lambda^{(1)} = 0, \qquad \mu^{(1)} = 1, \qquad \lambda^{(2)} = 0, \qquad \mu^{(2)} = 1$$

giving the semi-invariant $A_{0,0}$,

$$\lambda^{(1)} = 0, \qquad \mu^{(1)} = 1, \qquad \lambda^{(2)} = 1, \qquad \mu^{(2)} = 0$$

giving the semi-invariant $A_{0,1}$,

$$\lambda^{(1)} = 1, \qquad \mu^{(1)} = 0, \qquad \lambda^{(2)} = 0, \qquad \mu^{(2)} = 1$$

giving the semi-invariant $A_{1,0}$,

$$\lambda^{(1)} = 1, \qquad \mu^{(1)} = 0, \qquad \lambda^{(2)} = 1, \qquad \mu^{(2)} = 0$$

giving the semi-invariant $A_{1,1}$.

Analyzing possibilities for the higher differences we see that the ring $\mathrm{SI}(Q/I, \beta)$ is generated by the semi-invariants $A_{0,0}, A_{1,0}, A_{0,1}, A_{1,1}$ with one relation $A_{0,0}A_{1,1} = A_{1,0}A_{0,1}$.

The weights of generating semi-invariants are as follows:

$$\begin{aligned}\sigma(A_{0,0}) &= (m-1, n-m+1, -n-1), \\ \sigma(A_{1,0}) &= \sigma(A_{0,1}) = (m, n-m, -n), \\ \sigma(A_{1,1}) &= (m+1, n-m-1, -n+1).\end{aligned}$$

Expressing these weights in the form $\langle\langle \alpha, - \rangle\rangle$ gives

$$\begin{aligned}
\sigma(A_{0,0}) &= (m-1, n-m+1, -n-1)\\
&= (m-1 \quad m+n-1 \quad n-1)\begin{pmatrix}1 & -2 & 2\\ 0 & 1 & -2\\ 0 & 0 & 1\end{pmatrix},\\
\sigma(A_{1,0}) &= \sigma(A_{0,1}) = (m, n-m, -n)\\
&= (m \quad m+n \quad n)\begin{pmatrix}1 & -2 & 2\\ 0 & 1 & -2\\ 0 & 0 & 1\end{pmatrix},\\
\sigma(A_{1,1}) &= (m+1, n-m-1, -n+1)\\
&= (m+1 \quad m+n+1 \quad n+1)\begin{pmatrix}1 & -2 & 2\\ 0 & 1 & -2\\ 0 & 0 & 1\end{pmatrix}.
\end{aligned}$$

This means the generating semi-invariants come from the modules of dimensions

$$(m-1, m+n-1, n-1), \quad (m, m+n, n), \quad (m+1, m+n+1, n+1).$$

We will write down the projective resolutions of general modules of these dimensions. We know that the projective modules P'_x, P'_y, P'_z have dimensions $(1,2,2)$, $(0,1,2)$, $(0,0,1)$, respectively. Thus the general module M of dimension $(m-1, m+n-1, n-1)$ has a minimal resolution

$$0 \to (n+1)P'_z \to (m-1)P'_x \oplus (n-m+1)P'_y \to M \to 0$$

for $n \geqslant m-1$, and

$$0 \to (n+1)P'_z \oplus (m-1-n)P'_y \to (m-1)P'_x \to M \to 0$$

for $n < m-1$.

The general module of dimension $(m, m+n, n)$ has a minimal resolution

$$0 \to (n)P'_z \to (m)P'_x \oplus (n-m)P'_y \to M \to 0$$

for $m \leqslant n$, and

$$0 \to (n)P'_z \oplus (m-n)P'_y \to (m)P'_x \to M \to 0$$

for $n < m$. Finally the general module of dimension $(m+1, m+n+1, n+1)$ has a minimal resolution

$$0 \to (n-1)P'_z \to (m+1)P'_x \oplus (n-m-1)P'_y \to M \to 0$$

for $n \geqslant m+1$, and

$$0 \to (n-1)P'_z \oplus (m+1-n)P'_y \to (m+1)P'_x \to M \to 0$$

for $n < m+1$.

In all the cases we see that the semi-invariants come from determinants of resolutions of modules of projective dimension 1 over KQ/I as asserted in Theorem 1.

4. A counterexample to saturation

In this section we show that the rings of semi-invariants for quivers with relations do not have to have saturated sets of weights. Consider the quiver Q

$$x \underset{a_3}{\overset{a_1}{\underset{\xrightarrow{\;a_2\;}}{\rightrightarrows}}} y \underset{b_3}{\overset{b_1}{\underset{\xrightarrow{\;b_2\;}}{\rightrightarrows}}} z$$

with the relations

$$(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3)(\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3) = 0$$

i.e., $b_s a_s = 0$ for $s = 1, 2, 3$ and $b_s a_t + b_t a_s = 0$ for $s, t = 1, 2, 3$, $s \neq t$. Let β be the dimension vector $\beta(x) = 1$, $\beta(y) = \beta(z) = 3$.

Proposition 3. *There is a unique component* $\operatorname{Rep}_K(Q/I, \beta)_1$ *of* $\operatorname{Rep}_K(Q/I, \beta)$ *containing representations* W' *with* $\det(W'(a_1), W'(a_2), W'(a_3)) \neq 0$. *Let* σ *be the weight with* $\sigma(x) = 0$, $\sigma(y) = 1$, $\sigma(z) = -1$. *Then we have*

$$\operatorname{SI}\big(Q/I, \operatorname{Rep}_K(Q/I, \beta)_1\big)_\sigma = 0 \quad \textit{and} \quad \operatorname{SI}\big(Q/I, \operatorname{Rep}_K(Q/I, \beta)_1\big)_{n\sigma} \neq 0$$

for some positive integer n.

Proof. Let us look at the open set of representations $W' \in \operatorname{Rep}_K(Q/I, \beta)$ such that $\det(W'(a_1), W'(a_2), W'(a_3)) \neq 0$. Denote by 1 the basis vector in $W'(x)$. Choose the basis $e_1 := W'(a_1)(1)$, $e_2 := W'(a_2)(1)$, $e_3 := W'(a_3)(1)$ of $W'(y)$. Our conditions boil down to $W'(b_s)(e_s) = 0$ (for $s = 1, 2, 3$), $W'(b_s)(e_t) = -W'(b_t)(e_s)$ (for $s, t = 1, 2, 3$, $s \neq t$). These conditions clearly define an affine space. Its Zariski closure gives our component $\operatorname{Rep}_K(Q/I, \beta)_1$. The semi-invariants of the weight σ are spanned by the semi-invariants

$$\det\big(\lambda_1 W'(b_1) + \lambda_2 W'(b_2) + \lambda_3 W'(b_3)\big).$$

Each such semi-invariant is zero, since by definition the map $\lambda_1 W'(b_1) + \lambda_2 W'(b_2) + \lambda_3 W'(b_3)$ has a non-zero kernel (namely $\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$).

Define a semi-invariant f of weight 2σ by

$$f(W) := \det\begin{pmatrix} W(b_1) & W(b_2) \\ W(b_2) & W(b_3) \end{pmatrix}.$$

Let $W \in \operatorname{Rep}(Q/I, \beta)_1$ be the representation given by

$$W(a_1) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad W(a_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad W(a_3) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

and

$$W(b_1)=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix},\qquad W(b_2)=\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix},$$
$$W(b_3)=\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}.$$

Now f is non-zero on $\operatorname{Rep}(Q/I,\beta)_1$ because

$$f(W)=\det\begin{pmatrix}0&1&0&0&0&-1\\-1&0&0&0&0&0\\0&0&0&1&0&0\\0&0&-1&0&0&0\\0&0&0&0&0&1\\1&0&0&0&-1&0\end{pmatrix}=1. \qquad \square$$

Note added in proof

We were informed by the referee that M. Domokos proved similar results in the paper "Relative invariants for representations of finite dimensional algebras," which in the meantime appeared in [*Manuscripta Math.* 108 (1) (2002) 123–133].

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