Multimagic Squares

Harm Derksen, Christian Eggermont & Arno van den Essen

Multimagic Squares

Harm Derksen, Christian Eggermont, and Arno van den Essen

1. INTRODUCTION. Suppose that M is an $m \times m$ matrix whose entries are natural numbers. Then M is called a *magic square* (of order m) if the sum of all elements in each column, each row, and each of the two main diagonals gives the same number, the so-called magic number. If, in addition, its matrix elements consist of the consecutive integers $1, 2, \ldots, m^2$ the matrix M is called a *pure* (or *normal*) magic square. Pure magic squares have been studied for more than four thousand years. During this long period of time many subclasses have been introduced, such as panmagic squares, which have the property that their broken diagonals also add up to the same magic number (a *broken diagonal* is a diagonal parallel to one of the two main diagonals), and multimagic squares, the subject of this article. A *most-perfect* magic square is a normal magic square of order n, a multiple of four, with the following two additional properties: (1) each 2×2 subsquare sums to $2s$, where $s = n^2 + 1$, and (2) all pairs of integers along any diagonal and at distance $n/2$ from each other sum to s. Recently some exciting new results have been found concerning these squares. For instance, the first method of constructing all most-perfect magic squares and their enumeration appeared in [**10**] (see also [**14**]). Another highlight is the verification (by computer) of the nonexistence of an 8×8 magic knight tour in [**13**] (i.e., it is impossible for a knight to make a tour on a checkerboard starting somewhere with the number 1, then putting 2 after one jump, and so forth, and after completing its tour to obtain a magic square). Finally, we would like to mention that in [**2**] a new method is found to construct and enumerate all Franklin squares. Franklin squares have the following properties: (i) the sum of the numbers in half a row (respectively half a column) is constant, (ii) each 2×2 subsquare has the same sum, and (iii) each of the 4 'bent diagonals' has the same sum. Here a 'bent diagonal' is formed by joining half of one diagonal to half of another diagonal at a 90° angle. However, there are still many unsolved problems (see [**1**] and [**12**]) or, to put it in the words of Clifford Pickover ([**12**, p. 26]) "the field of magic square study is wide open."

In this paper we concentrate on one of the aforementioned open problems, namely, the existence and construction of so-called multimagic squares. To describe this class of magic squares, let M be an $m \times m$ matrix and d a positive integer. Then M^{*d} denotes the $m \times m$ matrix obtained by raising each entry of M to the power d. Now let n be a positive integer. Then M is called an *n-multimagic square* if M is a normal magic square and $M^{*2}, M^{*3}, \ldots, M^{*n}$ are all magic. The first 2-multimagic square was found by Pfeffermann in 1891. It is the following square of order 8:

56	34	8	57	18	47	9	31
33	20	54	48	7	29	59	10
26	43	13	23	64	38	4	49
19	5	35	30	53	12	46	60
15	25	63	2	41	24	50	40
6	55	17	11	36	58	32	45
61	16	42	52	27	1	39	22
44	62	28	37	14	51	21	3

In 1905 the first 3-multimagic square, a square of order 128, was constructed by Tarry. In 2001 both a 4- and a 5-multimagic squares were constructed by Boyer and Viricel, squares of order 512 and 1024, respectively (see [**4**] and [**5**], where a nice history of the subject is given). The record up to now has been a 6-multimagic square of order 4096 constructed by Pan Fengchu in October 2003 [**8**]. However, the following question remained open: Do there exist n-multimagic squares when $n \geq 7$?

In this paper we give an affirmative answer to this question by providing an explicit construction for each n greater than 2. In particular, we give the first proof of the existence of 7-multimagic squares (ours has order 13^7), the first 8-multimagic squares (order 17^8), etc. Our construction is based on elementary facts from linear algebra over finite fields and can easily be extended to construct n-multimagic cubes and hypercubes. At the end of this paper we comment on various generalizations of our construction.

2. CONSTRUCTION OF MULTIMAGIC SQUARES. Looking at the examples of multimagic squares described in the introduction, one observes that they all are matrices whose orders are powers of prime numbers. Not all multimagic squares need have prime power order. For example, Trump recently constructed the smallest 3-multimagic square, which has order 12 [**15**]. Nevertheless, each n-multimagic square we are going to construct will have order p^n, where p is an odd prime number (at the end of this paper we indicate how to construct n-multimagic squares of order q^n, where q can be any number greater than 2).

Fix an odd prime p. The starting point of our construction is the following observation: a square matrix M of order p^n whose entries comprise all integers from 1 to $(p^n)^2$ can be viewed as a bijection from the set of pairs (i, j) with $1 \leq i, j \leq p^n$ to the set of consecutive integers from 1 to $(p^n)^2$. To find such bijections we first consider the bijection $N : \mathbb{Z}/p\mathbb{Z} \to \{0, 1, \dots, p-1\}$ given by $N(\bar{i}) = i$ $(0 \leq i \leq p-1)$, where as usual $\bar{i}$ denotes the class of i modulo p. We use k to denote the field $\mathbb{Z}/p\mathbb{Z}$. It is easy to verify that $N(a) + N(-a-1) = p-1$ for all a in k.

More generally, for each positive integer m we obtain a bijection $N_m : k^m \to \{1, 2, \dots, p^m\}$ defined by means of the formula

$$N_m\left((a_1, \dots, a_m)\right) = 1 + \sum_{j=1}^{m} p^{j-1} N(a_j) \quad ((a_1, \dots, a_m) \in k^m).$$

Using the relation $N(a) + N(-a-1) = p-1$ one quickly deduces that

$$N_m(a) + N_m(-a+c) = p^m + 1 \tag{1}$$

for all a in k^m, where $c = (-1, -1, \dots, -1)$.

Next we choose a bijection $F : (k^n)^2 \to (k^n)^2$. The composition

$$M := N_{2n} \circ F \circ (N_n^{-1} \times N_n^{-1}) \tag{2}$$

gives a bijection from $\{1, 2, \dots, p^n\}^2$ to $\{1, 2, \dots, p^{2n}\}$. Clearly, not every choice of F gives rise to an n-multimagic square. To obtain such squares we first choose F to be an affine map given by an invertible $2n \times 2n$ matrix A over k and a vector t in k^{2n} (i.e.,

$$F\big((a, b)\big) = A\begin{pmatrix} a \\ b \end{pmatrix} + t \tag{3}$$

for (a, b) in $(k^n)^2$). Since A is invertible, F is bijective, hence so is M. In order to ensure that M is n-multimagic we impose on A the following condition: all $n \times n$

minors of P, Q, $P+Q$, and $P-Q$ are units in k, where P (respectively, Q) is the $2n \times n$ matrix formed by the first (respectively, last) n columns of A. (Of course, we could have simply said that the minors are nonzero. However, since we will later work with arbitrary commutative rings rather than fields, we chose to write "units in k" instead of "nonzero elements in k"). Such a matrix A is called an *n-multimagic generator matrix*. In the next section we will give explicit formulas for these matrices when $n \geq 3$ and $p \geq 2n-1$. Now we are ready to state the main result of this paper.

Theorem 2.1. *Let F be as in* (3)*, where A is an n-multimagic generator matrix. Then the matrix M defined by* (2) *is an n-multimagic square.*

Proof. Since, as observed, (2) is a bijection, the entries of the matrix M include all integers from 1 to p^{2n}. Thus M is normal.

(i) We write $A = (P|Q)$ as indicated, and let d be an integer with $1 \leq d \leq n$. We show that each column-sum of M^{*d} is the same constant that depends only on p and n. Therefore we fix an element b in k^n. To compute the sum of the $N_n(b)$th column of M^{*d}, which we denote by $S_d(b)$, we have to add the elements $M^d_{N_n(a),N_n(b)}$, where a runs through k^n (remember that N_n is a bijection). It follows from the definition of M that

$$S_d(b) = \sum_{a \in k^n} \left(1 + \sum_{j=1}^{2n} C_j(a,b)\right)^d, \tag{4}$$

where

$$C_j(a,b) = p^{j-1} N(P_{(j)}a + Q_{(j)}b + t_j) \quad (1 \leq j \leq 2n) \tag{5}$$

and $P_{(j)}$ (respectively, $Q_{(j)}$) denotes the jth row of P (respectively, Q). Now observe that if $x_1, x_2, \ldots, x_{2n}$ are variables, then $(1 + x_1 + \cdots + x_{2n})^d$ is equal to $1+g$, where g is a sum of terms of the form $\alpha x_{j_1}^{e_1} \ldots x_{j_s}^{e_s}$. Here α is a positive integer, $1 \leq j_1 < j_2 < \ldots < j_s \leq 2n$, $e_1, \ldots, e_s \geq 1$, and $e_1 + \cdots + e_s \leq d$. In particular $s \leq d \leq n$.

It will follow from (4) that $S_d(b)$ depends only on p and n if we can show that for each set of exponents $e_1, \ldots, e_s$ and indices $j_1, \ldots, j_s$ the sum

$$\sum_{a \in k^n} C_{j_1}(a,b)^{e_1} \cdots C_{j_s}(a,b)^{e_s} \tag{6}$$

depends only on p and n (and, of course, on the e_i and j_i). To see this, put $J = (j_1, \ldots, j_s)$ and consider the affine map $L : k^n \to k^s$ given by the formula

$$L(a) = P_{(J)}a + Q_{(J)}b + t_{(J)},$$

where $P_{(J)}$ (respectively, $Q_{(J)}$) is the $s \times n$ matrix formed by the rows $P_{(j_1)}, \ldots, P_{(j_s)}$ (respectively, $Q_{(j_1)}, \ldots, Q_{(j_s)}$) and $t_{(J)}$ is the column of length s with components $t_{j_1}, \ldots, t_{j_s}$. Since $s \leq n$ and all $n \times n$ minors of P are units in k, we infer that L is surjective. On the basis of (5) we know that $C_{j_i}(a,b) = p^{j_i-1} N\big(L(a)_i\big)$ for $i = 1, 2, \ldots, s$. Accordingly, the expression (6) is equal to

$$p^{e_1(j_1-1)+\cdots+e_s(j_s-1)} \sum_{a \in k^n} N\big(L(a)_1\big)^{e_1} \cdots N\big(L(a)_s\big)^{e_s}.$$

By Lemma 2.2, which we will establish shortly, the surjectivity of L implies that this expression equals

$$p^{n-s} p^{e_1(j_1-1)+\cdots+e_s(j_s-1)} \left(\sum_{i=0}^{p-1} i^{e_1}\right) \cdots \left(\sum_{i=0}^{p-1} i^{e_s}\right), \tag{7}$$

which indeed depends only on p and n, as desired. Interchanging the roles of a and b in the argument just given we find that all row sums of M^{*d} are also equal to the same constant.

(ii) To compute the sum of the main diagonal elements of M^{*d} we have to add all elements $M^d_{N_n(a),N_n(a)}$, where a runs through k^n. For this we repeat the preceding arguments with $b = a$ and define $L_1 : k^n \to k^s$ by the formula

$$L_1(a) = P_{(J)}a + Q_{(J)}a + t_{(J)} = (P + Q)_{(J)}a + t_{(J)}.$$

Then L_1 is surjective since all $n \times n$ minors of $P + Q$ are units of k. The rest of the proof is the same as that given in (i).

(iii) Finally, to compute the sum of all elements of the "second" main diagonal of M^{*d} we have to add the elements $M^d_{N_n(a),p^n+1-N_n(a)}$. Using (1) we can repeat the arguments in (i) with $b = -a + c$ and define $L_2 : k^n \to k^s$ by

$$L_2(a) = P_{(J)}a + Q_{(J)}(-a + c) + t_{(J)} = (P - Q)_{(J)} + (Q_{(J)}c + t_{(J)}).$$

Again L_2 is surjective because all $n \times n$ minors of $P - Q$ are units in k. The rest of the proof mimics that in (i). ■

To complete the proof of Theorem 2.1 it remains to fill in the following missing piece:

Lemma 2.2. *If $L : k^n \to k^s$ is a surjective affine map and $e_1, \ldots, e_s$ are nonnegative integers, then*

$$\sum_{a\in k^n} N\big(L(a)_1\big)^{e_1} \cdots N\big(L(a)_s\big)^{e_s} = p^{n-s} \left(\sum_{i=0}^{p-1} i^{e_1}\right) \cdots \left(\sum_{i=0}^{p-1} i^{e_s}\right).$$

Proof. Put $V(a) = N\big(L(a)_1\big)^{e_1} \cdots N\big(L(a)_s\big)^{e_s}$ for each a in k^n. Observe that k^n is a disjoint union of the fibers $L^{-1}(y)$, where $y = (y_1, \ldots, y_s)$ runs through k^s, and that V is constant (equal to $N(y_1)^{e_1} \cdots N(y_s)^{e_s}$) on each fiber $L^{-1}(y)$. Since by the surjectivity and linearity of L each fiber has p^{n-s} elements, we see that

$$\sum_{a\in k^n} V(a) = p^{n-s} \sum_{y\in k^s} N(y_1)^{e_1} \cdots N(y_s)^{e_s} = p^{n-s} \left(\sum_{y_1\in k} N(y_1)^{e_1}\right) \cdots \left(\sum_{y_s\in k} N(y_s)^{e_s}\right).$$

Because $N : k \to \{0, 1, \ldots, p - 1\}$ is a bijection, the lemma follows. ■

3. FINDING *n*-MULTIMAGIC GENERATOR MATRICES. In order to construct n-multimagic squares using Theorem 2.1, we need to show how to find n-multimagic generator matrices A over k, where as earlier k denotes the field of p elements for an odd prime p. Therefore we will assume that $n \geq 3$ and $p \geq 2n - 1$. Then when $0 \leq j < i \leq 2n - 2$ the differences $i - j$ are units in k. By appealing to Vandermonde determinants we conclude that all $n \times n$ minors of the following $2n \times n$ matrix P are units in k:

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 4 & \dots & 2^{n-1} \\ 1 & 3 & 9 & \dots & 3^{n-1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & (2n-2) & (2n-2)^2 & \dots & (2n-2)^{2n-1} \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Using this matrix it is easy to make an n-multimagic generator matrix A as follows: write $P = \binom{P_1}{P_2}$, where P_1 (respectively, P_2) is the $n \times n$ matrix formed by the first (respectively, last) n rows of P and set $Q = \binom{2P_1}{-2P_2}$. Then define $A = (P|Q)$.

Using elementary column operations one can reduce

$$A = \begin{pmatrix} P_1 & 2P_1 \\ P_2 & -2P_2 \end{pmatrix}$$

to the matrix

$$\begin{pmatrix} P_1 & 0 \\ P_2 & -4P_2 \end{pmatrix},$$

which is clearly invertible because both $\det P_1$ and $\det(-4P_2)$ are units in k. It follows without difficulty that A is an n-multimagic generator matrix.

4. THE ORDERS OF MULTIMAGIC SQUARES. Now that we know that n-multimagic squares exist for each positive integer n one might wonder whether for each such n n-multimagic squares exist of order m, say if m is sufficiently large. The next result shows that this is not the case. For example, taking $e = 1$ and $p = 2$ in Theorem 4.1 shows that 3-multimagic squares of order m such that $m \equiv 2 \pmod 4$ cannot exist. A very simple proof of this was communicated to us by Christian Boyer. It goes as follows: Let M be a 3-magic square of order m. Then its magic sum S_1 and the magic sum S_3 of M^{*3} are equivalent modulo 2, for every integer is equivalent to its cube modulo 2. On the other hand, it is well known that $S_1 = m(m^2+1)/2$ and $S_3 = mS_1^2$. Thus if $m \equiv 2 \pmod 4$, then S_3 is even, whereas S_1 is odd!

Theorem 4.1. *Let M be an n-multimagic square of order m. If the prime number p divides m with multiplicity $e \geq 1$, then $n \leq p^{e+1} - 2$.*

Proof. Consider the polynomial

$$f(x) = \binom{x-1}{n}$$

in $\mathbb{Q}[x]$. Then f has degree n. Since M is n-multimagic it follows that the matrix $f(M)$ obtained by applying f to each entry of M is a magic square consisting of (nonnegative) integers. In particular, the sum $f(1) + f(2) + \cdots + f(m^2)$ is m times the magic sum. The sum

$$\binom{0}{n} + \binom{1}{n} + \cdots + \binom{m^2-1}{n},$$

which is equal to $\binom{m^2}{n+1}$, is thus divisible by m. Hence the (rational) number

$$q := \frac{m(m^2-1)(m^2-2)\cdots(m^2-n)}{(n+1)!}$$

is an integer. Since an n-multimagic square of order m cannot exist if an $(n-1)$-multimagic square of order m cannot exist, it suffices to show that the existence of such a square in the case $n = p^{e+1} - 1$ leads to a contradiction. Let $n = p^{e+1} - 1$. Note that q is nonzero since by hypothesis p^e divides m, implying that $m^2 \geq p^{2e} > p^{e+1} - 1 = n$ (e is positive). To obtain a contradiction we compute the multiplicity of p in q. To this end, for any nonzero integer r we denote the multiplicity of p in r by $v_p(r)$ and for any nonzero rational number r/s we put $v_p(r/s) = v_p(r) - v_p(s)$. It is easy to verify that $v_p(q_1q_2) = v_p(q_1) + v_p(q_2)$ for any pair of nonzero rational numbers q_1 and q_2. Observe that q is a product of $m/(n+1)$ and the factors $(m^2 - i)/i$, for $i = 1, \ldots, n$. It follows that $v_p(q) = v_p(m/(n+1))$, for $v_p\big((m^2-i)/i\big) = 0$ when $1 \leq i \leq n = p^{e+1} - 1$. Recalling that $n = p^{e+1} - 1$, we obtain

$$v_p(q) = v_p(m) - v_p(p^{e+1}) = e - (e+1) = -1,$$

which contradicts the fact that q is an integer. ■

5. MULTIMAGIC CUBES AND HYPERCUBES. In this section we indicate how the method described earlier can be extended to construct multimagic cubes and hypercubes. Also, special subclasses of these magic cubes (hypercubes) can be constructed using the same method. We illustrate this for the class of "perfect" multimagic cubes.

We start with multimagic cubes. We remark at the outset that there is no consensus on the definition of multimagic cubes (or hypercubes or higher dimensional analogues) in the literature. The choice we make is the following (see [**9**] or [**16**]): an $m \times m \times m$ cube of natural numbers (respectively, the consecutive numbers $1, 2, \ldots, m^3$) is called *magic* (respectively, *normal magic*) if the sum of all elements in each row, column, and pillar is the same and is equal to the sum of all elements of each of the four space diagonals. Furthermore, if $n \geq 1$, such a cube is called *n-multimagic* if the cube obtained by raising each of its entries to the dth power ($1 \leq d \leq n$) is magic.

The construction of an n-multimagic cube goes as follows: Analogously to the construction of n-multimagic squares in (2) and (3), we define a $p^n \times p^n \times p^n$ cube M (where the prime number p has to be chosen appropriately) by the formula

$$M = N_{3n} \circ F \circ (N_n^{-1} \times N_n^{-1} \times N_n^{-1}),$$

where F is an affine map given by an invertible $3n \times 3n$ matrix A over k (the field of p elements) and a vector t in k^{3n} (i.e.,

$$F\big((a,b,c)\big) = A \begin{pmatrix} a \\ b \\ c \end{pmatrix} + t$$

for (a, b, c) in $(k^n)^3$). In order to make M n-multimagic we choose A to be a so-called n-multimagic 3-generator matrix. This means that $A = (P|Q|R)$, where P, Q, and R are $3n \times n$ matrices over k that exhibit the following three properties:

1. A is invertible over k (which guarantees that all the natural numbers $1, 2, \ldots, p^{3n}$ appear as entries in M);

2. all $n \times n$ minors of the matrices P, Q, and R are units in k (which guarantees that when $1 \leq d \leq n$ the sum of all elements in each column, row, and pilar of M^{*d} is the same, hence equal to the magic sum);
3. all $n \times n$ minors of the matrices $P + Q + R$, $-P + Q + R$, $P - Q + R$, and $P + Q - R$ are units in k (which guarantees that when $1 \leq d \leq n$ the sum of all elements of each of the four space diagonals of M^{*d} is equal to the magic sum).

If we want to construct n-multimagic cubes having additional properties, we can achieve this by imposing extra conditions on the matrices P, Q, and R.

To illustrate this approach we indicate how to construct so-called n-multimagic perfect cubes. A magic cube is *perfect* if the diagonals of each orthogonal slice have the magic sum property. Furthermore, such a cube is *n-multimagic perfect* if when $1 \leq d \leq n$ the cube obtained by raising each of its entries to the dth power is again perfect. To guarantee that an n-multimagic cube M arrived at via the foregoing construction is also n-multimagic perfect, we impose on the matrices P, Q, and R a fourth condition:

4. all $n \times n$ minors of the matrices $P + Q$, $P - Q$, $P + R$, $P - R$, $Q + R$, and $Q - R$ are units in k.

At this point it should be clear to the reader how to proceed in higher dimensions (i.e., how to construct hypercubes and hypercubes with additional properties). It remains only to show how to select a suitable prime number p and how to construct an n-multimagic 3-generator matrix A or, more generally, how to construct for each d (≥ 2) an n-multimagic d-generator matrix A (i.e., an invertible matrix $A = (A_1|\cdots|A_d)$ over k such that each A_i is a $dn \times n$ matrix with the property that for arbitrary $\delta_1, \ldots, \delta_n$ in $\{-1, 0, 1\}$, not all zero, all $n \times n$ minors of each combination $\delta_1 A_1 + \cdots + \delta_d A_d$ are units in k). Before we indicate the construction of such matrices we recall the following well-known fact (which is easy to prove by induction on m):

If f is a nonzero polynomial in m variables over the integers, then there exist integers $a_1, \ldots, a_m$ such that $f(a_1, \ldots, a_m)$ is nonzero. (8)

Now let $n \geq 1$ and $d \geq 2$ be given. To avoid complicating an easy matter we demonstrate only how to find n-multimagic d-generator matrices for the case $d = 2$. For other d the procedure is similar.

Let $A_u = (A_{ij})$ and $B_u = (B_{ij})$ be two universal $2n \times n$ matrices (i.e., the entries A_{ij} and B_{ij} are distinct variables). Then each $n \times n$ minor of A_u, B_u, $A_u + B_u$, and $A_u - B_u$ is a nonzero polynomial with integer coefficients in the $4n^2$ variables A_{ij} and B_{ij}. Let f denote the polynomial obtained by taking the product of all these minors and the polynomial $\det(A_u|B_u)$. By (8) we can find integers a_{ij} and b_{ij} such that $f(a_{ij}, b_{ij})$ is a nonzero integer. Finally, let p be a prime number that does not divide this integer. It turns out that over $k = \mathbb{Z}/p\mathbb{Z}$ the matrix $A = (A_1|A_2)$, where $A_1 = (a_{ij})$ and $A_2 = (b_{ij})$, leads to an n-multimagic 2-generator matrix when the preceding construction is carried out.

6. GENERALIZATIONS. In the previous sections we described how to construct multimagic squares, three-dimensional cubes, and cubes of arbitrary dimension of order p^n. A crucial ingredient was the field $\mathbb{Z}/p\mathbb{Z}$ of p elements. In this section we indicate how this construction can be generalized to more general situations. More precisely, for each integer q (≥ 2) we show how to construct multimagic squares and their higher dimensional counterparts of order q^n. Furthermore, our generalization even allows us to obtain more n-multimagic squares of order p^n. The generalization involves two variations on the earlier theme: first, instead of a field of p elements we consider arbitrary commutative rings R having precisely q elements; second, in place of the bijection $N : \mathbb{Z}/p\mathbb{Z} \to \{0, 1, \ldots, p-1\}$ we exploit more general bijections between R and $\{0, 1, \ldots, q-1\}$. A key observation is that the bijection N has the property that $N(a) + N(-a-1) = p-1$ for all a in $\mathbb{Z}/p\mathbb{Z}$. In the general setting we consider bijections $N : R \to \{0, 1, \ldots, q-1\}$ with the property that there exists an element c in R such that $N(a) + N(-a+c) = q-1$ for all a in R. (In Lemma 6.1 we show that for each commutative ring R with q elements such a bijection exists!) For each positive integer m we obtain a bijection $N_m : R^m \to \{1, 2, \ldots, q^m\}$:

$$N_m\big((a_1, \ldots, a_m)\big) = 1 + \sum_{j=1}^{m} q^{j-1} N(a_j).$$

Again, as in (1), we have a relation

$$N_n(a) + N_n(-a + c') = q^n + 1$$

for all a in R^n, where $c' = (c, c, \ldots, c)$ belongs to R^m.

The proof of Theorem 2.1 can be copied completely, simply by replacing k with R and p with q. Furthermore the construction of an n-multimagic generator matrix at the end of section 3 also works if we assume that all the elements $1, 2, \ldots, 2n-2$ are units in R. In case $n = 2$ we need to assume additionally that 3 is a unit in R. Finally, for the construction of n-multimagic d-generator matrices we take for R the ring $\mathbb{Z}/q\mathbb{Z}$ and repeat all the arguments at the end of section 5.

It remains only to indicate why for every commutative ring R having q elements there exists a bijection $N : R \to \{0, 1, \ldots, q-1\}$ with the property that, for some c in R, $N(a) + N(-a+c) = q-1$ holds for all a in R. We call such a bijection a *bijection of type c*.

Lemma 6.1. *If 2 is a unit in R, then for each c in R there exists a bijection N of type c. If 2 is not a unit in R, then for each unit c in R there exists a bijection N of type c.*

Proof. For c in R define $\varphi = \varphi_c : R \to R$ by $\varphi(a) = -a + c$. Then φ^2 is equal to the identity, so all orbits of φ have length one or two and an element a is a fixed point of φ if and only if $2a = c$. For a in R denote its orbit under φ by $O(a)$.

(i) Let 2 be a unit in R. Then φ has exactly one fixed point, namely, $a_0 := 2^{-1}c$. The orbit $O(a_0)$ of a_0 consists of one element. Let $O(a_1), \ldots, O(a_s)$ be the other orbits. Each consists of two elements. In particular, $q = 2s + 1$. Define $N(a_0) = s$, $N(a_i) = i - 1$, and $N\big(\varphi(a_i)\big) = q - 1 - N(a_i)$ $(= q - i)$ for $i = 1, 2, \ldots, s$. Then N is the desired bijection.

(ii) Let 2 be a nonunit in R. Then for c in R^*, the group of units in R, the map $\varphi = \varphi_c$ has no fixed point: indeed, if $\varphi(a) = a$ for some a, then $2a = c$. This implies that 2 is a unit in R, a contradiction. Consequently, we can write R as the disjoint union of orbits $O(a_1), \ldots, O(a_s)$, each of which consists of the two elements. In particular, $q = 2s$. Finally define $N(a_i) = i - 1$ and $N\big(\varphi(a_i)\big) = q - 1 - N(a_i)$ $(= q - i)$ for all i. Again N is a bijection of type c. ∎

7. EXAMPLES To conclude this paper we describe some examples of bimagic (i.e., 2-multimagic) squares obtained by the construction in section 3 and its generalizations in section 6. Recall that a magic square of order m is *associative* if the sum of any pair of its entries that are symmetric with respect to the center of the square is $m^2 + 1$.

Example 1 (A family of associative bimagic squares of order 16). Take

$$R = \mathbb{F}_2[x]/(x^2 + x + 1)$$

and the 2-multimagic generator matrix

$$A = \begin{pmatrix} x & 0 & 1 & x \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ x & 1 & 0 & x \end{pmatrix}.$$

Let $N : R \to \{0, 1, 2, 3\}$ be the bijection given by $N(0) = 0$, $N(1) = 2$, $N(x) = 1$, and $N(x + 1) = 3$. Then N is of type $x + 1$. For each t in R^4 we obtain an associative bimagic square of order 16. Taking $t = (0, 1, 1, 0)^T$ yields the following example from this family:

41	252	74	155	125	176	30	207	129	84	226	51	213	8	182	103
62	239	93	144	106	187	9	220	150	71	245	40	194	19	161	116
3	210	100	177	87	134	56	229	171	122	204	25	255	46	160	77
24	197	119	166	68	145	35	242	192	109	223	14	236	57	139	90
240	61	143	94	188	105	219	10	72	149	39	246	20	193	115	162
251	42	156	73	175	126	208	29	83	130	52	225	7	214	104	181
198	23	165	120	146	67	241	36	110	191	13	224	58	235	89	140
209	4	178	99	133	88	230	55	121	172	26	203	45	256	78	159
98	179	1	212	54	231	85	136	202	27	169	124	158	79	253	48
117	168	22	199	33	244	66	147	221	16	190	111	137	92	234	59
76	153	43	250	32	205	127	174	228	49	131	82	184	101	215	6
95	142	64	237	11	218	108	185	247	38	152	69	163	114	196	17
167	118	200	21	243	34	148	65	15	222	112	189	91	138	60	233
180	97	211	2	232	53	135	86	28	201	123	170	80	157	47	254
141	96	238	63	217	12	186	107	37	248	70	151	113	164	18	195
154	75	249	44	206	31	173	128	50	227	81	132	102	183	5	216

Example 2 (A family of bimagic squares of odd order). Take $R = \mathbb{Z}/q\mathbb{Z}$, where $q \geq 3$ and q is odd,

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 1 & 2 & 1 \\ 2 & 1 & 2 & 2 \end{pmatrix},$$

and t an arbitrary member of R^4. Let $N : R \to \{0, 1, \ldots, q - 1\}$ be the standard bijection: $N(\bar{i}) = i$ $(0 \leq i \leq q - 1)$. Then N is of type -1. Since A is a 2-multimagic generator matrix, our construction gives a family of bimagic squares of odd order. If we choose $q = 3$ and t the zero vector, we find the following square:

1	35	60	23	48	79	18	40	65
70	14	39	56	9	31	78	19	53
49	74	27	44	69	10	30	61	5
38	72	13	33	55	8	52	77	21
26	51	73	12	43	68	4	29	63
59	3	34	81	22	47	64	17	42
75	25	50	67	11	45	62	6	28
36	58	2	46	80	24	41	66	16
15	37	71	7	32	57	20	54	76

Furthermore, if we take $q = 3$ and $t = (2, 1, 0, 2)^T$ we recover the associative bimagic square of order nine constructed by R. V. Heath in 1933 (see [**3**, p. 212]).

Before we turn to our last example, we recall that a magic square is called *pandiagonal* if also the sum of the elements on each broken diagonal is equal to the magic sum.

Example 3 (An associative, pandiagonal, bimagic, magic square of order 25). Take $R = \mathbb{F}_5$, let $N : R \to \{0, 1, 2, 3, 4\}$ be the standard bijection (of type -1), and set

$$X = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}, \quad t = \begin{pmatrix} 0 \\ 4 \\ 0 \\ 2 \end{pmatrix}.$$

Then the corresponding 25×25 matrix is associative, pandiagonal, and bimagic, and it has the following properties:

(i) each of the twenty-five standard 5×5 submatrices is pandiagonal (with the same magic sum);

(ii) for each pair (i, j) $(1 \le i, j \le 25)$ the 5×5 matrix obtained by deleting each row with row number not equivalent to i modulo 5 and each column with column number not equivalent to j modulo 5 is pandiagonal!

103	350	567	164	381	291	513	235	452	74	584	176	423	20	362	147	494	86	308	530	440	32	254	621	218
167	389	106	328	575	460	52	299	516	238	23	370	587	184	401	311	533	130	497	94	604	221	443	40	257
331	553	175	392	114	524	241	463	60	277	187	409	1	373	595	480	97	319	536	133	43	265	607	204	446
400	117	339	556	153	63	285	502	249	466	351	598	195	412	9	544	136	483	80	322	207	429	46	268	615
564	156	378	125	342	227	474	66	288	510	420	12	359	576	198	83	305	547	144	486	271	618	215	432	29
134	476	98	320	537	447	44	261	608	205	115	332	554	171	393	278	525	242	464	56	591	188	410	2	374
323	545	137	484	76	611	208	430	47	269	154	396	118	340	557	467	64	281	503	250	10	352	599	191	413
487	84	301	548	145	30	272	619	211	433	343	565	157	379	121	506	228	475	67	289	199	416	13	360	577
526	148	495	87	309	219	436	33	255	622	382	104	346	568	165	75	292	514	231	453	363	585	177	424	16
95	312	534	126	498	258	605	222	444	36	571	168	390	107	329	239	456	53	300	517	402	24	366	588	185
290	507	229	471	68	578	200	417	14	356	141	488	85	302	549	434	26	273	620	212	122	344	561	158	380
454	71	293	515	232	17	364	581	178	425	310	527	149	491	88	623	220	437	34	251	161	383	105	347	569
518	240	457	54	296	181	403	25	367	589	499	91	313	535	127	37	259	601	223	445	330	572	169	386	108
57	279	521	243	465	375	592	189	406	3	538	135	477	99	316	201	448	45	262	609	394	111	333	555	172
246	468	65	282	504	414	6	353	600	192	77	324	541	138	485	270	612	209	426	48	558	155	397	119	336
441	38	260	602	224	109	326	573	170	387	297	519	236	458	55	590	182	404	21	368	128	500	92	314	531
610	202	449	41	263	173	395	112	334	551	461	58	280	522	244	4	371	593	190	407	317	539	131	478	100
49	266	613	210	427	337	559	151	398	120	505	247	469	61	283	193	415	7	354	596	481	78	325	542	139
213	435	27	274	616	376	123	345	562	159	69	286	508	230	472	357	579	196	418	15	550	142	489	81	303
252	624	216	438	35	570	162	384	101	348	233	455	72	294	511	421	18	365	582	179	89	306	528	150	492
597	194	411	8	355	140	482	79	321	543	428	50	267	614	206	116	338	560	152	399	284	501	248	470	62
11	358	580	197	419	304	546	143	490	82	617	214	431	28	275	160	377	124	341	563	473	70	287	509	226
180	422	19	361	583	493	90	307	529	146	31	253	625	217	439	349	566	163	385	102	512	234	451	73	295
369	586	183	405	22	532	129	496	93	315	225	442	39	256	603	388	110	327	574	166	51	298	520	237	459
408	5	372	594	186	96	318	540	132	479	264	606	203	450	42	552	174	391	113	335	245	462	59	276	523

More research into different properties and various examples can be found in the thesis of the second author [**6**]. The reader is also referred to the website [**7**].

ACKNOWLEDGMENT The authors would like to thank Michiel de Bondt for various stimulating and helpful discussions. The first author was partially supported by NSF grant DMS0349019.

REFERENCES

1. G. Abe, Unsolved problems on magic squares, *Discrete Math.* **127** (1994) 3–13.
2. M. Ahmed, How many squares are there Mr. Franklin: Constructing and enumerating Franklin squares, this MONTHLY **111** (2004) 394–410.
3. W. W. Rouse Ball and H. S. M. Coxeter, *Mathematical Recreations and Essays*, 12th ed., University of Toronto Press, Toronto, 1974.
4. C. Boyer, Les premiers carrés tétra et pentamagiques, *Pour la science* (August 2001) 98–102.
5. `http://www.multimagie.com`.
6. C. Eggermont, Master's thesis, Radboud University, Nijmegen (to appear).
7. ———, `http://www.puzzled.nl`.
8. P. Fengchu, announcements at `http://www.zhghf.net/China/` and `http://www.zhghf.net/` (Chinese).
9. M. Gardner, Magic squares and cubes, in *Time Travel and Other Mathematical Bewilderments*, W. H. Freeman, New York, 1988, pp. 213–225.
10. K. M. Ollerenshaw and D. S. Brée, *Most-Perfect Pandiagonal Magic Squares: Their Construction and Enumeration*, Institute for Mathematics and Applications, Southend-on-Sea, United Kingdom, 1998.
11. G. Pfeffermann, Les Tablettes du Chercheur, *Journal des jeux d'esprit et de combinaisons* (fortnightly magazine, various issues from 1891), Paris.
12. C. A. Pickover, *The Zen of Magic Squares, Circles, and Stars*, Princeton University Press, Princeton, NJ, 2002.
13. G. Stertenbrink, announcement (5 August 2003) at `http://magictour.free.fr`.
14. I. Stewart, Most-perfect magic squares, *Scientific American* **281** (1999) 122.
15. W. Trump, Story of the smallest trimagic square (January 2003), available at `http://www.multimagie.com/English/Tri12story.htm`.
16. E. W. Weisstein, Magic cube, available at `http://mathworld.wolfram.com/MagicCube.html`.

HARM DERKSEN studied at the University of Nijmegen and received his Ph.D in 1997 from the University of Basel under the direction of Hanspeter Kraft. After a year at Northeastern University and two years at MIT, he moved to the University of Michigan, where he is now an associate professor. His research interests include commutative algebra, invariant theory, and representation theory.
Dept. of Mathematics, Univ. of Michigan, East Hall, 530 Church Street, Ann Arbor, MI 48109-1043
hderksen@umich.edu

CHRISTIAN EGGERMONT is a metagrobologist, puzzle designer, and part-time webdeveloper. He studied mathematics at the University of Nijmegen. Besides mathematics and mechanical puzzles he enjoys origami, magic, and playing the piano.
Dept. of Mathematics, Univ. of Nijmegen, 6525 ED Nijmegen, The Netherlands
C.Eggermont@science.ru.nl

ARNO VAN DEN ESSEN received his Ph.D in 1979 from the University of Nijmegen. After one year at the University of Stockholm and one year at the University of Utrecht he held visiting positions at the University of Washington Seattle, the University of Reims, the Universidad Autònoma de Barcelona, and the University of Manitoba. He is currently professor at the university of Nijmegen. His main research interest is polynomial automorphisms (as described in his book *Polynomial Automorphisms and the Jacobian Conjecture* Birkhäuser, 2000). He is currently finishing a popular book on Magic Squares and propagandizing the relation between food and rheumatism.
Dept. of Mathematics, Univ. of Nijmegen, 6525 ED Nijmegen, The Netherlands
A.vandenEssen@math.ru.nl