1.) Given the two functions $f(x) = \dfrac{5-x}{x+5}$ and $g(x) = \dfrac{5}{3+5x}$.

a.) Write the function represented by the composite function $f \circ g(x)$.

b.) Write the function represented by the composite function $g(f(x))$.

c.) Reduce $f \circ g(x)$ to a single rational expression and simplify completely.

d.) Convert the "improper" rational expression in part c.) into a "mixed" rational expression (e.g. $\frac{x-1}{x} = 1 - \frac{1}{x}$).

Solution:

a.) $f \circ g(x) = \dfrac{5 - \frac{5}{5x+3}}{\frac{5}{5x+3} + 5}$

b.) $g(f(x)) = \dfrac{5}{3 + 5 \cdot \frac{5-x}{x+5}}$

c.) $\dfrac{5 - \frac{5}{5x+3}}{\frac{5}{5x+3} + 5} = \dfrac{25x+10}{20+25x} = \dfrac{5x+2}{4+5x}$

d.) $\dfrac{5x+2}{5x+4} = 1 - \dfrac{2}{5x+4}$

Scoring Rubric:

a.) 2 points

b.) 2 points

c.) 3 points

d.) 2 points

2.) According to an article by the Mayo Clinic, newborn babies often triple their weight in their first year.

a.) If this trend should continue, give a formula for $w(t)$ the weight in pounds of a 10 pound newborn baby t years after it is born (give units).

b.) If this trend should continue, how many years would it take for that 10 pound newborn to reach one ton? (1 ton = 2000 pounds). Give an exact answer (your answer should contain logs).

c.) Another (unusual) newborn baby starts at 100 pounds, but doubles its weight every year. After how many years would the 10 pound newborn in part a) first surpass the weight of the 100 pound newborn assuming they were born on the same day? Give an exact answer (your answer should contain logs).

Solution:

a.) $w(t) = 10(3^t)$ lbs.

b.)

$$\begin{aligned} 2000 &= 10(3^t) \\ 200 &= 3^t \\ t &= \frac{\ln 200}{\ln 3} \text{ years} \end{aligned}$$

c.)

$$\begin{aligned} 100(2^t) &= 10(3^t) \\ 10(2^t) &= 3^t \\ 10 &= \left(\frac{3}{2}\right)^t \\ \ln 10 &= t\ln(1.5) \\ t &= \frac{\ln 10}{\ln 1.5} \text{ years} \end{aligned}$$

Scoring Rubric:

a.) 2 points:
exponential (1 point)
accuracy and units (1 point)

b.) 3 points:
set-up (1 point)
solving using log or other appropriate method (1 point)
accuracy and units (1 points)

c.) 4 points:
set-up (1 points)
solving using log or other appropriate method (2 points)
accurancy and units (1 point)

3.) Luke is watching a "lightening ferris wheel" rotate at a constant speed. There are many lights around the Ferris wheel and Luke watches a particular blue light closely. The height of the blue light after t seconds is given by

$$H(t) = 10\sin\left(\frac{\pi}{50}(t-25)\right) + 12 \text{ meters.}$$

a.) At what height will the blue light be after 50 seconds?

b.) How long does it take for the wheel to complete a full circle?

c.) At 2 minutes, is the blue light rising or descending? Explain your reasoning.

d.) At what times in the first three minutes will the blue light be 17 meters above the ground? Explain your reasoning.

Solution:

a.) $H(50) = 10\sin\left(\frac{\pi}{2}\right) + 12 = 22$ meters.

b.) Period$=\dfrac{2\pi}{\frac{\pi}{50}} = 100$ seconds.

c.) 2 minutes=120 seconds.
Remark that $100 < 120 < 125$.
The lowest height is reached at 100 seconds: $H(100) = 10\sin\left(-\frac{\pi}{2}\right) + 12 = 2$ meters.
The highest height is reached at 150 seconds: $H(50) = H(150) = 22$ meters.
$H(125) = 10\sin(2\pi) + 12 = 12$ meters.
The blue light is rising at 2 minutes.

d.)

$$\begin{aligned} 10\sin\left(\frac{\pi}{50}(t-25)\right) + 12 &= 17 \\ 10\sin\left(\frac{\pi}{50}(t-25)\right) &= 5 \\ \sin\left(\frac{\pi}{50}(t-25)\right) &= \frac{1}{2}. \end{aligned}$$

Two solutions are given by:

$$\begin{aligned} \frac{\pi}{50}(t-25) &= \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \\ t-25 &= \frac{50}{6} \text{ or } \frac{250}{6} \\ t &= \frac{100}{3} \text{ or } \frac{200}{3} \text{ seconds} \end{aligned}$$

3 minutes=180 seconds.
Using the periodicity, the times when the blue light is 17 meters above the ground are: $\frac{100}{3}$, $\frac{200}{3}$, $\frac{400}{3}$, and $\frac{500}{3}$ seconds.

Scoring Rubric:

a.) 1 point.

b.) 1 point.

c.) 3 points:
lowest and highest height (2 point)
or any appropriate method
answer (1 point).

d.) 4 points:
setting up the equation (1 point)
solving it (1 point)
all answers (2 points)