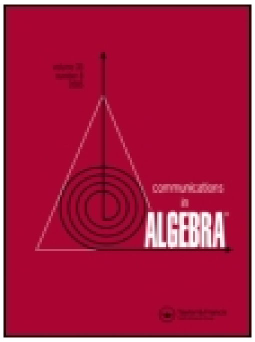

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Inverse degrees and the Jacobian conjecture

Harm Derksen
University of Nijmegen
Department of Mathematics
Toernooiveld
6525 ED Nijmegen
The Netherlands
E-mail: hderksen@sci.kun.nl

Abstract

Bass, Connell and Wright proved that if the Jacobian Conjecture is true, then there exists a number $C(d)$ such that for every k-algebra A and every invertible polynomial map $F : A^n \to A^n$ with $\det(J(F)) = 1$ the degree of F^{-1} is bounded by $C(d)$. A year later Bass proved the converse. In this paper we give a short proof of this last result.

In this paper k is a field of characteristic 0. If $F = (F_1, \ldots, F_n) : k^n \to k^n$ is a polynomial map, then we define the degree of F as the maximum degree of the polynomials $F_1, \ldots, F_n$. This paper is about the equivalence of the following two conjectures:

Conjecture 1 JC(n): *If $F : k^n \to k^n$ is a polynomial map with $\det(J(F)) = 1$ then F is invertible, e.g. there exists a polynomial map $G : k^n \to k^n$ such that $G \circ F = F \circ G = \text{id}$.*

This conjecture is known as the Jacobian conjecture (c.f, [3, 4, 6]), and it was first formulated by O.H. Keller in [5].

Conjecture 2 BI(n): *For every finite dimensional k-algebra A the following holds:*
If $F : A^n \to A^n$ is an invertible polynomial map such that $\det(J(F)) = 1$, then the degree of F^{-1} is bounded by a constant $C(d)$, depending only on d (and n), and not on the choice of A.

If A is a field, then one can take $C(d) = d^{n-1}$. This result is due to Gabber (c.f. [2, Ch. I, Cor. (1.4).]). if A is a reduced ring, then it is easy to reduce it to the case that A is a field. For a reduced ring A we can also take $C(d) = d^{n-1}$ as the upper bound. If A is allowed to have nilpotent elements then it is not known whether there can be given an upper bound for the degree of the inverse. In [2, Ch. I, Prop. (1.2).] Bass, Connell and Wright proved that JC(n) implies BI(n). Bass proved in [1] the converse. We will now give a short proof of this implication:

Theorem 1 *If the conjecture* BI(n) *is true then the jacobian conjecture* JC(n) *is true.*

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Proof: Assume BI(n) holds and assume that $F : k^n \to k^n$ is a polynomial map satisfying $\det(J(F)) = 1$. Without loss of generality we may assume that $F(0) = 0$. Now F is locally invertible in the neigbourhood of 0, e.g. there exist $G_1, G_2, \ldots, G_n \in k[[X_1, \ldots, X_n]]$ such that $F_i(G_1, \ldots, G_n) = X_i$ for all i. Write

$$F = F_{(1)} + F_{(2)} + \ldots + F_{(d)} \text{ and } G = G_{(1)} + G_{(2)} + G_{(3)} + \ldots$$

where $F_{(i)}$ and $G_{(i)}$ are homogeneous of degree d for all i. Let us define $\widehat{F} : k[t]^n \to k[t]^n$ by

$$\widehat{F} = t^{-1}F(tX) = F_{(1)} + tF_{(2)} + \ldots + t^{d-1}F_{(d)}$$

and likewise define

$$\widehat{G} = t^{-1}G(tX) = G_{(1)} + tG_{(2)} + t^2G_{(3)} + \ldots$$

Since $J(\widehat{F}) = J(t^{-1}F(tX)) = J(F)(tX)$, we have $\det(J(\widehat{F})) = \det(J(F))(tX) = 1$. Now $\widehat{G}$ is the formal inverse of $\widehat{F}$ because

$$\widehat{F} \circ \widehat{G} = t^{-1}F(tt^{-1}G(tX)) = t^{-1}F(G(tX)) = t^{-1}tX = X$$

Choose $l > C(d)$ arbitrary. We now calculate modulo t^l. Define $\overline{F} : (k[t]/(t^l))^n \to (k[t]/(t^l))^n$ and $\overline{G}$ to be $\widehat{F}$ respectively $\widehat{G}$ modulo t^l. Again we get $\det(J(\overline{F})) = 1$ and $\overline{F} \circ \overline{G} = X$. So $\overline{G}$ is the inverse of $\overline{F}$, hence the degree of $\overline{G}$ is bounded by $C(d)$. Since

$$\overline{G} = G_{(1)} + tG_{(2)} + \ldots + t^{l-1}G_{(l)}$$

it follows that $t^{l-1}G_{(l)} = 0$. The fact that $t^{l-1} \neq 0$ (in $k[t]/(t^l)$) forces $G_{(l)} = 0$. We can conclude that $G_{(l)} = 0$ for alle $l > C(d)$. So G is also a polynomial map and this proves that F is invertible. □

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