



# Counterexamples to Okounkov’s log-concavity conjecture

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## ABSTRACT

We give counterexamples to Okounkov’s log-concavity conjecture for Littlewood–Richardson coefficients.

## 1. Introduction

Motivated by physical considerations, Okounkov [Oko03, Conjecture 1] has conjectured that the Littlewood–Richardson coefficients  $c_{\mu,\nu}^\lambda$  are log-concave in  $(\lambda, \mu, \nu)$ . A particular version of this conjecture would be (see also [Ful00, pp. 239]) as follows.

CONJECTURE 1.1 (Okounkov’s log-concavity conjecture). Let  $\lambda, \mu, \nu$  be three partitions. Then

$$c_{(N+1)\mu, (N+1)\nu}^{(N+1)\lambda} \cdot c_{(N-1)\mu, (N-1)\nu}^{(N-1)\lambda} \leq (c_{N\mu, N\nu}^{N\lambda})^2,$$

for every integer  $N \geq 1$ .

Important implications of this conjecture are also discussed in [Oko03]. It is easy to see that Conjecture 1.1, if true, would immediately imply a conjecture of Fulton on Littlewood–Richardson coefficients (see [Bel05] or [KTW04]). Moreover, the log-concavity of the Littlewood–Richardson coefficients as a function of highest weights would imply the saturation conjecture for Littlewood–Richardson coefficients (see [DW00] or [KT99]) and the Schur-log-concavity conjecture for skew-Schur functions (see [LPP05]). We should also point out that the results in [TZ04] give some evidence for the log-concavity conjecture. However, the conjecture turns out to be false in general.

In this paper, we construct infinite families of counterexamples to Conjecture 1.1 (and hence to Okounkov’s original conjecture), as follows.

THEOREM 1.2. Let  $n \geq 1$  be an integer and let  $\lambda(n), \mu(n)$  be two partitions defined by

$$\lambda(n) = (4^n, 3^{2^n}, 2^n) \quad \text{and} \quad \mu(n) = (3^n, 2^n, 1^n).$$

Then

$$c_{\mu(n), \mu(n)}^{\lambda(n)} = \binom{n+2}{2} \quad \text{and} \quad c_{2\mu(n), 2\mu(n)}^{2\lambda(n)} = \binom{n+5}{5}.$$

Consequently, when  $n \geq 21$ , Conjecture 1.1 fails for  $\lambda = \lambda(n)$ ,  $\mu = \nu = \mu(n)$ , and  $N = 1$ .

The details of our notation can be found in the notation paragraph at the end of this section. We would like to point out that Conjecture 1.1 is true *asymptotically*. This fact was proved by Okounkov in [Oko03, § 3.5]. A different proof can be found herein § 3.4 (Remark 3.5 and Example 3.6).

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The layout of this paper is as follows. In §2, we give a direct proof of Theorem 1.2 by using the Littlewood–Richardson rule. A different approach to Okounkov’s conjecture is based on quiver theory. In §3, we review some tools from quiver invariant theory and explain why the log-concavity conjecture is bound to fail (see §3.4). In §4, we give another proof of Theorem 1.2 and present more counterexamples. In particular, Proposition 4.4 provides counterexamples to the log-concavity conjecture [Kir04, Conjecture 6.17] for *parabolic* Kostka numbers.

*Notation.* A partition is a sequence  $\lambda = (\lambda_1, \dots, \lambda_r)$  of integers such that  $\lambda_1 \geq \dots \geq \lambda_r \geq 0$ . The length of a partition is defined to be the number of its non-zero parts. If  $\lambda$  is a partition, we define  $|\lambda|$  to be the sum of its parts. The Young diagram of a partition  $\lambda = (\lambda_1, \dots, \lambda_r)$  is a collection of boxes, arranged in left-justified rows with  $\lambda_i$  boxes in row  $i$ . For a partition  $\lambda$ , we denote by  $\lambda'$  the partition conjugate to  $\lambda$ , i.e., the Young diagram of  $\lambda'$  is the Young diagram of  $\lambda$  reflected with respect to its main diagonal.

If  $\lambda = (\lambda_1, \dots, \lambda_r)$  is a partition then we define  $N\lambda$  by  $N\lambda = (N\lambda_1, \dots, N\lambda_r)$ . By  $\lambda = (\lambda_1^{m_1}, \dots, \lambda_k^{m_k})$ , we denote the partition that has  $m_i$  parts equal to  $\lambda_i$ ,  $1 \leq i \leq k$ . For a partition  $\lambda$  of length at most  $r$ ,  $S^\lambda(V)$  denotes the irreducible polynomial representation of  $GL(V)$  with highest weight  $\lambda$ , where  $V$  is an  $r$ -dimensional complex vector space. Let  $\lambda, \mu, \nu$  be three partitions of length at most  $r$ . Then we define the Littlewood–Richardson coefficient  $c_{\mu, \nu}^\lambda$  to be the multiplicity of  $S^\lambda(V)$  in  $S^\mu(V) \otimes S^\nu(V)$ , i.e.,

$$c_{\mu, \nu}^\lambda = \dim_{\mathbb{C}}(S^\lambda(V)^* \otimes S^\mu(V) \otimes S^\nu(V))^{\text{GL}(V)},$$

where  $S^\lambda(V)^*$  is the dual representation. More generally, if  $\gamma, \lambda(1), \dots, \lambda(m)$  are partitions of length at most  $r$ , we define

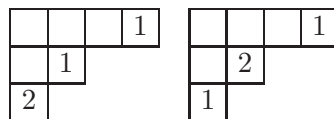
$$c_{\lambda(1), \dots, \lambda(m)}^\gamma = \dim_{\mathbb{C}}(S^\gamma(V)^* \otimes S^{\lambda(1)}(V) \otimes \dots \otimes S^{\lambda(m)}(V))^{\text{GL}(V)}.$$

### 2. A direct proof by Littlewood–Richardson rule

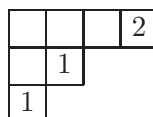
Our main references for Young tableau and Littlewood–Richardson rule are [Ful97] and [Mac95] (see also [Ful00]). If  $\lambda, \mu, \nu$  are three partitions, the Littlewood–Richardson coefficient  $c_{\mu, \nu}^\lambda$  can be described as the cardinality of the set  $LR(\lambda, \mu, \nu)$  of diagrams  $D$  of skew shape  $\lambda/\mu$ , filled with  $\nu_1$  1s,  $\nu_2$  2s, etc., subject to the following conditions.

- (1) Diagram  $D$  is a semistandard Young tableau, i.e., the entries in rows are weakly increasing from left to right and the entries in columns are strictly increasing from top to bottom.
- (2) Diagram  $D$  is a lattice permutation, i.e., when the entries are listed, from right to left in rows, starting with the top row, the resulting word  $w(D)$  is a lattice permutation. This last condition means that for any integer  $1 \leq r \leq |\nu|$ , and any positive integer  $i$ , the number of occurrences of  $i$  in the first  $r$  entries of  $w(D)$  is no less than the number of occurrences of  $i + 1$  in these first  $r$  entries.

*Example 2.1.* For  $\lambda = (4, 2, 1)$ ,  $\mu = (3, 1, 0)$ , and  $\nu = (2, 1, 0)$ , there are only two diagrams  $D$  that satisfy conditions (1) and (2) above, as given below.



Note that the diagram



is not a lattice permutation. Reversing the roles of  $\mu, \nu$ , we get the diagrams given below.

		1	1
	1		
2			

		1	1
	2		
1			

*Proof of Theorem 1.2.* For our purposes it will be convenient to work with conjugate partitions. This is always possible since the Littlewood–Richardson coefficients are invariant when passing to conjugate partitions. First, we show that

$$c_{\mu(n)', \mu(n)'}^{\lambda(n)'} = \binom{n+2}{2},$$

where  $\lambda(n)' = (4n, 4n, 3n, n)$  and  $\mu(n)' = (3n, 2n, n)$ . We look at the cases  $n = 1, 2$  and then describe the general pattern. For  $n = 1$ , the multiplicity is 3 because the only three skew diagrams satisfying the requirements of the Littlewood–Richardson rule are those below.

			1
		1	2
	1	2	
3			

			1
		1	2
	1	3	
2			

			1
		1	2
	2	3	
1			

Let us call these tableaux  $S_3, S_2, S_1$  respectively, i.e., we label them by the content of the last row.

For  $n = 2$ , we get the following six tableaux.

					1	1	
				1	1	2	2
		1	1	2	2		
3	3						

					1	1	
				1	1	2	2
		1	1	2	3		
2	3						

						1	1
				1	1	2	2
		1	2	2	3		
1	3						

					1	1	
				1	1	2	2
		1	1	3	3		
2	2						

					1	1	
				1	1	2	2
		1	2	3	3		
1	2						

						1	1
				1	1	2	2
		2	2	3	3		
1	1						

Looking at their last row, these clearly correspond to the monomials of degree 2 in  $S_1, S_2, S_3$ .

So, for general  $n$  we define a bijection between the set of monomials

$$S^a := S_1^{a_1} S_2^{a_2} S_3^{a_3}$$

of degree  $n$  and the set  $LR(\lambda(n)', \mu(n)', \mu(n)')$  of tableaux of the shape  $\lambda(n)'/\mu(n)'$  satisfying the conditions (1) and (2) above, whose cardinality is equal to  $c_{\mu(n)', \mu(n)'}^{\lambda(n)'}$ . To achieve this, we associate to each monomial  $S^a$  a tableau  $E(a)$  from  $LR(\lambda(n)', \mu(n)', \mu(n)')$  as follows. The first two rows of each  $E(a)$  are the same; they contain in each column the number 1 for columns with numbers  $2n + 1, \dots, 3n$ , and the numbers 1, 2 for columns with numbers  $3n + 1, \dots, 4n$ .

The filling of the last row of  $E(a)$  is a tableau of shape  $(n)$  and we just define it to have  $a_1$  1s,  $a_2$  2s and  $a_3$  3s. Now it is clear that the remaining third row of  $\lambda(n)'/\mu(n)'$  can be uniquely filled by the remaining available numbers to get the tableau  $E(a)$  from  $LR(\lambda(n)', \mu(n)', \mu(n)')$ . Indeed, the first (from left to right)  $n$  boxes in the third row have to be filled by the remaining 1s and 2s and the last  $n$  boxes by the remaining 2s and 3s, in weakly increasing order. This assures semistandardness. The lattice permutation condition between 1s and 2s is satisfied because there

are  $2n$  1s already in the columns  $2n + 1, \dots, 4n$ . The lattice permutation condition between 2s and 3s is also satisfied because there are already  $n$  2s in columns  $3n + 1, \dots, 4n$ .

This gives us an injection from the set of monomials  $S^a$  to the set of tableaux  $LR(\lambda(n)', \mu(n)', \mu(n)')$ . It is clearly surjective because to each diagram  $E$  from the set  $LR(\lambda(n)', \mu(n)', \mu(n)')$  we can associate the monomial  $S_1^{u_1} S_2^{u_2} S_3^{u_3}$  of degree  $n$  by taking  $u_i$  to be the number of occurrences of  $i$  in the last row of  $E$ . This shows that

$$c_{\mu(n)', \mu(n)'}^{\lambda(n)'} = \binom{n+2}{2}.$$

Let us turn to the second statement. We need to show that

$$c_{\sigma(n), \sigma(n)}^{\rho(n)} = \binom{n+5}{5},$$

where  $\rho(n) = (2\lambda(n))' = (4n, 4n, 4n, 4n, 3n, 3n, n, n)$  and  $\sigma(n) = (2\mu(n))' = (3n, 3n, 2n, 2n, n, n)$ .

Let us exhibit the case  $n = 1$  below.

			1			1
			2			2
		1	3			1
		2	4			2
	1	3				1
	2	4				5
5						6
6						4

			1			1
			2			2
		1	3			1
		2	4			2
	1	3				5
	4	5				6
2						3
6						4

			1			1
			2			2
		1	3			1
		2	4			2
	1	3				5
	4	5				6
2						3
6						4

Let us label these tableaux by the content of the first column, i.e.,  $T_{5,6}$ ,  $T_{4,6}$ ,  $T_{3,4}$ ,  $T_{2,6}$ ,  $T_{2,4}$ , and  $T_{1,2}$ , respectively. We also order them by a total order respecting the lexicographic order of the indices, i.e.,

$$T_{1,2} < T_{2,4} < T_{2,6} < T_{3,4} < T_{4,6} < T_{5,6}.$$

We define a bijection between the set of monomials

$$T^a := T_{1,2}^{a_{1,2}} T_{2,4}^{a_{2,4}} T_{2,6}^{a_{2,6}} T_{3,4}^{a_{3,4}} T_{4,6}^{a_{4,6}} T_{5,6}^{a_{5,6}}$$

of degree  $n$  and the set  $LR(\rho(n), \sigma(n), \sigma(n))$  of tableaux of the shape  $\rho(n)/\sigma(n)$  satisfying the conditions (1) and (2) above, whose cardinality is equal to  $c_{\sigma(n), \sigma(n)}^{\rho(n)}$ . To achieve this, we associate to each monomial  $T^a$  a tableau  $D(a)$  from  $LR(\rho(n), \sigma(n), \sigma(n))$  as follows. The first four rows of each  $D(a)$  are the same; they contain in each column the numbers 1, 2 for columns with numbers  $2n + 1, \dots, 3n$ , and the numbers 1, 2, 3, 4 for columns with numbers  $3n + 1, \dots, 4n$ .

The filling of the last two rows of  $D(a)$  form a tableau of shape  $(n^2)$ . We start with a tableau having  $a_{i,j}$  columns of type

$$\begin{array}{|c|} \hline i \\ \hline j \\ \hline \end{array}$$

We order them according to the order on  $T_{i,j}$ , so columns

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}$$

are the first ones and columns

$$\begin{array}{|c|} \hline 5 \\ \hline 6 \\ \hline \end{array}$$

are the last ones. The only problem is that the columns

$$\begin{array}{|c|} \hline 3 \\ \hline 4 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|} \hline 2 \\ \hline 6 \\ \hline \end{array}$$

cannot be standard in any order. So, every occurrence of the columns

$$\begin{array}{|c|} \hline 3 \\ \hline 4 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|} \hline 2 \\ \hline 6 \\ \hline \end{array}$$

has to be replaced by

$$\begin{array}{|c|c|} \hline 2 & 3 \\ \hline 4 & 6 \\ \hline \end{array}$$

This defines the filling of the last two rows of  $D(a)$ . Now, we claim that the remaining fifth and sixth rows of  $\rho(n)/\sigma(n)$  can be uniquely filled by the remaining available numbers to get the tableau  $D(a)$  from  $LR(\rho(n), \sigma(n), \sigma(n))$ . The point is that 1s have to appear in the fifth row at the beginning, and 2s cannot appear in that row after 1s because the lattice permutation condition would be violated. Similarly, 6s have to appear at the end of the sixth row, but there has to be a 5 above each 6, otherwise the lattice permutation condition is violated. The rest of the 5s have to appear before the 6s. The remaining part of the diagram can be uniquely filled with 3s and 4s to complete it to a standard diagram. Indeed, the number 4 cannot appear in the fifth row, because the lattice permutation condition would be violated (the number of 3s and 4s in the first four rows is the same). Semistandardness and the lattice permutation condition easily follow. This gives us an injection from the set of monomials  $T^a$  to the set  $LR(\rho(n), \sigma(n), \sigma(n))$ . This is enough for the counterexample, because we showed that the coefficient  $c_{\sigma(n), \sigma(n)}^{\rho(n)}$  is at least  $\binom{n+5}{5}$ . The fact that the defined map is surjective is not difficult to prove, so we leave it to the reader.  $\square$

### 3. Quiver theory

In this section we review the main tools from quiver invariant theory that will be used to study Littlewood–Richardson coefficients.

#### 3.1 Generalities

A quiver  $Q = (Q_0, Q_1, t, h)$  consists of a finite set of vertices  $Q_0$ , a finite set of arrows  $Q_1$  and two functions  $t, h : Q_1 \rightarrow Q_0$  that assign to each arrow  $a$  its tail  $ta$  and its head  $ha$ , respectively. We write  $ta \xrightarrow{a} ha$  for each arrow  $a \in Q_1$ .

For simplicity, we will be working over the field of complex numbers  $\mathbb{C}$ . A representation  $V$  of  $Q$  over  $\mathbb{C}$  is a family of finite dimensional  $\mathbb{C}$ -vector spaces  $\{V(x) \mid x \in Q_0\}$  together with a

family  $\{V(a) : V(ta) \rightarrow V(ha) \mid a \in Q_1\}$  of  $\mathbb{C}$ -linear maps. If  $V$  is a representation of  $Q$ , we define its dimension vector  $\underline{d}_V$  by  $\underline{d}_V(x) = \dim_{\mathbb{C}} V(x)$  for every  $x \in Q_0$ . Thus the dimension vectors of representations of  $Q$  lie in  $\Gamma = \mathbb{Z}^{Q_0}$ , the set of all integer-valued functions on  $Q_0$ . For every vertex  $x$ , we denote by  $e_x$  the simple dimension vector corresponding to  $x$ , i.e.  $e_x(y) = \delta_{x,y}$ , for all  $y \in Q_0$ , where  $\delta_{x,y}$  is the Kronecker symbol.

Given two representations  $V$  and  $W$  of  $Q$ , we define a morphism  $\phi : V \rightarrow W$  to be a collection of linear maps  $\{\phi(x) : V(x) \rightarrow W(x) \mid x \in Q_0\}$  such that, for every arrow  $a \in Q_1$ , we have  $\phi(ha)V(a) = W(a)\phi(ta)$ . We denote by  $\text{Hom}_Q(V, W)$  the  $\mathbb{C}$ -vector space of all morphisms from  $V$  to  $W$ . In this way, we obtain the abelian category  $\text{Rep}(Q)$  of all quiver representations of  $Q$ . Let  $V$  and  $W$  be two representations of  $Q$ . We say that  $V$  is a subrepresentation of  $W$  if  $V(x)$  is a subspace of  $W(x)$  for all vertices  $x \in Q_0$  and  $V(a)$  is the restriction of  $W(a)$  to  $V(ta)$  for all arrows  $a \in Q_1$ .

If  $\alpha, \beta$  are two elements of  $\Gamma$ , we define the Euler inner product

$$\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha). \tag{1}$$

From now on, we will assume that our quivers are without oriented cycles.

### 3.2 Semi-invariants for quivers

Let  $\beta$  be a dimension vector of  $Q$ . The representation space of  $\beta$ -dimensional representations of  $Q$  is defined by

$$\text{Rep}(Q, \beta) = \bigoplus_{a \in Q_1} \text{Hom}(\mathbb{C}^{\beta(ta)}, \mathbb{C}^{\beta(ha)}).$$

If  $\text{GL}(\beta) = \prod_{x \in Q_0} \text{GL}(\beta(x))$  then  $\text{GL}(\beta)$  acts algebraically on  $\text{Rep}(Q, \beta)$  by simultaneous conjugation, i.e., for  $g = (g(x))_{x \in Q_0} \in \text{GL}(\beta)$  and  $V = \{V(a)\}_{a \in Q_1} \in \text{Rep}(Q, \beta)$ , we define  $g \cdot V$  by

$$(g \cdot V)(a) = g(ha)V(a)g(ta)^{-1} \quad \text{for each } a \in Q_1.$$

In this way,  $\text{Rep}(Q, \beta)$  is a rational representation of the linearly reductive group  $\text{GL}(\beta)$  and the  $\text{GL}(\beta)$ -orbits in  $\text{Rep}(Q, \beta)$  are in one-to-one correspondence with the isomorphism classes of  $\beta$ -dimensional representations of  $Q$ . As  $Q$  is a quiver without oriented cycles, one can show that there is only one closed  $\text{GL}(\beta)$ -orbit in  $\text{Rep}(Q, \beta)$  and hence the invariant ring  $I(Q, \beta) = \mathbb{C}[\text{Rep}(Q, \beta)]^{\text{GL}(\beta)}$  is exactly the base field  $\mathbb{C}$ .

Now, consider the subgroup  $\text{SL}(\beta) \subseteq \text{GL}(\beta)$  defined by

$$\text{SL}(\beta) = \prod_{x \in Q_0} \text{SL}(\beta(x)).$$

Although there are only constant  $\text{GL}(\beta)$ -invariant polynomial functions on  $\text{Rep}(Q, \beta)$ , the action of  $\text{SL}(\beta)$  on  $\text{Rep}(Q, \beta)$  provides us with a highly non-trivial ring of semi-invariants. Note that any  $\sigma \in \mathbb{Z}^{Q_0}$  defines a rational character of  $\text{GL}(\beta)$  by

$$\{g(x) \mid x \in Q_0\} \in \text{GL}(\beta) \mapsto \prod_{x \in Q_0} (\det g(x))^{\sigma(x)}.$$

In this way, we can identify  $\Gamma = \mathbb{Z}^{Q_0}$  with the group  $X^*(\text{GL}(\beta))$  of rational characters of  $\text{GL}(\beta)$ , assuming that  $\beta$  is a sincere dimension vector (i.e.,  $\beta(x) > 0$  for all vertices  $x \in Q_0$ ). We also refer to the rational characters of  $\text{GL}(\beta)$  as weights.

Let  $\text{SI}(Q, \beta) = \mathbb{C}[\text{Rep}(Q, \beta)]^{\text{SL}(\beta)}$  be the ring of semi-invariants. As  $\text{SL}(\beta)$  is the commutator subgroup of  $\text{GL}(\beta)$  and  $\text{GL}(\beta)$  is linearly reductive, we have

$$\text{SI}(Q, \beta) = \bigoplus_{\sigma \in X^*(\text{GL}(\beta))} \text{SI}(Q, \beta)_{\sigma},$$

where

$$\text{SI}(Q, \beta)_\sigma = \{f \in \mathbb{C}[\text{Rep}(Q, \beta)] \mid gf = \sigma(g)f \text{ for all } g \in \text{GL}(\beta)\}$$

is the space of semi-invariants of weight  $\sigma$ . If  $\alpha \in \Gamma$ , we define  $\sigma = \langle \alpha, \cdot \rangle$  by

$$\sigma(x) = \langle \alpha, e_x \rangle, \quad \text{for all } x \in Q_0.$$

Similarly, one can define the weight  $\tau = \langle \cdot, \alpha \rangle$ .

LEMMA 3.1 (Reciprocity property [DW00, Corollary 1]). *Let  $\alpha$  and  $\beta$  be two dimension vectors. Then*

$$\dim \text{SI}(Q, \beta)_{\langle \alpha, \cdot \rangle} = \dim \text{SI}(Q, \alpha)_{-\langle \cdot, \beta \rangle}.$$

Now, we can define  $(\alpha \circ \beta)_Q$  by

$$(\alpha \circ \beta)_Q = \dim \text{SI}(Q, \beta)_{\langle \alpha, \cdot \rangle} = \dim \text{SI}(Q, \alpha)_{-\langle \cdot, \beta \rangle}.$$

(When no confusion arises, we shall drop the subscript  $Q$ .)

### 3.3 Exceptional sequences

A dimension vector  $\beta \in \mathbb{N}^{Q_0}$  is said to be a *Schur root* if there exists a  $\beta$ -dimensional representation  $W \in \text{Rep}(Q, \beta)$  such that  $\text{End}_Q(W) \cong \mathbb{C}$ .

DEFINITION 3.2. A sequence of dimension vectors  $\varepsilon_1, \dots, \varepsilon_r$  is called an *exceptional sequence* if:

- (i) each  $\varepsilon_i$  is a real Schur root, i.e.,  $\varepsilon_i$  is a Schur root and  $\langle \varepsilon_i, \varepsilon_i \rangle = 1$ , for all  $1 \leq i \leq r$ ;
- (ii)  $(\varepsilon_i \circ \varepsilon_j)_Q \neq 0$ , for all  $1 \leq i < j \leq r$ .

The following theorem will be quite useful for us (for a more general version, see [DW06]).

THEOREM 3.3 [DW06, Theorem 2.39]. *Let  $\varepsilon_1, \varepsilon_2$  be an exceptional sequence for a quiver  $Q$  without oriented cycles. Assume that  $\langle \varepsilon_2, \varepsilon_1 \rangle = -l$ , where  $l$  is some non-negative integer. Define a new quiver  $\theta(l)$  with set of vertices  $\theta(l)_0 = \{1, 2\}$  and  $l$  arrows from vertex 2 to vertex 1. Consider the linear transformation*

$$I : \mathbb{N}^{\theta(l)_0} = \mathbb{N}^2 \longrightarrow \mathbb{N}^{Q_0}$$

defined by

$$I(\beta_1, \beta_2) = \beta_1 \varepsilon_1 + \beta_2 \varepsilon_2,$$

for all dimension vectors  $\beta = (\beta_1, \beta_2) \in \mathbb{N}^{\theta(l)_0}$ .

If  $\alpha, \beta \in \mathbb{N}^{\theta(l)_0}$  are such that  $(\alpha \circ \beta)_{\theta(l)} \neq 0$  then

$$(\alpha \circ \beta)_{\theta(l)} = (I(\alpha) \circ I(\beta))_Q.$$

The quiver  $\theta(l)$  that appears in Theorem 3.3 is called *the generalized Kronecker quiver*. As we will see in § 4, this particular quiver will be our main source of Littlewood–Richardson coefficients. It has been proved in [DW06] that the map  $I$  in the theorem above allows one to ‘embed’ much of the combinatorics of the ‘new quiver’  $\theta(l)$  into the combinatorics of the original quiver  $Q$ . For this reason, we refer to Theorem 3.3 as the ‘embedding theorem’.

### 3.4 Polynomiality for semi-invariants and (non-)log-concavity

We are interested in how the dimensions  $N\alpha \circ \beta = \dim_{\mathbb{C}} \text{SI}(Q, \beta)_{N\langle \alpha, \cdot \rangle}$  and  $\alpha \circ N\beta = \dim_{\mathbb{C}} \text{SI}(Q, \alpha)_{-N\langle \cdot, \beta \rangle}$  vary as  $N \in \mathbb{Z}_{\geq 0}$  varies.

PROPOSITION 3.4 [DW02, Corollary 1]. Let  $\alpha, \beta$  be two dimension vectors such that  $\alpha \circ \beta \neq 0$ . There exist polynomials  $P, Q \in \mathbb{Q}[X]$  (both depending on  $\alpha$  and  $\beta$ ) with  $P(0) = Q(0) = 1$ , and

$$N\alpha \circ \beta = P(N), \quad \text{for all } N \geq 0,$$

and

$$\alpha \circ N\beta = Q(N), \quad \text{for all } N \geq 0.$$

Remark 3.5. Note that there is a sufficiently large integer  $N_0 > 0$  such that  $p(t) = P(t + 1)/P(t)$  and  $q(t) = Q(t + 1)/Q(t)$  are weakly decreasing functions on  $[N_0, \infty)$ . In other words, we have

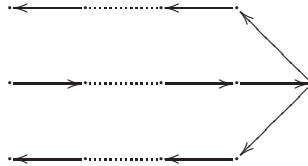
$$((N + 1)\alpha \circ \beta) \cdot ((N - 1)\alpha \circ \beta) \leq (N\alpha \circ \beta)^2, \tag{2}$$

$$(\alpha \circ (N + 1)\beta) \cdot (\alpha \circ (N - 1)\beta) \leq (\alpha \circ N\beta)^2, \tag{3}$$

for every  $N > N_0$ .

Thus the dimensions of spaces of semi-invariants are *asymptotically* log-concave (in each argument).

Example 3.6. For an integer  $r \geq 1$ , let  $T_{r,r,r}$  be the following triple flag quiver with arms of length  $r$ .



Now, given a triple  $(\lambda, \mu, \nu)$  of partitions of length at most  $r$ , one can construct dimension vectors  $\alpha$  and  $\beta$  (see for example [DW00]) such that

$$N\alpha \circ \beta = c_{N\mu, N\nu}^{N\lambda},$$

for all  $N \geq 1$ . This calculation together with inequality (2) shows that the Littlewood–Richardson coefficients are *asymptotically* log-concave (compare with [Oko03, § 3.5]).

Next, we are going to show that the log-concavity property for semi-invariants fails in many cases. For  $\beta \in \mathbb{N}^{Q_0}$  a dimension vector and  $\sigma \in \mathbb{Z}^{Q_0}$  a weight of  $Q$ , we define

$$\sigma(\beta) = \sum_{x \in Q_0} \sigma(x)\beta(x).$$

DEFINITION 3.7 [Kin94, Proposition 3.1]. Let  $\beta$  be a dimension vector and  $\sigma$  be a weight such that  $\sigma(\beta) = 0$ . A  $\beta$ -dimensional representation  $W \in \text{Rep}(Q, \beta)$  is said to be:

- (i)  $\sigma$ -semi-stable if  $\sigma(\underline{d}_{W'}) \leq 0$  for every subrepresentation  $W'$  of  $W$ ;
- (ii)  $\sigma$ -stable if  $\sigma(\underline{d}_{W'}) < 0$  for every proper subrepresentation  $0 \neq W' \subsetneq W$ .

We say that a dimension vector  $\beta$  is  $\sigma$ -(semi-)stable if there exists a  $\sigma$ -(semi-)stable representation  $W \in \text{Rep}(Q, \beta)$ .

Let  $\beta$  be a  $\sigma$ -semi-stable dimension vector. The set of  $\sigma$ -semi-stable representations in  $\text{Rep}(Q, \beta)$  is denoted by  $\text{Rep}(Q, \beta)_\sigma^{\text{s.s.}}$  while the set of  $\sigma$ -stable representations in  $\text{Rep}(Q, \beta)$  is denoted by  $\text{Rep}(Q, \beta)_\sigma^{\text{s}}$ . The one-dimensional torus

$$T = \{(t \text{Id}_{\beta(x)})_{x \in Q_0} \mid t \in \mathbb{C}^*\} \subseteq \text{GL}(\beta)$$

acts trivially on  $\text{Rep}(Q, \beta)$  and so there is a well-defined action of  $\text{PGL}(\beta) = \text{GL}(\beta)/T$  on  $\text{Rep}(Q, \beta)$ .



Using methods from geometric invariant theory, one can construct the following GIT-quotient of  $\text{Rep}(Q, \beta)$ :

$$\mathcal{M}(Q, \beta)_{\sigma}^{\text{s.s.}} = \text{Proj} \left( \bigoplus_{n \geq 0} \text{SI}(Q, \beta)_{n\sigma} \right).$$

It was proved by King [Kin94] that  $\mathcal{M}(Q, \beta)_{\sigma}^{\text{s.s.}}$  is a categorical quotient of  $\text{Rep}(Q, \beta)_{\sigma}^{\text{s.s.}}$  by  $\text{PGL}(\beta)$ . Note that  $\mathcal{M}(Q, \beta)_{\sigma}^{\text{s.s.}}$  is an irreducible projective variety, called the moduli space of  $\beta$ -dimensional  $\sigma$ -semi-stable representations (for more details, see [Kin94]).

For the remainder of this section, we assume that  $\beta$  is a  $\sigma$ -stable dimension vector. Then there is a non-empty open subset  $\mathcal{M}(Q, \beta)_{\sigma}^{\text{s}} \subseteq \mathcal{M}(Q, \beta)_{\sigma}^{\text{s.s.}}$  which is a geometric quotient of  $\text{Rep}(Q, \beta)_{\sigma}^{\text{s}}$  by  $\text{PGL}(\beta)$ . Now, a  $\sigma$ -stable representation must be a Schur representation and so its stabilizer in  $\text{PGL}(\beta)$  is zero-dimensional. It follows that

$$\dim \mathcal{M}(Q, \beta)_{\sigma}^{\text{s.s.}} = 1 - \langle \beta, \beta \rangle.$$

Let us further assume that  $\langle \beta, \beta \rangle < 0$  (that is to say,  $\beta$  is *imaginary and non-isotropic*). Then it is known that  $m\beta$  stays  $\sigma$ -stable (see for example [DW06, Proposition 3.16]) and hence

$$\dim \mathcal{M}(Q, m\beta)_{\sigma}^{\text{s.s.}} = 1 - m^2 \langle \beta, \beta \rangle,$$

for every integer  $m \geq 1$ . Now, write  $\sigma = \langle \alpha, \cdot \rangle$  for some dimension vector  $\alpha$ . If we fix  $m$  then  $n\alpha \circ m\beta$  has degree  $1 - m^2 \langle \beta, \beta \rangle$  as a polynomial in  $n$ . Therefore, when  $n > 0$  is sufficiently large, we must have

$$n\alpha \circ 2\beta > (n\alpha \circ \beta)^2. \tag{4}$$

Indeed, the left-hand side of the above inequality is a polynomial in  $n$  of degree  $1 - 4\langle \beta, \beta \rangle$  while the right-hand side is a polynomial of degree  $2 - 2\langle \beta, \beta \rangle$  and  $1 - 4\langle \beta, \beta \rangle > 2 - 2\langle \beta, \beta \rangle$ .

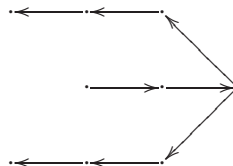
Note that inequality (4) gives counterexamples to the log-concavity property for semi-invariants.

### 4. Counterexamples

In this section, we first give a different proof of Theorem 1.2 and then present more counterexamples. In particular, we provide counterexamples to Kirillov's  $q$ -log-concavity conjecture for parabolic Kostka polynomials (see Proposition 4.4).

#### 4.1 Littlewood–Richardson coefficients from star and generalized Kronecker quivers

It is well known that the Littlewood–Richardson coefficients can be viewed as dimensions of spaces of semi-invariants of star quivers (see for example [Chi04, DW00]). Now, let us consider the star quiver  $T_{4,3,4}$  with the orientation given below.



We are going to reduce the problem of computing semi-invariants of  $T_{4,3,4}$  to that of computing semi-invariants of (rather small) generalized Kronecker quivers.

Let us recall that, for every integer  $n \geq 1$ , we define

$$\lambda(n) = (4^n, 3^{2n}, 2^n) \quad \text{and} \quad \mu(n) = (3^n, 2^n, 1^n).$$

PROPOSITION 4.1. *Let  $\theta(3)$  be the generalized Kronecker quiver with three arrows and vertices labelled 1, 2:*

$$\theta(3) : 1 \rightleftarrows 2.$$

Then

$$\dim \text{SI}(\theta(3), (n, n))_{(-m, m)} = c_{m\mu(n), m\mu(n)}^{m\lambda(n)},$$

for every  $m, n \geq 1$ .

*Proof.* Let us consider the exceptional sequence of  $T_{4,3,4}$  given by

$$\varepsilon_1 = \begin{matrix} 1 & 2 & 3 \\ & 0 & 3 & 4 \\ & 1 & 2 & 3 \end{matrix}$$

and

$$\varepsilon_2 = \begin{matrix} 0 & 0 & 0 \\ & 1 & 0 & 0. \\ & 0 & 0 & 0 \end{matrix}$$

Since  $\langle \varepsilon_2, \varepsilon_1 \rangle = -3$ , we know that the generalized Kronecker quiver  $\theta(3)$  can be embedded in  $T_{4,3,4}$  by Theorem 3.3. In particular, if  $\alpha = (n, n)$  and  $\beta = (m, 2m)$  are dimension vectors for  $\theta(3)$  then

$$(\alpha \circ \beta)_{\theta(3)} = (I(\alpha) \circ I(\beta))_{T_{4,3,4}},$$

where

$$I(\alpha) = \begin{matrix} n & 2n & 3n \\ & n & 3n & 4n \\ & n & 2n & 3n \end{matrix}$$

and

$$I(\beta) = \begin{matrix} m & 2m & 3m \\ & 2m & 3m & 4m. \\ & m & 2m & 3m \end{matrix}$$

Next, computing with Schur functors (see [Chi04] or [DW00] for explicit computations) we obtain

$$I(\alpha) \circ I(\beta) = c_{m\mu(n), m\mu(n)}^{m\lambda(n)},$$

and so we are done. □

*Another proof of Theorem 1.2.* By Proposition 4.1, we only need to compute the dimensions of the spaces  $\text{SI}(\theta(3), (n, n))_{(-m, m)}$  when  $m = 1, 2$ . For this, we first decompose the affine coordinate ring of  $\text{Rep}(\theta(3), (n, n))$  as a direct sum in which the summands are tensor products of irreducible representations of the  $\text{GL}(n)$ . For convenience, let us write  $V = \mathbb{C}^n, W = \mathbb{C}^n$ . Then we have

$$\begin{aligned} \mathbb{C}[\text{Rep}(\theta(3), (n, n))] &= \mathbb{C}[\text{Hom}(W, V) \oplus \text{Hom}(W, V) \oplus \text{Hom}(W, V)] \\ &= S(W \otimes V^*) \otimes S(W \otimes V^*) \otimes S(W \otimes V^*). \end{aligned}$$

Using Cauchy's formula [Ful97, p. 121], we obtain that

$$S(W \otimes V^*) = \bigoplus S^\mu W \otimes S^\mu V^*$$

as  $\text{GL}(V) \times \text{GL}(W)$ -modules, where the sum is over all partitions  $\mu$  with at most  $n$  non-zero parts. Hence, we have

$$\begin{aligned} &\mathbb{C}[\text{Rep}(\theta(3), (n, n))]^{\text{SL}(V) \times \text{SL}(W)} \\ &= \bigoplus (S^{\mu(1)} V^* \otimes S^{\mu(2)} V^* \otimes S^{\mu(3)} V^*)^{\text{SL}(V)} \otimes (S^{\mu(1)} W \otimes S^{\mu(2)} W \otimes S^{\mu(3)} W)^{\text{SL}(W)}, \end{aligned}$$

where the sum is over all partitions  $\mu(1), \mu(2), \mu(3)$  with at most  $n$  non-zero parts. Sorting out those semi-invariants of weight  $(-m, m)$ , it is easy to see that

$$\text{SI}(\theta(3), (n, n))_{(-m, m)} = \bigoplus \left( \det_V^m \otimes \bigotimes_{i=1}^3 S^{\mu(i)} V^* \right)^{\text{GL}(V)} \otimes \left( \det_W^{-m} \otimes \bigotimes_{i=1}^3 S^{\mu(i)} W \right)^{\text{GL}(W)},$$

where the sum is over all partitions  $\mu(1), \mu(2), \mu(3)$  with at most  $n$  non-zero parts. For our purposes it is useful to work with conjugate partitions in the identity above. So, we can write

$$\dim \text{SI}(\theta(3), (n, n))_{(-m, m)} = \sum (c_{\lambda(1), \lambda(2), \lambda(3)}^{(n^m)})^2, \tag{5}$$

where the sum is over all partitions  $\lambda(1), \lambda(2), \lambda(3)$  (with at most  $m$  non-zero parts).

Next, it is easy to see that

$$c_{\lambda(1), \lambda(2), \lambda(3)}^{(n^m)} \leq 1, \tag{6}$$

for  $m \in \{1, 2\}$ . Indeed, one can either check this directly with the Littlewood–Richardson rule or view these coefficients as dimensions of spaces of semi-invariants for a quiver of type  $\mathbb{D}_4$ .

Therefore,  $\dim \text{SI}(\theta(3), (n, n))_{(-1, 1)}$  is simply the number of monomials in three (commuting) variables of degree  $n$ , and so

$$c_{\mu(n), \mu(n)}^{\lambda(n)} = \binom{n+2}{2}.$$

Now, let  $\lambda(i) = (\lambda_1(i), \lambda_2(i))$ ,  $1 \leq i \leq 3$ , be three partitions with at most two non-zero parts. We claim that

$$|\lambda(1)| + |\lambda(2)| + |\lambda(3)| = 2n, \tag{7}$$

$$n - \lambda_1(i) - \lambda_2(j) - \lambda_2(k) \geq 0, \quad \text{where } \{i, j, k\} = \{1, 2, 3\}, \tag{8}$$

give a (minimal) list of necessary and sufficient Horn inequalities for the non-vanishing of the Littlewood–Richardson coefficient  $c_{\lambda(1), \lambda(2), \lambda(3)}^{(n^2)}$ . This follows from [Ful00, Theorem 17]. Alternatively, one can deduce this claim from the description of the so-called cone of effective weights for a type  $\mathbb{D}_4$  quiver.

From (5)–(8), we obtain that  $\dim \text{SI}(\theta(3), (n, n))_{(-2, 2)}$  equals the cardinality of the set  $\mathcal{S}$  of all triples  $(\lambda(1), \lambda(2), \lambda(3))$  of partitions with at most two non-zero parts satisfying the conditions (7) and (8).

Note that every  $(\lambda(1), \lambda(2), \lambda(3)) \in \mathcal{S}$  gives rise to a monomial  $X_1^{n_1} \cdot X_2^{n_2} \cdot X_3^{n_3} \cdot X_4^{n_4} \cdot X_5^{n_5} \cdot X_6^{n_6}$  of degree  $n$ , where

$$\begin{aligned} n_1 &= n - \lambda_1(1) - \lambda_2(2) - \lambda_2(3), & n_2 &= \lambda_2(1), \\ n_3 &= n - \lambda_1(2) - \lambda_2(3) - \lambda_2(1), & n_4 &= \lambda_2(2), \\ n_5 &= n - \lambda_1(3) - \lambda_2(1) - \lambda_2(2), & n_6 &= \lambda_2(3). \end{aligned}$$

It is clear that in this way we get a bijection from  $\mathcal{S}$  to the set of all monomials in six (commuting) variables of degree  $n$ . So, we have

$$\dim \text{SI}(\theta(3), (n, n))_{(-2, 2)} = \binom{n+5}{5},$$

and this finishes the proof. □

*Remark 4.2.* It is worth pointing out that using the same ideas as above one can construct non-log-concave Littlewood–Richardson coefficients for every star quiver  $T_{p,q,r}$  of wild representation type.

### 4.2 Non-log-concave parabolic Kostka numbers

In this section, we consider some rather special Littlewood–Richardson coefficients. Let  $\lambda$  be a partition and let  $R = ((m_1^{l_1}), \dots, (m_k^{l_k}))$  be a sequence of rectangular partitions. Then the parabolic Kostka number  $K_{\lambda,R}$  associated to  $\lambda$  and  $R$  is defined by

$$K_{\lambda,R} = \dim_{\mathbb{C}}(S^{\lambda}(V)^* \otimes S^{(m_1^{l_1})}(V) \otimes \dots \otimes S^{(m_k^{l_k})}(V))^{\text{GL}(V)},$$

where  $V$  is a complex vector space of sufficiently large dimension. In general, it is well known that  $K_{\lambda,R}$  is the value at  $q = 1$  of the corresponding parabolic Kostka polynomial (see [Kir04, ch. 4] and the reference therein).

If  $R = ((m_1^{l_1}), \dots, (m_k^{l_k}))$  is a sequence of rectangles and  $N \geq 1$  is an integer, we define  $NR$  to be the sequence of rectangles  $NR = (((Nm_1)^{l_1}), \dots, ((Nm_k)^{l_k}))$ . The log-concavity conjecture for parabolic Kostka numbers (compare with the more general version [Kir04, Conjecture 6.17]) is as follows.

CONJECTURE 4.3. Let  $\lambda$  be a partition and  $R$  be a sequence of rectangular partitions. Then

$$K_{(N+1)\lambda,(N+1)R} \cdot K_{(N-1)\lambda,(N-1)R} \leq (K_{N\lambda,NR})^2,$$

for every integer  $N \geq 1$ .

Our next proposition shows that Conjecture 4.3 fails in general.

PROPOSITION 4.4. For every  $n \geq 1$ , consider

$$\lambda(n) = (2^n, 1^{2n})$$

and

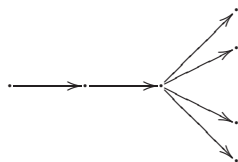
$$R(n) = ((1^n), (1^n), (1^n), (1^n)).$$

Then

$$K_{\lambda(n),R(n)} = \binom{n+2}{2} \quad \text{and} \quad K_{2\lambda(n),2R(n)} = \binom{n+5}{5}.$$

Consequently, when  $n \geq 21$ , Conjecture 4.3 fails for  $\lambda = \lambda(n)$ ,  $R = R(n)$  and  $N = 1$ .

*Proof.* To obtain parabolic Kostka numbers, we work with the following star quiver  $Q$ .



Let  $\varepsilon_1, \varepsilon_2$  be the exceptional sequence of  $Q$  shown below.

$$\varepsilon_1 = \begin{matrix} & & & & 1 & & & & & & & & & & & 0 \\ & & & & 1 & & & & & & & & & & & 0 \\ \varepsilon_1 = & 0 & 3 & 4 & & & & & & & & & & & & \\ & & & & 1 & & & & & & & & & & & 0 \\ & & & & 1 & & & & & & & & & & & 0 \end{matrix} \quad \text{and} \quad \varepsilon_2 = \begin{matrix} & & & & & & & & & & & & & & & 0 \\ & & & & & & & & & & & & & & & 0 \\ & & & & & & & & & & & & & & & \\ & & & & & & & & & & & & & & & 0 \\ & & & & & & & & & & & & & & & 0 \end{matrix}$$

Reasoning as in Proposition 4.1, we get

$$\dim \text{SI}(\theta(3), (n, n))_{(-m,m)} = K_{m\lambda(n),mR(n)},$$

for every  $m, n \geq 1$ . The proof follows from that of Theorem 1.2. □

*Remark 4.5.* Note that the parabolic Kostka numbers appearing in Proposition 4.4 can be written as Littlewood–Richardson coefficients:

$$K_{m\lambda(n),mR(n)} = c_{((3m)^n, (2m)^n, m^n), ((4m)^n, (3m)^n, (2m)^n, m^n)},$$

for every integer  $m \geq 1$ . Indeed, this follows immediately from [Zel99, Proposition 9].

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