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Hilbert series of subspace arrangements

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Abstract

The vanishing ideal I of a subspace arrangement $V_1 \cup V_2 \cup \cdots \cup V_m \subseteq V$ is an intersection $I_1 \cap I_2 \cap \cdots \cap I_m$ of linear ideals. We give a formula for the Hilbert polynomial of I if the subspaces meet transversally. We also give a formula for the Hilbert series of the product ideal $J = I_1 I_2 \cdots I_m$ without any assumptions about the subspace arrangement. It turns out that the Hilbert series of J is a combinatorial invariant of the subspace arrangement: it only depends on the intersection lattice and the dimension function. The graded Betti numbers of J are determined by the Hilbert series, so they are combinatorial invariants as well. We will also apply our results to generalized principal component analysis (GPCA), a tool that is useful for computer vision and image processing.

MSC: Primary: 13D40; secondary: 13D02; 68T45

1. Introduction

Suppose that V is an n-dimensional K-vector space. A subspace arrangement is a union

$$\mathcal{A} = V_1 \cup \cdots \cup V_m$$

where V_i is a subspace of V for all i. Interestingly, various algebraic and topological properties of the arrangement $\mathcal{A}$ only depend on the dimensions $n_S := \dim_K \bigcap_{i \in S} V_i$, $S \subseteq \{1, 2, \ldots, m\}$. Such properties are called *combinatorial invariants* of the subspace arrangement. For example, if $K = \mathbb{R}$, the number of regions of the complement of a hyperplane arrangement is a combinatorial invariant (see [20]). More generally, the topological Betti numbers of the complement $V \backslash \mathcal{A}$ of a subspace arrangement $\mathcal{A}$ are combinatorial invariants (see [8]). If $K = \mathbb{C}$, then the cohomology ring of $V \backslash \mathcal{A}$ is a combinatorial invariant (see [11,2]). For more on subspace arrangements and hyperplane arrangements, see [13].

Let $I_j \subseteq K[V]$ be the vanishing ideal $V_j \subseteq V$ for $j = 1, 2, \ldots, m$. The vanishing ideal I of $\mathcal{A}$ is equal to the intersection $I_1 \cap I_2 \cap \cdots \cap I_m$. We also define $J = I_1 I_2 \cdots I_m$ as the product ideal. We give a formula for the Hilbert series of J (Theorem 3.1). We also will give a formula for the Hilbert polynomial of I if all subspaces meet transversally (Theorem 3.3). The Hilbert series of J is a combinatorial invariant (Theorem 3.1), but the Hilbert polynomial and Hilbert series of I are not. The Betti numbers (and graded Betti numbers) of J are also combinatorial invariants.

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The original motivation for this paper comes from computer vision. A generalization of principal component analysis naturally leads to the question of recovering the dimensions $n_i := \dim V_i$, $i = 1, 2, \ldots, m$, given the Hilbert polynomial of the subspace arrangement. For more on generalized principal component analysis, see [17] and [21].

2. Hilbert functions, series and polynomials

Suppose that V is an n-dimensional vector space over a field K. We identify the coordinate ring $R := K[V]$ with the polynomial ring $K^{[n]} := K[x_1, x_2, \ldots, x_n]$ in n variables by choosing a basis in V. There is a natural grading $R = \bigoplus_{d\in\mathbb{N}} R_d$ where $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of natural numbers and R_d denotes the space of homogeneous polynomials of degree d. Let $\mathbb{Z}$ be the integers and suppose that $M = \bigoplus_{d\in\mathbb{Z}} M_d$ is a finitely generated graded R-module. We have $M_d = 0$ for $d \ll 0$ because M is finitely generated. The *Hilbert function* h_M of M is

$$h_M(d) = h(M, d) = \dim_K M_d, \quad d \in \mathbb{Z}.$$

The *Hilbert series* of M is defined by

$$\mathcal{H}(M, t) := \sum_{d\in\mathbb{Z}} h(M, d)t^d.$$

It is a Laurent series because $h(M, d) = 0$ for $d \ll 0$. Let M be again a finitely generated graded R-module. For $r \in \mathbb{Z}$ we define the shifted module $M[r] = \bigoplus_{d\in\mathbb{Z}} M[r]_d$ by $M[r]_d := M_{r+d}$, $d \in \mathbb{Z}$. Shifting the degrees affects the Hilbert series as follows:

$$\mathcal{H}(M[r], t) = t^{-r}\mathcal{H}(M, t).$$

The module M has a minimal finite free graded resolution

$$0 \to \bigoplus_{j\in\mathbb{Z}} R[-j]^{\beta_{r,j}} \to \bigoplus_{j\in\mathbb{Z}} R[-j]^{\beta_{1,j}} \to \cdots \to \bigoplus_{j\in\mathbb{Z}} R[-j]^{\beta_{0,j}} \to M \to 0 \tag{1}$$

by Hilbert's Syzygy Theorem (see for example [9],[19, Section 13],[5, Section 19.2]). The nonnegative integers $\beta_{i,j}$ are called the *graded Betti numbers*. For all but finitely many pairs (i, j) we have $\beta_{i,j} = 0$. The *Betti numbers* are defined by $\beta_i = \sum_{j\in\mathbb{Z}} \beta_{i,j}$ (not to be confused with topological Betti numbers of the complement of the subspace arrangement mentioned earlier). Without loss of generality we may assume that $\beta_r \neq 0$. The nonnegative integer $\mathrm{pd}(M) := r$ is the *projective dimension* of the module M and is at most n. The *Castelnuovo–Mumford regularity* of M is

$$\mathrm{reg}(M) := \max\{j - i \mid 0 \le i \le r, \beta_{i,j} \neq 0\}.$$

From the exactness of (1) follows that

$$\mathcal{H}(M, t) = \frac{\sum_{i=0}^{r} (-1)^i \sum_{j\in\mathbb{Z}} \beta_{i,j} t^j}{(1 - t)^n}.$$

3. Subspace arrangements

For the remainder of this paper, let V be an n-dimensional vector space and suppose that $V_1, V_2, \ldots, V_m$ are subspaces of V. For a subset $X \subseteq V$, let $\mathcal{I}(X) \subseteq R = K[V]$ be its vanishing ideal. Define $I_j = \mathcal{I}(V_j)$ for $j = 1, 2, \ldots, m$. The union

$$\mathcal{A} = V_1 \cup V_2 \cup \cdots \cup V_m$$

is a subspace arrangement. Its vanishing ideal is

$$I := \mathcal{I}(\mathcal{A}) = I_1 \cap I_2 \cap \cdots \cap I_m.$$

Define

$$J := I_1 I_2 \cdots I_m.$$

Conca and Herzog proved $\mathrm{reg}(J) = m$ in [1]. Sidman and the author proved $\mathrm{reg}(I) \le m$ in [3,4]. Easy considerations show that these regularity bounds imply the existence of *polynomials* $\widetilde{h}_I(d)$ and $\widetilde{h}_J(d)$ such that

$$h_I(d) = \widetilde{h}_I(d) \quad \text{and} \quad h_J(d) = \widetilde{h}_J(d)$$

for $d \ge m$ ($\widetilde{h}_I(d)$ and $\widetilde{h}_J(d)$ are called the *Hilbert polynomials* of I and J respectively).

For $S \subseteq \{1, 2, \ldots, m\}$, define $I_S = \bigcap_{s\in S} I_s$ and $J_S = \prod_{s\in S} I_s$. Note that $I = I_{\{1,2,\ldots,m\}}$ and $J = J_{\{1,2,\ldots,m\}}$. We use the convention $I_\emptyset = J_\emptyset = R$. For $S \subseteq \{1, 2, \ldots, m\}$ define $V_S = \bigcap_{i\in S} V_i$, $n_S = \dim V_S$ and $c_S = n - n_S$ is the codimension of V_S in V. We also set $n_i = n_{\{i\}} = \dim V_i$ and $c_i = c_{\{i\}} = n - n_i$ for $i = 1, 2, \ldots, m$.

We define polynomials $p_S(t)$ recursively as follows. First we define

$$p_\emptyset(t) = 1.$$

If $S \neq \emptyset$ and $p_X(t)$ is already defined for all proper subsets $X \subset S$, then $p_S(t)$ is uniquely determined by

$$\sum_{X\subseteq S} (-t)^{|X|} p_X(t) \equiv 0 \bmod (1-t)^{c_S}, \qquad \deg(p_S(t)) < c_S.$$

Here $\deg(p_X(t))$ is the degree of the polynomial $p_X(t)$ and $|X|$ is the cardinality of the set X. It is clear from this recursive definition that $p_S(t)$ is completely determined by the numbers c_X, $X \subseteq S$.

Theorem 3.1. *We have*

$$\mathcal{H}(J, t) = \frac{p(t)t^m}{(1-t)^n},$$

where $p(t) = p_{\{1,2,\ldots,m\}}(t)$*. In particular,* $\mathcal{H}(J, t)$ *is a combinatorial invariant, i.e. it is completely determined by all numbers* n_X*,* $X \subseteq \{1, 2, \ldots, n\}$*.*

Note that Theorem 3.1 still makes sense if $V_i \subseteq V_j$ for some $i \neq j$ or when there are repetitions among $V_1, V_2, \ldots, V_m$.

The ideal J is generated in degree m, and it is m-regular. This implies that it has a *linear* minimal free resolution (see [6, Proposition]), i.e., $\beta_{i,j} \neq 0$ only if $j = i + m$. From this follows that

$$p(t) = \beta_0 - \beta_1 t + \beta_2 t^2 - \cdots + (-1)^r \beta_r t^r$$

with $r = \mathrm{pd}(J) \le n - 1$. So the graded Betti numbers are combinatorial invariants as well.

Definition 3.2. The subspaces $V_1, \ldots, V_m$ are called *transversal* or *linearly general* (see [1, Proposition 3.4]) if

$$c_S = \min\left(n, \sum_{i\in S} c_i\right)$$

for all $S \subseteq \{1, 2, \ldots, m\}$, where min denotes the minimum.

Note that we always have $c_S \le \min(n, \sum_{i\in S} c_i)$. So the subspaces are transversal if any intersection of some of the subspaces has the smallest possible dimension. From [1, Proposition 3.4] follows, that if $V_1, \ldots, V_m$ are transversal, then $h_I(d) = h_J(d)$ for $d \ge m$. This is equivalent to saying that $\mathcal{H}(I, t) - \mathcal{H}(J, t)$ is a polynomial of degree $< d$.

Theorem 3.3. *Suppose that* $V_1, \ldots, V_m$ *are transversal. Then* $\mathcal{H}(J, t) - f(t)$ *is a polynomial, where*

$$f(t) := \frac{\prod_{i=1}^{m} (1 - (1-t)^{c_i})}{(1-t)^n}.$$

The theorem implies the following formula for the Hilbert polynomials.

Corollary 3.4. *If $V_1, \ldots, V_m$ are transversal, then*

$$\widetilde{h}_I(d) = \widetilde{h}_J(d) = \sum_S (-1)^{|S|} \binom{d+n-1-c_S}{n-1-c_S}$$

where $c_S = \sum_{i \in S} c_i$ and the sum is over all $S \subseteq \{1, 2, \ldots, m\}$ for which $c_S < n$.

We have seen that the Hilbert function and the Hilbert series of J are combinatorial invariants. The Hilbert function and Hilbert series of I are *not* combinatorial invariants. In fact, the Hilbert function $h_I(d)$ depends on the geometry of the arrangement in a very subtle way. Take for example the configuration $\mathcal{A} = \bigcup_{i=1}^m V_i$ where $V_1, \ldots, V_m$ are distinct 1-dimensional subspaces of V. This configuration is transversal. We can view this configuration as a set of m points in the projective space $\mathbb{P}^{n-1}$. We have $h_I(d) > 0$ if and only if all m points lie on a hypersurface of degree d. From this it is clear that the values of $h_I(d)$ for small d depend on more than just the combinatorial data n_S, $S \subseteq \{1, 2, \ldots, m\}$. It is quite a hard problem to determine the possible Hilbert functions for point configurations in $\mathbb{P}^{n-1}$. See for example [7] and the references there. For $d \geq m$, we have that $h_I(d) = h_J(d)$, so $h_I(d)$ is a combinatorial invariant. In particular the Hilbert polynomial $\widetilde{h}_I = \widetilde{h}_J$ is a combinatorial invariant. This is not so surprising, because $\widetilde{h}_I = \widetilde{h}_R - \widetilde{h}_{R/I}$ and $\widetilde{h}_{R/I}$ is the constant function with value m.

For *non-transversal* arrangements, the Hilbert *polynomial* $\widetilde{h}_I$ is *not* a combinatorial invariant either. For an arrangement $\mathcal{A} = \bigcup_{i=1}^m V_i \subseteq V$ and an s-dimensional vector space W define another arrangement by

$$\mathcal{A} \times W := \bigcup_{i=1}^m V_i \times W \subseteq V \times W.$$

Then we have

$$H(\mathcal{I}(\mathcal{A} \times W), t) = \frac{H(\mathcal{I}(\mathcal{A}), t)}{(1-t)^s}.$$

We have seen that there exist arrangements $\mathcal{A}, \mathcal{A}'$ with the same combinatorial invariants but not the same Hilbert series. If $s = \dim W$ is large enough, then

$$H(\mathcal{I}(\mathcal{A}' \times W), t) - H(\mathcal{I}(\mathcal{A} \times W), t) = \frac{H(\mathcal{I}(\mathcal{A}'), t) - H(\mathcal{I}(\mathcal{A}), t)}{(1-t)^s}$$

is not a polynomial. So $\mathcal{A} \times W$ and $\mathcal{A}' \times W$ have the same combinatorial invariants but *not* the same Hilbert polynomial.

4. Complexes of product ideals and intersection ideals

Theorem 4.1 (*See Chapter IV of [14]*). *There exist complexes*

$$0 \to I \to \bigoplus_{|S|=m-1} I_S \to \bigoplus_{|S|=m-2} I_S \to \cdots \to \bigoplus_{|S|=1} I_S \to R \to 0$$

and

$$0 \to J \to \bigoplus_{|S|=m-1} J_S \to \bigoplus_{|S|=m-2} J_S \to \cdots \to \bigoplus_{|S|=1} J_S \to R \to 0$$

whose homologies are killed by $\mathfrak{a} = \sum_{j=1}^m I_j$.

To describe the maps in the complexes in Theorem 4.1 it suffices to define maps $I_T \to I_S$ and $J_T \to J_S$ for all subsets $S, T \subseteq \{1, 2, \ldots, m\}$ with $|T| = |S| + 1$. If $T = \{i_1, i_2, \ldots, i_r\}$ with $i_1 < i_2 < \cdots < i_r$ and $S = \{i_1, i_2, \ldots, i_{s-1}, i_{s+1}, \ldots, i_r\}$ then the maps $I_T \to I_S$ and $J_T \to J_S$ in the complexes in Theorem 4.1 are given by $f \mapsto (-1)^s f$. All other maps are equal to 0.

Corollary 4.2. *If $V_{\{1,2,\ldots,m\}} = \bigcap_{i=1}^m V_i = (0)$, then*

$$\sum_{S \subseteq \{1,2,\ldots,m\}} (-1)^{|S|} \mathcal{H}(I_S, t)$$

and

$$\sum_{S\subseteq\{1,2,\ldots,m\}} (-1)^{|S|}\mathcal{H}(J_S,t)$$

are polynomials in t.

Proof. The ideal

$$\sum_{j=1}^{m} I_j = \mathcal{I}\left(\bigcap_{i=1}^{m} V_i\right) = \mathcal{I}(\{0\}) = \mathfrak{m}$$

is the maximal homogeneous ideal of R.

Suppose that

$$0 \xrightarrow{\partial_{r+1}} C_r \xrightarrow{\partial_r} C_{r-1} \xrightarrow{\partial_{r-1}} \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

is a complex of finitely generated graded R-modules. The i-th homology group is

$$H_i = \ker(\partial_i)/\mathrm{im}(\partial_{i+1}).$$

We have

$$\sum_{i=0}^{r}(-1)^i\mathcal{H}(C_i,t) = \sum_{i=0}^{r}(-1)^i\mathcal{H}(H_i,t).$$

If $\mathfrak{m}H_i = 0$, then H_i is finite dimensional, and $\mathcal{H}(H_i,t)$ is a polynomial for all i.

We now apply this to the complexes in Theorem 4.1. □

Proof of Theorem 3.1. Because J has a linear resolution, we can write

$$\mathcal{H}(J,t) = \frac{\sum_{i=0}^{r}(-1)^i\beta_i t^{i+m}}{(1-t)^n} = \frac{t^m p(t)}{(1-t)^n},$$

where

$$p(t) = \beta_0 - \beta_1 t + \cdots + (-1)^r\beta_r t^r \tag{2}$$

is a polynomial of degree $r \le \mathrm{pd}(J) \le n-1$. Similarly we can write

$$\mathcal{H}(J_S,t) = \frac{t^{|S|}p_S(t)}{(1-t)^n}$$

with

$$\deg(p_S(t)) \le n-1 \tag{3}$$

for all $S \subseteq \{1,2,\ldots,m\}$.

Let $W = V/V_S$ and define $W_X = V_X/V_S$ for $X \subseteq S$. Let $\overline{J}_i \subseteq K[W] \cong K^{[n-n_S]}$ be the vanishing ideal of W_i for all $i \in S$. Define $\overline{J}_X = \prod_{i\in X}\overline{J}_i$ for all $X \subseteq S$.

We have

$$\overline{J}_X \otimes_K K^{[n_S]} = J_X$$

inside $K^{[n]} = K^{[n-n_S]} \otimes_K K^{[n_S]}$. From this follows that

$$\frac{t^{|X|}p_X(t)}{(1-t)^n} = \mathcal{H}(J_X,t) = \frac{\mathcal{H}(\overline{J}_X,t)}{(1-t)^{n_S}}.$$

In particular, we have

$$\mathcal{H}(\overline{J}_S,t) = \frac{t^{|S|}p_S(t)}{(1-t)^{n-n_S}}.$$

From this it follows that $\deg(p_S(t)) \le \dim W - 1 = n - n_S - 1$ (see (3)). Since $\bigcap_{i \in S} W_i = 0$ in W, Corollary 4.2 implies that

$$\sum_{X \subseteq S} (-1)^{|X|} \mathcal{H}(\overline{J}_X, t) = \sum_{X \subseteq S} \frac{(-t)^{|X|} p_X(t)}{(1-t)^{n-n_S}}$$

is a polynomial in t. Multiplying with $(1-t)^{n-n_S}$ gives

$$\sum_{X \subseteq S} (-t)^{|X|} p_X(t) \equiv 0 \bmod (1-t)^{n-n_S}. \quad \square$$

Proof of Theorem 3.3. Define $\widetilde{p}_\emptyset(t) := 1 = p_\emptyset(t)$. For $X \neq \emptyset$, let $\widetilde{p}_X(t)$ be the unique polynomial of degree $< n$ such that

$$\widetilde{p}_X(t) \equiv \prod_{i \in X} \left(\frac{1 - (1-t)^{c_i}}{t} \right) \bmod (1-t)^n.$$

From the transversality follows that

$$\deg(\widetilde{p}_X(t)) < c_X = \max \left\{ \sum_{i \in X} c_i, n \right\}.$$

To show that $\widetilde{p}_S(t) = p_S(t)$ for all $S \subseteq \{1, 2, \ldots, n\}$ it suffices to show that $\widetilde{p}_S(t)$ satisfies the same recursion formula:

For every $X \subseteq S$ we have

$$\widetilde{p}_X(t) \equiv \prod_{i \in X} \left(\frac{1 - (1-t)^{c_i}}{t} \right) \bmod (1-t)^n,$$

so

$$\sum_{X \subseteq S} (-t)^{|X|} p_X(t) \equiv \sum_{X \subseteq S} (-1)^{|X|} \prod_{i \in X} (1 - (1-t)^{c_i})$$
$$\equiv \prod_{i \in S} \big(1 - (1 - (1-t)^{c_i})\big) \equiv (1-t)^{(\sum_{i \in S} c_i)} \equiv 0 \bmod (1-t)^{c_S}.$$

For $S = \{1, 2, \ldots, m\}$, the Hilbert series

$$H(J, t) = \frac{t^m p_S(t)}{(1-t)^n} = \frac{t^m \widetilde{p}_S(t)}{(1-t)^n}$$

is equal to $f(t)$ up to a polynomial. $\square$

5. Application to generalized principal component analysis

The object of principal component analysis (PCA) is to approximate a data set inside a vector space V by a subspace of smaller dimension. In generalized principal component analysis (GPCA) one tries to approximate a data set inside a vector space V by a *union* of subspaces spaces (in other words, a *subspace arrangement*). Some applications of GPCA are motion segmentation (see [18,16]), image segmentation (see [15]), image compression (see [10]) and hybrid control systems ([12]). For an overview of GPCA, see [17].

A first start in GPCA is to decide on the number of subspaces and the dimensions of the subspaces of the subspaces arrangement that will approximate the data.

Suppose that $v_1, v_2, \ldots, v_r \in V$ are data points. Here r is fairly large. Suppose that $v_1, \ldots, v_r$ are contained in some subspace arrangement $\mathcal{A} = V_1 \cup \cdots \cup V_m$, unknown to us. We would like to recover $n_1, \ldots, n_m$ where $n_i = \dim V_i$. Let $\mathfrak{a}_j$ be the vanishing ideal of the ray through v_j. Then we have that

$$h(\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_r, d) = h(I, d)$$

for small values of d, where $I = \mathcal{I}(\mathcal{A})$ as before. Now

$$h(\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_r, d)$$

can be computed using linear algebra for small values of d. Therefore, we can determine $h(I, d)$ for small values of d. So an important question is, given $h(I, d)$ for small values of d, can we determine the dimensions $n_1, n_2, \ldots, n_m$? Proposition 5.1 gives an affirmative answer if the subspaces are transversal. Of course, in real applications the data are approximated by the subspaces arrangement, but not contained in it. In that case, using the PCA method in $K[V]_d$ one still can estimate the value $h(\mathcal{I}(\mathcal{A}), d)$.

Proposition 5.1. *Assume that the arrangement is transversal. Suppose that $c_1, \ldots, c_m$ are unknown, but we know the values of the Hilbert polynomial*

$$h_I(d)$$

for $d = m, m + 1, \ldots, m + n - 1$, then we can recover $c_1, \ldots, c_m$.

Proof. Note that $h_I(d) = \widetilde{h}_I(d)$ for $d \geq m$. Since we know $\widetilde{h}_I(d)$ for $d = m, m + 1, \ldots, m + n - 1$ and $\widetilde{h}_I$ has degree $\leq n - 1$, $\widetilde{h}_I$ is uniquely determined. From this, we can determine $\mathcal{H}(I, t)$, up to a polynomial. Suppose that $\mathcal{H}(I, t)$ is equal to $a(t)/(1 - t)^n$ up to a polynomial. Let $b(t)$ be the remainder of division of $a(t)$ by $(1 - t)^n$. Then $b(t)$ has degree $< n$ and $\mathcal{H}(I, t)$ is equal to $b(t)/(1 - t)^n$ modulo a polynomial. So we have that

$$b(t) \equiv \prod_{i=1}^{d} (1 - (1 - t)^{c_i}) \bmod (1 - t)^n$$

and

$$b(1 - t) \equiv \prod_{i=1}^{d} (1 - t^{c_i}) \bmod t^n.$$

Let r_i be the number of the c_j's equal to i. Then we have

$$b(1 - t) \equiv \prod_{i=1}^{n-1} (1 - t^i)^{r_i} \bmod t^n.$$

From this we can easily determine $r_1, r_2, \ldots, r_{n-1}$ (in that order). Indeed, if we already know $r_1, \ldots, r_s$, then the Taylor series of

$$\frac{b(1 - t)}{\prod_{i=1}^{s} (1 - t^i)^{r_i}}$$

is

$$1 - r_{s+1} t^{s+1} + \text{ higher order terms.}$$

So we find the value of r_{s+1}. □

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