On Global Degree Bounds for Invariants

Harm Derksen* and Gregor Kemper

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Abstract

Let G be a linear algebraic group over a field K of characteristic 0. An integer m is called a global degree bound for G if for every linear representation V the invariant ring $K[V]^G$ is generated by invariants of degree at most m. We prove that if G has a global degree bound, then G must be finite. The converse is well known from Noether's degree bound.

Introduction

A classical topic in invariant theory is the question of degree bounds: Is it possible to generate an invariant ring $K[V]^G$ by homogeneous invariants of degree at most m, and can any a priori upper bound for such a number m be given? Perhaps the most prominent example of such a bound is Noether's degree bound [8], which states that for G finite and K of characteristic 0, every invariant ring is generated in degree at most $|G|$. Upper bounds for linearly reductive groups were given by Popov [9, 10] and then improved by Derksen [2]. It is remarkable that these bounds, in contrast to Noether's bound, do not only depend on G, but also involve properties of the representation V, such as its dimension. The same is true for an a priori bound given by Derksen and Kemper [3, Theorem 3.9.11] for finite groups (where the characteristic of K may divide $|G|$).

This observation leads to the following question. If G is infinite does there exist any upper bound at all which only depends on G and not on the representation? In this note we answer this question for the case that $\operatorname{char}(K) = 0$. The answer is as expected from observations: A global bound only exists if G is finite. This is stated in Theorem 2.1.

In the first section we establish the result for the case that G is linearly reductive. The second section deals with the general case of a linear algebraic group over an algebraically closed field of characteristic 0.

Let us fix some notation. Throughout the paper, G is a linear algebraic group over an algebraically closed field K. By a **G-module** we mean a finite-dimensional vector space V over K with a linear action of G given by a morphism $G \times V \to V$ of varieties. Recall that there always exists a faithful G-module (see Borel [1, Proposition I.1.10]). If V is a G-module, then G also acts on the polynomial ring $K[V]$ on V, and the invariant ring is denoted by $K[V]^G$. The ring $K[V]^G$ is a graded algebra.

If A is any graded algebra over $K = A_0$, we write

$$\beta(A) = \min\{d \in \mathbb{N} \mid A \text{ is generated by elements of degree } \leq d\},$$

where by convention the minimum over an empty set is ∞. Moreover, define

$$\beta(G) := \sup\left\{\beta\left(K[V]^G\right) \mid V \text{ } G\text{-module}\right\} \in \mathbb{N} \cup \{\infty\}.$$

We say that G has a **global degree bound** if $\beta(G) < \infty$, i.e., there exists an integer m such that $\beta\left(K[V]^G\right) \leq m$ for all G-modules V.

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1 Linearly reductive groups

If G is linearly reductive, then $K[V]$ has a unique isotypical decomposition, i.e.,

$$K[V] = \bigoplus_{\lambda \in \Lambda} K[V]_\lambda, \tag{1.1}$$

where Λ is the set of all isomorphism classes of irreducible G-modules and $K[V]_\lambda$ is a direct sum of irreducible modules which lie in the class λ (see Springer [14]).

Lemma 1.1. *Suppose that G is linearly reductive and V is a faithful G-module. Assume that only finitely many components appear in the isotypical decomposition (1.1) of $K[V]$, i.e.,*

$$|\{\lambda \in \Lambda \mid K[V]_\lambda \neq 0\}| < \infty.$$

Then $|G| < \infty$.

Proof. For every λ, $K[V]_\lambda$ is a finitely generated module over $K[V]^G$ (see Springer [14, III, Satz 4.2]). If there are only finitely many λ such that $K[V]_\lambda \neq 0$, then $K[V]$ is a finitely generated $K[V]^G$-module. Let $K(V)$ be the quotient field of $K[V]$. The field of invariant rational functions $K(V)^G$ contains the quotient field of $K[V]^G$. It follows that $K(V) : K(V)^G$ is an algebraic extension. Since G acts faithfully, it follows from Galois theory that G must be finite. □

Proposition 1.2. *Let G be linearly reductive and infinite. Then G has no global degree bound.*

Proof. Let U be a faithful G-module, and let k be an arbitrary non-negative integer. We write $K[U]_i$ for the homogeneous part of degree i of the polynomial ring. By Lemma 1.1 there exists an isomorphism class λ of irreducible G-modules such that $K[U]_\lambda \neq 0$ but $(K[U]_i)_\lambda = 0$ for all $i < k$. Let m be the least integer with $(K[U]_m)_\lambda \neq 0$. Choose a representative W from λ and set $V = W \oplus U$. Then $K[V] = K[W] \otimes_K K[U]$ has a G-invariant bigrading by putting $K[V]_{i,j} = K[W]_i \otimes K[U]_j$. For the part of bidegree $(1, j)$ we have

$$K[V]^G_{1,j} = (W^* \otimes K[U]_j)^G \cong \operatorname{Hom}_G(W, K[U]_j).$$

Hence $K[V]^G_{1,j} = 0$ for $j < m$, and there exists an $f \in K[V]^G_{1,m} \setminus \{0\}$. The total degree of f is $m+1$, and by using the bigrading we see that f cannot be written as a polynomial in invariants of smaller total degree. Hence

$$\beta\left(K[V]^G\right) \geq m + 1 > k.$$

Since k was chosen arbitrarily, there exists no global bound. □

2 The general case

Let G be a linear algebraic group. It is obvious but noteworthy that for a closed normal subgroup $N \trianglelefteq G$ we have

$$\beta(G/N) \leq \beta(G). \tag{2.1}$$

We will also use a result of Schmid [12, Proposition 5.1] which states that if $H \leq G$ is a subgroup of finite index, then

$$\beta(H) \leq \beta(G). \tag{2.2}$$

Schmid only states this result for finite groups, but the proof (which works by inducing representations from H to G) only uses that the index is finite.

We can now prove the main result.

Theorem 2.1. *Let G be a linear algebraic group over an algebraically closed field K of characteristic 0. Then G has a global degree bound if and only if it is finite.*

Proof. If G is finite, then the Noether bound [8] says $\beta(G) \le |G|$.

On the other hand, assume that G is infinite. Let $U \trianglelefteq G$ be the unipotent radical. G/U is reductive and therefore linearly reductive (this uses $\mathrm{char}(K) = 0$, see Springer [14, V, Satz 1.1]). If G/U is infinite, then the result follows from Proposition 1.2 and the inequality (2.1). If, on the other hand, G/U is finite, then by (2.2) it suffices to prove that $\beta(U) = \infty$. It follows from Humphreys [6, Corollary 17.5, Proposition 17.4, and Lemma 15.1C] that U has a closed normal subgroup N such that U/N is isomorphic to the additive group G_a. By (2.1) we are reduced to showing that $\beta(G_a) = \infty$. This is done in the following lemma. □

Lemma 2.2. *If $G = G_a$ is the additive group over an algebraically closed field K of characteristic 0, then $\beta(G) = \infty$.*

Proof. We use Roberts' isomorphism. This states that for an SL_2-module V (on which G_a acts by the matrices $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$) we have an isomorphism

$$\Phi\colon (K[U] \otimes_K K[V])^{\mathrm{SL}_2} \xrightarrow{\sim} K[V]^{G_a}, \tag{2.3}$$

where U is the natural 2-dimensional SL_2-module. A good reference for (2.3) is Kraft [7, page 191] (where a more general situation is considered) for the case $K = \mathbb{C}$, and Seshadri [13] for general K. The isomorphism is given by $\Phi\left(\sum_i f_i \otimes g_i\right) = \sum_i f_i(v) g_i$ with $v = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in U$. We have a natural bigrading on $(K[U] \otimes K[V])^{\mathrm{SL}_2}$, and if $f \in (K[U] \otimes K[V])^{\mathrm{SL}_2}$ has bidegree (i, j), then $\Phi(f)$ is homogeneous of degree j. Also consider the epimorphism

$$K[U] \otimes K[V] \twoheadrightarrow K[V], \ \sum_i f_i \otimes g_i \mapsto \sum_i f_i(0) g_i.$$

Since SL_2 is linearly reductive, this restricts to an epimorphism

$$\pi\colon (K[U] \otimes K[V])^{\mathrm{SL}_2} \twoheadrightarrow K[V]^{\mathrm{SL}_2}.$$

If $f \in (K[U] \otimes K[V])^{\mathrm{SL}_2}$ has bidegree (i, j), then $\pi(f)$ has degree j.

Now let V be an SL_2-module and set $k := \beta\left(K[V]^{G_a}\right)$. We can take preimages under Φ of homogeneous generating invariants for $K[V]^{G_a}$ and decompose them into their bi-homogeneous parts. It follows that $(K[U] \otimes K[V])^{\mathrm{SL}_2}$ is generated by bi-homogeneous invariants of degrees (i, j) with $j \le k$. By applying π we obtain that $K[V]^{\mathrm{SL}_2}$ is generated by homogeneous invariants of degree at most k, so $\beta\left(K[V]^{\mathrm{SL}_2}\right) \le k$. This argument shows that

$$\beta(\mathrm{SL}_2) \le \beta(G_a).$$

But $\beta(\mathrm{SL}_2) = \infty$ by Proposition 1.2. This finishes the proof. □

Unfortunately, we were unable to extend this or a similar result to positive characteristic. We conjecture the following.

Conjecture 2.3. *Let G be a linear algebraic group over an algebraically closed field K. Then the following are equivalent:*

(a) G has a global degree bound.

(b) G is finite and $\mathrm{char}(K)$ *does not divide the group order $|G|$.*

The implication "(b) ⇒ (a)" is given by the Noether bound, which was recently proved to hold also if $\mathrm{char}(K) < |G|$ but $\mathrm{char}(K) \nmid |G|$ independently by Fleischmann [4] and Fogarty [5]. It is also known from Richman [11] that a finite group with $|G|$ divisible by $\mathrm{char}(K)$ does not have a global degree bound. Both results can also be found in the book by Derksen and Kemper [3].

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Harm Derksen
Department of Mathematics
University of Michigan
East Hall
525 East University
Ann Arbor, MI 48109-1109
USA
hderksen@umich.edu

Gregor Kemper
Technische Universität München
Zentrum Mathematik - M11
Boltzmannstr. 3
85 748 Garching
Germany
Gregor.Kemper@iwr.uni-heidelberg.de