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## Newton polytopes of invariants of additive group actions

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### Abstract

It is shown that the vertices of Newton polytopes of invariants of an algebraic group action of the additive group of a field  $k$  of arbitrary characteristic on affine  $n$ -space over  $k$  lie on the coordinate hyperplanes. Furthermore, let  $E$  be the set of all edges of these Newton polytopes whose vertices lie in different coordinate hyperplanes. It is shown that one of these polytopes has edges with all directions represented in  $E$ . © 2001 Elsevier Science B.V. All rights reserved.

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### 0. Introduction

The ring of polynomials  $B = k[x_1, \dots, x_m]$  over an arbitrary field  $k$  is the coordinate ring of affine  $m$ -space. Any polynomial in this ring can be written as a finite sum  $\alpha = \sum_{u \in \mathbf{Z}^m} \alpha_u x^u$ . The set of  $u$ 's such that  $\alpha_u \neq 0$  is called the *support* of  $\alpha$ , and its convex hull in  $\mathbf{R}^m$  is called the *Newton polytope* of  $\alpha$ . If  $\alpha$  does not involve one of the  $x_i$ 's, then the Newton polytope collapses into one of the coordinate hyperplanes.

Any algebraic group action of the additive group of  $k$  on the affine  $m$ -space induces an action on  $B$  by automorphisms. In Section 3 we show that the Newton polytopes of the invariants of these actions have all their vertices on the coordinate hyperplanes (though generally not all vertices in the same hyperplane).

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One consequence is that if  $y_1, \dots, y_m$  be another set of generators of  $B$ , that is  $k[x_1, \dots, x_m] = k[y_1, \dots, y_m]$ , then the Newton polytope of a polynomial in  $k[y_1, \dots, y_{m-1}]$  has all its vertices in the coordinate hyperplanes. This generalizes the main result of Hadas [1] to arbitrary base field (our Corollary 3.3). In characteristic zero, the constants of a locally nilpotent derivation are the invariants of a corresponding additive group action, so the vertices of a constant of a locally nilpotent derivation are all on the coordinate hyperplanes (our Corollary 3.4).

Let  $A$  be the ring of invariants of an additive group action on affine  $m$ -space. Let  $E$  be the set of all directions of edges that are not contained in coordinate hyperplanes of the Newton polytopes of all the elements of  $A$ . In Section 4 we show that  $E$  is finite. Moreover, there is a nontrivial ideal  $I$  in  $A$ , such that the Newton polytope of any nonzero element of  $I$  has edges in all directions of  $E$ . Some information is obtained about the faces of these Newton polytopes.

## 1. $\mathbf{G}_a$ -actions on affine space and their invariants

The purpose of this section is to recall basic definitions, and establish notations. Let  $k$  be any field (of arbitrary characteristic). The coordinate ring of affine  $m$ -space is the algebra of polynomials  $B = k[x_1, \dots, x_m]$ . A  $\mathbf{G}_a$ -action is an algebraic group action of the additive group  $\mathbf{G}_a$  of the field  $k$  on an algebraic variety, in our case affine  $m$ -space. Going over to the level of coordinate rings, a  $\mathbf{G}_a$ -action is a homomorphism of  $k$ -algebras  $\sigma: B \rightarrow B[t]$  satisfying certain axioms arising from the axioms of a group action on a set. To write down these axioms, we write the action of  $\sigma$  in exponential form, so  $\beta^\sigma$  is the image of  $\beta \in B$  in  $B[t]$ . We also use the same letter  $\sigma$  to denote any extension of  $\sigma$  arising from extension of the base ring  $k$ . In particular, we need the extension  $\sigma: B[s] \rightarrow B[s, t]$  fixing  $s$ . The two axioms of group actions on sets translate into

- (a)  $\beta^\sigma(0) = \beta$ ,
- (b)  $(\beta^\sigma(s))^\sigma(t) = \beta^\sigma(s + t)$

for all  $\beta \in B$ , where the second identity is in the ring  $B[s, t]$ . Note that since  $\sigma$  and substitutions are algebra homomorphisms, it is sufficient to verify (a) and (b) on a set of generators. We can write

$$\beta^\sigma(t) = \sum_{n=0}^{\infty} t^n \delta^{[n]}(\beta),$$

where  $\delta^{[n]}: B \rightarrow B$  are linear operators. Then the two action axioms translate into

- (a)  $\delta^{[0]} = 1$ ,
- (b)  $\delta^{[i]} \delta^{[j]} = \binom{i+j}{i} \delta^{[i+j]}$ .

The fact that  $\beta^\sigma(t)$  is a polynomial in  $t$  translates into

$$\forall \beta \in B, \quad \exists N = N(\beta), \quad \forall n > N, \quad \partial^{[n]}(\beta) = 0$$

and the fact that  $\sigma$  is a homomorphism of rings translates into the “Leibniz rule”

$$\partial^{[n]}(\alpha\beta) = \sum_{i+j=n} \partial^{[i]}(\alpha)\partial^{[j]}(\beta).$$

In characteristic zero these facts imply that  $\partial^{[1]}$  is a locally nilpotent derivation and all the  $\partial^{[n]}$ 's are uniquely determined by  $\partial^{[1]}$ .

An element  $\beta \in B$  is called an *invariant* of the action  $\sigma$  if  $\beta^\sigma = \beta$ . If  $\beta^\sigma \in B$  then by axiom (a) this  $\beta$  is an invariant. The invariants form a subalgebra  $A = B^\sigma \cap B$  of  $B$ . The subalgebra of invariants is *factorially closed* in the sense that if  $\beta \in B$  is a factor of  $\alpha \in A$  then  $\beta \in A$ . This is so because  $A$  is the preimage of  $B$  under  $\sigma$ , and  $B$  is factorially closed in  $B[t]$ . Note that the trivial action is given by the standard embedding of  $B$  in  $B[t]$ , and this is the only case when  $B$  is the ring of invariants of a  $\mathbf{G}_a$ -action. In all other cases the transcendence degree of the ring of invariants is one less than the transcendence degree of  $B$ . An algebraic proof of this fact is provided in Section 2. Geometrically, this is so because a generic orbit of the action is one dimensional.

## 2. Homogenization of an action

Let  $B$  be an  $R$ -graded algebra, where  $R$  is an additive monoid. This means that  $B = \bigoplus_{\varepsilon \in R} B_\varepsilon$  and  $B_\varepsilon B_{\varepsilon'} \subseteq B_{\varepsilon+\varepsilon'}$ . Every  $\beta \in B$  decomposes into a sum of homogeneous components  $\beta = \sum_{\varepsilon \in R} \beta_\varepsilon$  with  $\beta_\varepsilon \in B_\varepsilon$ . If  $R$  is a linearly ordered monoid then every nonzero  $\beta \in B$  has a nonzero homogeneous component of maximal degree, called the *leading component* of  $\beta$ . We denote it by  $\text{gr}(\beta)$  and its degree by  $D(\beta)$  (so  $\text{gr}(\beta) = \beta_{D(\beta)}$ ). We also write  $D(0) = -\infty$  and  $\text{gr}(0) = 0$ . If  $B$  is a domain, then  $D$  is a *degree function*, in the sense that  $D(\alpha\beta) = D(\alpha) + D(\beta)$  and  $D(\alpha + \beta) \leq \max\{D(\alpha), D(\beta)\}$  for every  $\alpha, \beta \in B$ . If  $A$  is a subring of  $B$ , we denote by  $\text{gr}(A)$  the subring generated by the leading components of the elements of  $A$ .

We assume now that  $R$  is a group (actually we only use subgroups of vector spaces over the real numbers as  $R$ ). Any mapping  $\sigma : B \rightarrow B'$  between  $R$ -graded algebras can be formally decomposed into a sum of homogeneous mappings  $\sigma = \sum_{t \in R} \sigma_t$ , where  $\sigma_t$  maps every  $B_\varepsilon$  into  $B'_{\varepsilon+t}$ . This sum is generally not finite, and need not have a maximal nonzero component (as is the case in the proof of Proposition 2.3 below). If it has one, we denote it by  $\text{gr}(\sigma)$ . Let  $\Delta(\beta) = D(\beta^\sigma) - D(\beta)$  for every non-zero  $\beta \in B$  (for  $\beta = 0$  it is undefined). Let  $D(\sigma) = \sup\{\Delta(\beta) \mid \beta \in B\}$ . Then it is clear that if  $\text{gr}(\sigma)$  exists, it is  $\sigma_{D(\sigma)}$ . A straightforward computation shows that if  $\sigma$  is an algebra homomorphism between unital domains, and if  $\Delta(\beta) \leq 0$  for all  $\beta$  in a set of homogeneous generators of  $B$ , then  $D(\sigma) = 0$ .

Now let us concentrate on the case of a polynomial algebra  $B = k[x_1, \dots, x_m]$ . Any monoid homomorphism  $\lambda : \mathbf{Z}^m \rightarrow R$  induces an  $R$ -grading on  $B$ , defined by

$B_\varepsilon = \text{Span}_k \{x^v \mid \lambda(v) = \varepsilon\}$  for all  $\varepsilon \in R$ , and thus a degree function  $D$ , and all other related notions. One important observation, used in Section 4, is that if  $\lambda$  is injective, then the leading forms are monomials, and we might as well replace  $R$  by  $\mathbf{Z}^m$  with the appropriate linear order. Our main goal in this section, is to show that given a nontrivial  $\mathbf{G}_a$ -action  $\sigma: B \rightarrow B[t]$  on affine  $m$ -space, a homomorphism  $\lambda: \mathbf{Z}^m \rightarrow R$  can be extended to define a grading of  $B[t]$  so that  $\text{gr}(\sigma)$  is a nontrivial  $\mathbf{G}_a$ -action. Clearly, an extension of  $\lambda$  to  $B[t]$  is determined uniquely by assigning a value to  $D(t)$ , and any choice of  $D(t) \in R$  defines an  $R$ -grading on  $B[t]$ . Once  $D(t)$  is fixed, we have  $D(\sum_n \beta_n t^n) = \max\{D(\beta_n) + nD(t) \mid n = 0, 1, 2, \dots\}$ .

**Lemma 2.1.**  *$\lambda$  can be extended so that*

- (a)  $D(\sigma) = 0$ .
- (b)  $\exists \beta \in B, D(\beta^\sigma - \beta) = D(\beta)$ .

**Proof.** Take  $X = \{x_1, \dots, x_m\}$ . This is a set of homogeneous generators of  $B$ . We assign

$$D(t) = \min \left\{ \frac{D(\beta) - D(\partial^{[n]}\beta)}{n} \mid \beta \in X, n = 1, 2, 3, \dots \right\}.$$

Then we have

$$\begin{aligned} \Delta(\beta) &= D(\beta^\sigma) - D(\beta) \\ &= D\left(\sum_{n=0}^{\infty} t^n \partial^{[n]}\beta\right) - D(\beta) \\ &= \max\{D(\partial^{[n]}(\beta)) - D(\beta) + nD(t) \mid n = 0, 1, 2, 3, \dots\} \leq 0 \end{aligned}$$

for all  $\beta \in X$  by our choice of  $D(t)$ , and therefore  $D(\sigma) = 0$ . Moreover, for at least one  $\beta \in X$  and one natural number  $n$ , we have  $D(t) = (D(\beta) - D(\partial^{[n]}\beta))/n$  so for this  $\beta$  we have

$$\begin{aligned} D(\beta^\sigma - \beta) &= D\left(\sum_{n=1}^{\infty} t^n \partial^{[n]}\beta\right) \\ &= \max\{D(\partial^{[n]}(\beta)) + nD(t) \mid n = 1, 2, 3, \dots\} = D(\beta). \quad \square \end{aligned}$$

**Proposition 2.2.** *If  $\sigma$  is a nontrivial  $\mathbf{G}_a$ -action on affine  $m$ -space, and  $\lambda: \mathbf{Z}^m \rightarrow R$  defines an  $R$ -grading of  $B = k[x_1, \dots, x_m]$ , then  $\lambda$  can be extended to define a grading of  $B[t]$  so that  $\text{gr}(\sigma)$  is a nontrivial  $\mathbf{G}_a$ -action on affine  $m$ -space. Moreover, if  $A$  is the ring of invariants of  $\sigma$ , then  $\text{gr}(A)$  is contained in the ring of invariants of  $\text{gr}(\sigma)$ .*

**Proof.** We extend  $\lambda$  as in the previous lemma. We have  $\Delta(\beta) = 0$  for all nonzero  $\beta \in B$ , so  $\text{gr}(\beta)^{\text{gr}(\sigma)} = \text{gr}(\beta^\sigma)$ , since

$$\text{gr}(\beta^\sigma) = \sum_{n \in S(\beta)} t^n \text{gr}(\partial^{[n]}\beta),$$

where

$$S(\beta) = \{n \mid nD(t) + D(\partial^{[n]}\beta) = D(\beta)\}.$$

The rest of the proof is straightforward verification. First we show that  $\text{gr}(\sigma)$  is a homomorphism of algebras. This follows from

$$\begin{aligned} (\text{gr}(\alpha)\text{gr}(\beta))^{\text{gr}(\sigma)} &= \text{gr}(\alpha\beta)^{\text{gr}(\sigma)} = \text{gr}(\alpha\beta)^\sigma \\ &= \text{gr}(\alpha^\sigma\beta^\sigma) = \text{gr}(\alpha^\sigma)\text{gr}(\beta^\sigma) = \text{gr}(\alpha)^{\text{gr}(\sigma)}\text{gr}(\beta)^{\text{gr}(\sigma)}. \end{aligned}$$

Then we have to show that the two action axioms hold. Since  $0 \in S(\beta)$  for any nonzero  $\beta \in B$ , we have

$$\text{gr}(\beta)^{\text{gr}(\sigma)}(0) = \text{gr}(\beta^\sigma)(0) = \text{gr}(\partial^{[0]}\beta) = \text{gr}(\beta),$$

so the first axiom holds. To show that the second axiom holds we need to show that

$$(\text{gr}(\beta)^{\text{gr}(\sigma)}(s))^{\text{gr}(\sigma)}(t) = \text{gr}(\beta)^{\text{gr}(\sigma)}(s + t).$$

Here we should think of  $D$  as being further extended to  $B[s, t]$  so that  $D(s) = D(t)$ . Then

$$\text{gr}(\beta)^{\text{gr}(\sigma)}(s) = \text{gr}(\beta^\sigma)(s) = \sum_{i \in S(\beta)} s^i \text{gr}(\partial^{[i]}\beta),$$

so,

$$\begin{aligned} (\text{gr}(\beta)^{\text{gr}(\sigma)}(s))^{\text{gr}(\sigma)}(t) &= \sum_{i \in S(\beta)} s^i \text{gr}(\partial^{[i]}\beta)^{\text{gr}(\sigma)}(t) \\ &= \sum_{i \in S(\beta)} s^i \text{gr}(\partial^{[i]}\beta)^\sigma(t) \\ &= \sum_{i \in S(\beta)} s^i \sum_{j \in S(\partial^{[i]}\beta)} t^j \text{gr}(\partial^{[j]}\partial^{[i]}\beta) \\ &= \sum_{\substack{i \in S(\beta) \\ j \in S(\partial^{[i]}\beta)}} s^i t^j \text{gr}(\partial^{[j]}\partial^{[i]}\beta). \end{aligned}$$

On the other hand,

$$\begin{aligned} \text{gr}(\beta)^{\text{gr}(\sigma)}(s + t) &= \text{gr}(\beta^\sigma)(s + t) \\ &= \sum_{n \in S(\beta)} (s + t)^n \text{gr}(\partial^{[n]}\beta) \\ &= \sum_{i+j \in S(\beta)} s^i t^j \binom{i+j}{i} \text{gr}(\partial^{[i+j]}\beta) \\ &= \sum_{i+j \in S(\beta)} s^i t^j \text{gr}(\partial^{[j]}\partial^{[i]}\beta). \end{aligned}$$

Thus, to show that the second axiom holds we need to show that  $i \in S(\beta)$  and  $j \in S(\partial^{[i]}\beta)$  if and only if  $i + j \in S(\beta)$ . Now if  $i \in S(\beta)$  and  $j \in S(\partial^{[i]}\beta)$  then

$$iD(t) + D(\partial^{[i]}\beta) = D(\beta)$$

and

$$jD(t) + D(\partial^{[j]}\partial^{[i]}\beta) = D(\partial^{[i]}\beta)$$

so,

$$(i+j)D(t) + D(\partial^{[j]}\partial^{[i]}\beta) = D(\beta).$$

In particular,  $\binom{i+j}{i}\partial^{[i+j]}\beta = \partial^{[j]}\partial^{[i]}\beta \neq 0$  so

$$(i+j)D(t) + D(\partial^{[i+j]}\beta) = D(\beta)$$

and therefore  $i+j \in S(\beta)$ . On the other hand, if  $i \notin S(\beta)$  then

$$iD(t) + D(\partial^{[i]}\beta) < D(\beta)$$

and if  $j \notin S(\partial^{[i]}\beta)$  then

$$jD(t) + D(\partial^{[j]}\partial^{[i]}\beta) < D(\partial^{[i]}\beta)$$

and in either case we have

$$(i+j)D(t) + D(\partial^{[i+j]}\beta) < D(\beta)$$

so  $i+j \notin S(\beta)$ . This concludes the proof that  $\text{gr}(\sigma)$  is an action.

To see that  $\text{gr}(\sigma)$  is a nontrivial action, we apply part (b) of the lemma. It implies that for some  $\beta \in B$  we have  $S(\beta) \neq \{0\}$ , and for this  $\beta$  we have  $\text{gr}(\beta)^{\text{gr}(\sigma)} = \text{gr}(\beta^\sigma) \neq \text{gr}(\beta)$ , so this  $\text{gr}(\beta)$  is not an invariant of  $\text{gr}(\sigma)$ .

Finally, if  $\beta \in A$ , that is,  $\beta$  is an invariant of  $\sigma$ , then  $\text{gr}(\beta)^{\text{gr}(\sigma)} = \text{gr}(\beta^\sigma) = \text{gr}(\beta)$ , so  $\text{gr}(\beta)$  is an invariant of  $\text{gr}(\sigma)$ . This shows that the ring of invariants of  $\text{gr}(\sigma)$  contains a set of generators of  $\text{gr}(A)$ , so it contains  $\text{gr}(A)$ .  $\square$

Before proceeding to the main result, let us prove the following fact. It is used in Section 4.

**Proposition 2.3.** *The transcendence degree of the ring of invariants of a nontrivial  $\mathbf{G}_a$ -action on affine  $m$ -space is  $m - 1$ .*

**Proof.** We use the trivial grading on  $B$  (the one induced by  $\lambda = 0$ ), and extend it to  $B[t]$  by having  $D(t) = 1$ . This means that  $D$  is the ordinary degree of polynomials in  $t$ . Since the transcendence degree of  $B$  is  $m$ , and  $\sigma$  is injective, the transcendence degree of  $B^\sigma$ , the image of  $\sigma$ , is  $m$ . Therefore, the transcendence degree of  $\text{gr}(B^\sigma)$  is  $m$  (this follows from an extension<sup>3</sup> of Proposition 3.2 in [1]). If we show that  $\text{gr}(B^\sigma) \subseteq A[t]$ , where  $A$  is the ring of invariants of  $\sigma$ , it follows that the transcendence degree of  $A$  is at least  $m - 1$ . It cannot be  $m$ , since  $A$  is factorially closed, hence algebraically closed in  $B$ , and the action is nontrivial, so  $A \neq B$ .

<sup>3</sup> There it was proved with an injective  $\lambda: \mathbf{Z}^m \rightarrow \mathbf{R}$ , but the same proof works with any injective  $\lambda$ , into any linearly ordered monoid, and then it is an easy consequence that the same holds when  $\lambda$  is not injective. A slight modification of the proof given there shows that the same proposition holds for subalgebras of any field of rational functions.

To show that  $\text{gr}(B^\sigma) \subseteq A[t]$ , let  $\beta \in B$ , and consider  $\beta^\sigma = \sum_{n=0}^N t^n \partial^{[n]} \beta$ , where  $N = D(\beta^\sigma)$ . Then  $\text{gr}(\beta^\sigma) = t^N \partial^{[N]} \beta$ . Now  $\partial^{[n]}(\partial^{[N]} \beta) = \binom{n+N}{n} \partial^{[n+N]} \beta = 0$  for all positive  $n$ , so  $(\partial^{[N]} \beta)^\sigma = \sum_n t^n \partial^{[n]}(\partial^{[N]} \beta) = \partial^{[N]} \beta$ , so  $\partial^{[N]} \beta \in A$ , and thus  $\text{gr}(\beta^\sigma) = t^N \partial^{[N]} \beta \in A[t]$ .  $\square$

### 3. Vertices of Newton polytopes of invariants

Let  $\alpha \in B = k[x_1, \dots, x_m]$ . The *Newton polytope*  $N(\alpha)$  is defined as the convex hull of the support of  $\alpha$  (see the introduction). We call a vertex of  $N(\alpha)$  an *intruder* if it is not contained in any of the coordinate hyperplanes.

**Theorem 3.1.** *Invariants of  $\mathbf{G}_a$ -actions on affine space have no intruders.*

**Proof.** Let  $\alpha$  be an invariant of a  $\mathbf{G}_a$ -action on affine  $m$ -space. If  $u$  is a vertex of the Newton polytope of  $\alpha$ , then there is a linear functional  $\lambda: \mathbf{R}^m \rightarrow \mathbf{R}$  such that  $\lambda(u) > \lambda(v)$  for any  $v \in N(\alpha) \setminus \{u\}$ . This  $\lambda$  induces a grading of  $B$  with  $D(\alpha) = \lambda(u)$ , and we may assume that  $\text{gr}(\alpha) = x^u$ .

By Proposition 2.2,  $\text{gr}(\alpha) = x^u$  is an invariant of a nontrivial  $\mathbf{G}_a$ -action on affine  $m$ -space, and since the ring of invariants of this action is factorially closed, and does not contain all the  $x_i$ 's, not all the  $x_i$ 's are factors of  $x^u$ . Thus  $u$  must be on one of the coordinate hyperplanes.  $\square$

**Corollary 3.2.** *If  $k[x_1, \dots, x_m] = k[y_1, \dots, y_m]$ , then any polynomial not involving  $y_m$ , that is  $\alpha(y_1, \dots, y_{m-1}) \in k[y_1, \dots, y_{m-1}]$ , has no intruders.*

**Proof.**  $A = k[y_1, \dots, y_{m-1}]$  is the ring of invariants of the  $\mathbf{G}_a$ -action defined by  $y_i^\sigma = y_i$  for  $i < m$  and  $y_m^\sigma = y_m + t$ .  $\square$

Now we obtain the main theorem of Hadas [1] without the restriction on the characteristic that was imposed there.

**Corollary 3.3.** *For any algebra automorphism  $\varphi$  of  $k[x_1, \dots, x_m]$ , all the vertices of the Newton polytope of  $\varphi x_1$  lie on the coordinate hyperplanes.*

**Proof.** Take  $y_i = \varphi x_i$  and  $\alpha = y_1$  in Corollary 3.2.  $\square$

Finally, we apply our theorem to constants of locally nilpotent derivations.

**Corollary 3.4.** *If the characteristic of  $k$  is zero, and if  $\partial$  is a locally nilpotent derivation on  $B = k[x_1, \dots, x_m]$ , then the constants of  $\partial$  have no intruders.*

**Proof.** As was mentioned in Section 1,  $\partial$  gives rise to a  $\mathbf{G}_a$  action defined by  $\partial^{[n]} = (1/n!) \partial^n$ , and the constants of  $\partial$ , that is the elements annihilated by  $\partial$ , are exactly the invariants of this action.  $\square$

#### 4. Edges and faces of Newton polytopes

Theorem 3.1 provides information on the *vertices* of the Newton polytopes of elements of  $A$ , where  $A$  is the invariant ring of a  $\mathbf{G}_a$ -action on affine  $m$ -space. In this section we show how to extract from Theorem 3.1 information on the edges and on the higher dimensional faces of these polytopes. For this purpose we will need to translate Theorem 3.1 into a more technical result involving gradings of  $B = k[x_1, \dots, x_m]$ .

We consider gradings arising from injective homomorphisms  $\lambda: \mathbf{Z}^m \rightarrow R$  into a linearly ordered monoid  $R$ . In this case we may replace  $R$  by  $\mathbf{Z}^m$  with the linear order induced by  $\lambda$ , so we may assume that  $D(\beta) \in \mathbf{Z}^m$  for every nonzero  $\beta \in B$ . When we consider more than one  $\lambda$ , we will index the resulting degree functions as  $D = D_\lambda$ . We denote the subspace of  $\mathbf{R}^m$  spanned by  $\{D(\alpha) \mid \alpha \in A\}$  by  $SD(A)$ . As its generators are in  $\mathbf{Z}^m$ , there is no harm in considering  $SD(A)$  as a subspace of  $\mathbf{Q}^m$ .

**Proposition 4.1.** *Let  $A$  be a subalgebra of  $B = k[x_1, \dots, x_m]$ . Then the following two statements are equivalent.*

- (a)  $A$  has no intruders.
- (b) For any injective  $\lambda$ ,  $SD_\lambda(A)$  is contained in a coordinate hyperplane.

**Proof.** If (b) does not hold, then for some  $\lambda$ , and some  $\alpha \in A$ ,  $D_\lambda(\alpha)$  is an intruder, so (a) does not hold. If (a) does not hold, then as in the proof of Theorem 3.1, an intruder  $u$  can be realized as  $D_\lambda(\alpha)$  for some injective  $\lambda$ , so  $SD_\lambda(A)$  is not contained in a coordinate hyperplane.  $\square$

In the rest of this section,  $A$  denotes a subalgebra of  $B = k[x_1, \dots, x_m]$  satisfying the equivalent conditions of Proposition 4.1. An edge of a Newton polytope is called a *trespasser* if it is not contained in any coordinate hyperplane.

Let  $\alpha \in A$ , let  $e = e_\alpha$  be an edge of the Newton polytope  $N(\alpha)$  that is a trespasser, and let  $u = u_\alpha$  and  $v = v_\alpha$  be the two vertices of  $e$ . Choose a linear mapping  $\lambda_1: \mathbf{Q}^m \rightarrow \mathbf{R}$  such that  $\lambda_1(w) < \lambda_1(u) = \lambda_1(v)$  for any  $w \in N(\alpha) \setminus e$ . There are many possibilities of choosing such  $\lambda_1$ , and of course none of them produces an injective  $\lambda_1$ . It is possible, however, to choose  $\lambda_1$  so that its kernel is one dimensional. Fix one such  $\lambda_1$ . Then for any  $w_0 \in \mathbf{Q}^m$  the set  $\{w \in \mathbf{Q}^m \mid \lambda_1(w) = \lambda_1(w_0)\}$  is a line parallel to  $e$ . In other words, the fibers of  $\lambda_1$  are the lines parallel to  $e$ .

This  $\lambda_1$  does not define a linear order on  $\mathbf{Z}^m$ , but it does define a partial order ( $w > 0$  if and only if  $\lambda_1(w) > 0$ ), and there are two possible ways to extend this partial order to a linear order, either with  $u > v$  or with  $u < v$ . To do that, let  $\lambda_2: \mathbf{Q}^m \rightarrow \mathbf{R}$  be a linear mapping with  $\lambda_2(u) \neq \lambda_2(v)$ , and define an order on  $\mathbf{Z}^m$  (and also on  $\mathbf{Q}^m$ ) by  $w > 0$  if and only if either  $\lambda_1(w) > 0$ , or  $\lambda_1(w) = 0$  and  $\lambda_2(w) > 0$ . This is equivalent to defining an injection  $\lambda: \mathbf{Q}^m \rightarrow \mathbf{R}^2$  by  $\lambda = (\lambda_1, \lambda_2)$ , and ordering  $\mathbf{R}^2$  lexicographically. If we choose  $\lambda_2^u$  such that  $\lambda_2^u(u) > \lambda_2^u(v)$ , then  $u > v$  and we have  $D_{\lambda^u}(\alpha) = u$  (for  $\lambda^u = (\lambda_1, \lambda_2^u)$ ). Similarly, we can choose  $\lambda_2^v$  so that we have  $\lambda_2^v(u) < \lambda_2^v(v)$  and therefore  $D_{\lambda^v}(\alpha) = v$ .



Now let  $\beta \in A$ , and let  $u_\beta = D_{\lambda^u}(\beta)$  and  $v_\beta = D_{\lambda^v}(\beta)$  (recall that  $u = u_\alpha = D_{\lambda^u}(\alpha)$  and  $v = v_\alpha = D_{\lambda^v}(\alpha)$ ). It is easy to see that  $\lambda_1(u_\beta) = \lambda_1(v_\beta) = \max \lambda_1(N(\beta))$ , so  $u_\beta$  and  $v_\beta$  lie on the same fiber of  $\lambda_1$ , and by our choice of  $\lambda_1$ , this is a line parallel to  $e = e_\alpha$ . Thus we have two possibilities. Either  $u_\beta = v_\beta$ , or else the Newton polytope  $N(\beta)$  has an edge  $e_\beta = u_\beta v_\beta$  that is parallel to  $e_\alpha$ .

An element  $\beta \in A$  is called a *g-element* of  $A$  if each vertex of the Newton polytope  $N(\beta)$  lies on no more than one coordinate hyperplane.

**Theorem 4.2.** *Let  $A$  be a subalgebra of  $B = k[x_1, \dots, x_m]$  satisfying the equivalent conditions of Proposition 4.1. Let  $\beta$  be a g-element of  $A$ . Then any trespasser of  $A$  is parallel to an edge of  $N(\beta)$ .*

**Proof.** Let  $\alpha \in A$  and let  $e = e_\alpha$  be a trespasser of  $N(\alpha)$ . By the above discussion we get vertices  $u_\beta$  and  $v_\beta$  of  $N(\beta)$ .

By the assumption on  $A$ , the subspaces  $SD_{\lambda^u}(A)$  and  $SD_{\lambda^v}(A)$  are contained in two coordinate hyperplanes,  $H^u$  and  $H^v$ , respectively. We have  $H^u \neq H^v$ , since  $e = uv$  is not contained in a coordinate hyperplane. We have  $u_\beta \in H^u$  and  $v_\beta \in H^v$ , and since  $H^u \neq H^v$ , the definition of a *g-element* rules out the possibility that  $u_\beta = v_\beta$ . Therefore  $e_\beta = u_\beta v_\beta$  is an edge of  $N(\beta)$  that is parallel to  $e$ .  $\square$

**Example.** Let  $A = \mathbf{C}[x + y, z]$  in  $B = \mathbf{C}[x, y, z]$ . Since  $\mathbf{C}[x, y, z] = \mathbf{C}[x + y, y, z]$ , Corollary 3.2 shows that  $A$  satisfies the conditions of Proposition 4.1. The element  $\beta = (x + y)z$  is clearly a *g-element* of  $A$ . It follows that any edge of any Newton polyhedron  $N(\alpha)$  for  $\alpha \in A$  is either contained in a coordinate plane or parallel to the single edge of  $N(\beta)$ .

Actually, it is rather easy to describe the Newton polyhedra of the elements of this subalgebra. If  $\alpha = P(x + y, z) \in A$  then  $f_x = N(P(x, z))$  is a face of  $N(\alpha)$  contained in the  $xz$ -coordinate plane. Also  $f_y = N(P(y, z))$  is a face of  $N(\alpha)$  contained in the  $yz$ -coordinate plane, and it is congruent to  $f_x$ . These two faces are symmetric with respect to the plane  $x = y$ . Now  $N(\alpha) = \text{conv}(f_x \cup f_y)$  is formed from  $f_x$  and  $f_y$  by adjoining corresponding points of  $f_x$  and  $f_y$  with line segments that are perpendicular to the symmetry plane  $x = y$ , and thus are parallel to the single edge of  $N(\beta)$ .

What can we say about the faces of  $N(\alpha)$ ? The two faces  $f_x$  and  $f_y$  are contained in the coordinate planes, and one of them can be arbitrarily chosen (thereby fixing  $N(\alpha)$ ). The other faces are either isosceles triangles or trapezoids, with equal sides on the  $xz$ - and  $yz$ -coordinate planes, and bases parallel to the single edge of  $N(\beta)$ . Any such triangle or trapezoid (whose vertices have integral coordinates) is the face of some  $N(\alpha)$ .

This example shows that we cannot have a finite set  $P$  of planes, such that any two-dimensional face of a Newton polytope of an element of  $A$  is either parallel to an element of  $P$  or contained in a coordinate hyperplane. We can, however, apply our method to obtain some information on the higher-dimensional faces.

Let  $\beta \in A$  be a  $g$ -element, and let  $\alpha \in A$  be any nonzero element. Let  $f_\alpha$  be a face of  $N(\alpha)$  (having any dimension). Let  $e'_\alpha, e''_\alpha, \dots$  be those edges of  $f_\alpha$ , that are trespassers. By Theorem 4.2, there are edges  $e'_\beta, e''_\beta, \dots$  of  $N(\beta)$  that are parallel to  $e'_\alpha, e''_\alpha, \dots$ , respectively. We show now, that these can be taken from a single face of  $N(\beta)$  (whose dimension might be smaller than the dimension of  $f_\alpha$ ).

Indeed, fix some  $\lambda_1 : \mathbf{Q}^m \rightarrow \mathbf{R}$  whose fibers are parallel to  $f_\alpha$ , and have the same dimension as  $f_\alpha$ , and such that  $\lambda_1(f_\alpha) = \max \lambda_1(N(\alpha))$ . Next, for the edge  $e'_\alpha$ , choose  $\lambda'_2 : \mathbf{Q}^m \rightarrow \mathbf{R}$  such that the fibers of  $(\lambda_1, \lambda'_2)$  are lines parallel to  $e'_\alpha$ , and such that  $\lambda'_2(e'_\alpha) = \max \lambda'_2(f_\alpha)$ . Now let  $u'_\alpha$  and  $v'_\alpha$  be the two vertices of  $e'_\alpha$ , and choose  $\lambda''_3$  and  $\lambda'_3$  such that  $\lambda^{u'} = (\lambda_1, \lambda'_2, \lambda''_3)$  and  $\lambda^{v'} = (\lambda_1, \lambda'_2, \lambda'_3)$  define two injections of  $\mathbf{Q}^m$  into the lexicographically ordered  $\mathbf{R}^3$ , yielding  $D_{\lambda^{u'}}(\alpha) = u'_\alpha$  and  $D_{\lambda^{v'}}(\alpha) = v'_\alpha$ . Similarly define  $\lambda^{u''}$  and  $\lambda^{v''}$  for  $e''_\alpha$ , etc.

Just as in the proof of Theorem 4.2, we get  $u'_\beta = D_{\lambda^{u'}}(\beta)$  and  $v'_\beta = D_{\lambda^{v'}}(\beta)$ , two different vertices connected by an edge  $e'_\beta$  of  $N(\beta)$  that is parallel to  $e'_\alpha$ . In the same way we obtain  $e''_\beta$ , etc.

However, by our construction of  $\lambda^{u'}$ ,  $\lambda^{v'}$ ,  $\lambda^{u''}$ ,  $\lambda^{v''}$ , etc., it is clear that  $\lambda_1(e'_\beta) = \lambda_1(e''_\beta) = \dots = \max \lambda_1(N(\beta))$ , so all the edges  $e'_\beta, e''_\beta, \dots$  are contained in one fiber of  $\lambda_1$ , which is parallel to, and has the same dimension as  $f_\alpha$ . The intersection of this fiber with  $N(\beta)$  is a face  $f_\beta$  of  $N(\beta)$ , whose dimension does not exceed the dimension of  $f_\alpha$ . Thus we have proved the following theorem.

**Theorem 4.3.** *Let  $A$  and  $\beta$  be as in Theorem 4.2. Let  $\alpha \in A$  and let  $f_\alpha$  be a  $d$ -dimensional face of  $N(\alpha)$ . Let  $e'_\alpha, e''_\alpha, \dots$  be the edges of  $f_\alpha$  that are trespassers. Then  $N(\beta)$  has a face  $f_\beta$  whose dimension does not exceed  $d$ , that is parallel to  $f_\alpha$ , and  $f_\beta$  has edges  $e'_\beta, e''_\beta, \dots$  that are parallel to  $e'_\alpha, e''_\alpha, \dots$ , respectively.  $\square$*

**Remark.** (1) The dimension of the face  $f_\beta$  is at least the dimension of the subspace of  $\mathbf{Q}^m$  spanned by vectors parallel to  $e'_\alpha, e''_\alpha, \dots$ .

(2) The fact that in the above example ( $A = \mathbf{C}[x + y, z]$ ) the two-dimensional faces not contained in the coordinate planes are either isosceles triangles or trapezoids, as described above, is an immediate consequence of Theorem 4.3, since any such face, and its edges that are trespassers, must be parallel to the line segment  $N((x + y)z)$ .

Now that we have seen that the  $g$ -elements of  $A$  hold quite a lot of information, let us see that they really exist when  $A$  is the ring of invariants of a  $\mathbf{G}_a$ -action.

**Proposition 4.4.** *Let  $A$  be a subalgebra of  $B = k[x_1, \dots, x_m]$  whose transcendence degree is  $m - 1$ . Then the  $g$ -elements of  $A$  (together with 0) form a nontrivial ideal in  $A$ .*

**Proof.** Let  $H^u$  and  $H^v$  be two different coordinate hyperplanes. Let  $I^{uv}$  be the set of all elements of  $A$  that have no point of the intersection  $H^u \cap H^v$  in their supports. We will show that  $I^{uv}$  is a nontrivial ideal in  $A$ , and then the intersection of all such ideals is a nontrivial ideal, the nonzero elements of which are exactly the  $g$ -elements of  $A$ .

Let  $\lambda_1 : \mathbf{Q}^m \rightarrow \mathbf{R}$  be a linear mapping such that  $\lambda_1(H^u \cap H^v) = 0$  and  $\lambda_1(w) < 0$  for any  $w \in \mathbf{Q}^m \setminus (H^u \cap H^v)$  that has nonnegative coordinates. This  $\lambda_1$  defines a partial order on  $\mathbf{Q}^m$ , that can be extended to a linear order. Consider the corresponding degree function  $D$ . For any element  $\alpha \in A$  we have  $\lambda_1(D(\alpha)) \leq 0$ . For any  $w \in \text{supp}(\alpha)$  we have  $w \leq D(\alpha)$  so  $\lambda_1(w) \leq \lambda_1(D(\alpha))$ . It follows that  $\alpha \in I^{uv}$  if and only if  $\lambda_1(D(\alpha)) < 0$ . This implies that  $I^{uv}$  is an ideal in  $A$ .

Now  $\text{tr-deg}(A) = m - 1$  and therefore also  $\text{tr-deg}(\text{gr}(A)) = m - 1$  (by an extension of Proposition 3.2 in [1], see the proof of Proposition 2.3 and the footnote there). Thus  $SD(A) = \text{span}(D(A))$  is a hyperplane. Therefore, since  $H^u \cap H^v$  is  $m - 2$  dimensional, there exists  $\alpha \in A$  such that  $D(\alpha) \notin H^u \cap H^v$ . Then  $\lambda_1(D(\alpha)) < 0$  so  $\alpha \in I^{uv}$ , and therefore  $I^{uv} \neq 0$ .  $\square$

By Proposition 2.3, invariant rings of  $\mathbf{G}_a$ -actions on affine  $m$ -space have transcendence degree  $m - 1$ , so by Theorem 3.1 and Propositions 4.2 and 4.4 we have

**Corollary 4.5.** *If  $A$  is the ring of invariants of a nontrivial  $\mathbf{G}_a$ -action on affine  $m$ -space, then there is an element  $\beta \in A$ , such that every trespasser of  $A$  is parallel to an edge of  $N(\beta)$ .  $\square$*

## References

- [1] O. Hadas, On the vertices of Newton polytopes associated with an automorphism of the ring of polynomials, J. Pure Appl. Algebra 76 (1991) 81–86.