Disappointment aversion and income risk: implications for portfolio allocation

Revansiddha Basavaraj Khanapure*

This draft: August, 2015

Abstract

I solve the life-cycle portfolio allocation problem of a disappointing averse (DA) agent. Unlike expected utility investors, the risky allocations relate positively with income risk over later phase of DA investor’s working life. The changing comovement between returns and disappointment/elation realization drives the effect. The heterogeneous Epstein-Zin model with disappointment aversion generates conditional portfolio shares that are a better match to the empirical pattern. Sufficiently disappointment-averse agents abstain from investing in stocks after retirement and the decline in income risk also drives stock allocations lower in the later phase of DA investor’s working life.

*Contact: University of Delaware, Department of Finance, Purnell Hall, Room # 310, 42 Amstel Ave, Newark, Delaware, 19716, USA. Phone: +1-302-831-1920, Fax: +1-302-831-3061, Email: khanapur@udel.edu. Website: http://sites.udel.edu/khanapure/. I thank Lars P. Hansen, John C. Heaton, Ralph S.J. Koijen, Juhani T. Linnainmaa and Pietro Veronesi for numerous discussions and comments. I have benefited from the comments of John H. Cochrane, George M. Constantinides, Ram Chivukula, Valentin Haddad, Tarek A. Hassan, Kenneth L. Judd, Faruk Gul, Stavros Panageas, Bryan Routledge, Shrihari Sontosh, Alexi Savov, Robert W. Vishny and seminar participants at the Board of Governors at Federal Reserve System, Eastern Finance Association 2014 meetings, Erasmus University Rotterdam School of Management, The Federal Reserve Bank of Boston, the HEC Montreal, Indian School of Business, Midwest Finance Association 2013 meetings, Southwestern Finance Association 2013 meetings, University of Chicago Booth School of Business, University of Delaware Department of Finance and Department of Economics. I thank He Yan for excellent research assistance with the empirical work related to the PSID dataset.
1 Introduction

The uncertain income is an important source of uninsurable risk affecting risky investment behavior (Guiso, Jappelli, and Terlizzese (1996), Heaton and Lucas (2000b), Angerer and Lam (2009), Betermier, Jansson, Parlour, and Walden (2012)). The standard models of portfolio choice imply a decline in risky investment with the rise in income risk (Cocco, Gomes, and Maenhout (2005), Gomes and Michaelides (2005), Polkovnichenko (2007)). The data, however, suggest a mixed pattern for the relation between income risk and risky investment. In this article I solve the life-cycle portfolio allocation problem for disappointment aversion (DA) preference model, a non-expected utility model. I find that the relationship between income risk and risky portfolio share is dependent on age under the DA model and in line with empirical observations. The DA agent increases risky allocation with income risk in late working life, a pattern opposite to that obtained under the standard models. In contrast the DA agent lowers risky allocation with income risk in the earlier phase of working life, a pattern similar to that obtained with the standard models. The DA preferences have a curvature component similar to that in expected utility (EU) models and an additional parameter that determines the extra weight on disappointing states. This additional parameter that I refer to as the disappointment-aversion parameter accounts for the counterintuitive pattern.

The age dependent impact of uninsurable risk on risky allocation also affects the age pattern in risky allocation. In the case of standard models the decline in human capital acts as a strong motivating force producing the age related decline in risky shares. The DA model creates an added motive. The decline in uninsurable income risk also adds to the decline in risky portfolio shares in the later phase of working life. The literature, however, does not provide conclusive estimates for effects of age on risky allocation (Ameriks, Caplin, and Leahy (2007)). Hence I focus on the relation between income risk and risky portfolio share in this article.

I solve the life-cycle problem using both homoscedastic and heteroscedastic income processes. In the case of heteroscedastic process the per-period income shocks are smaller in size in the later phase of working life. The often used homoscedastic process implies an unlikely large and sharp decline in uncertainty once the agent steps into retirement. The proposed heteroscedastic process provides a parsimonious way to avoid such a pattern. Further, the investment pattern generated by the heteroscedastic process is in fact a better match to the observed equity allocation data in the case of DA preferences. The improved match is a result of the positive relation between risky investment and income risk in the later phase of life-cycle. This positive relation is noticeable starting in late forties and early fifties with the heteroscedastic process.
The positive income risk and risky investment relation in the later phase of life-cycle also helps lower conditional risky allocations over this phase in a heterogeneous Epstein-Zin model with disappointment aversion (EZ-DA). Gomes and Michaelides (2005) use a heterogeneous Epstein-Zin (EZ) model to match market participation rates and risky allocations conditional on participation. However, in their model the market participants with low elasticity of intertemporal substitution (EIS) and low risk aversion counterfactually invest all of the savings in equities. The EZ-DA model mitigates such pattern. It generates an equity investment pattern that avoids the all-equity allocation for low EIS and low risk averse group in the later phase of life-cycle.

The counterintuitive effects of income risk on risky allocation also have implications for attempts made at resolving asset pricing puzzles using DA preferences. The homogeneous representative agent framework with these preferences has been proposed as an alternative towards this end. However, Chapman and Polkovnichenko (2009) point that the models with heterogeneous DA agents generate substantially lower equity premium. Their model questions the improvements brought about by the homogeneous DA model. Their arguments hinge on market non-participation by sufficiently disappointment averse agents. However, I prove that when sure income turns risky then the sufficiently disappointment averse agent is persuaded into purchasing risky asset. The result holds even if the expected income declines provided the income are returns are independent or conditionally independent. Thus unlike the conclusion in Chapman and Polkovnichenko (2009), the homogeneous model of DA agents is likely sufficient to examine the asset pricing implications of this alternative preference model. However, the homogeneous model setup must explicitly account for uncertain labor income.

I also prove that the DA agent behaves as if he is risk neutral if he is not sufficiently disappointment averse and the curvature component of the preferences is linear. The agent invests all his savings in the risky asset independent of the relation between income and asset returns at all levels of income risk. The pattern, however, does not apply once the disappointment-aversion parameter is set above a threshold. Further, I prove that the welfare for DA preferences in presence of non-tradable risky income is concave in wealth. The proof is non-trivial. The sufficient conditions are: (1) the curvature parameter must be positive and (2) the disappointment-aversion parameter must be non-negative. I also include bequest motive in the DA formulation in a manner that retains the concavity of welfare in wealth.

Curcuru, Heaton, Lucas, and Moore (2010) conclude that explaining stock market non-participation is a challenge for quantitative theories. A one-time entry cost helps explain

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non-participation of the young households (Gomes and Michaelides (2005)). However once the entry costs are paid the agent continues to participate in the asset market in such models. A recent survey indicates that the asset income for those aged 65 and above was less than 13% of their total income (Purcell (2009)). Such low asset income suggests low stock market participation rates for older households. The DA preferences can provide preference based explanation for non-participation by the elderly. The non-participation in retirement occurs if disappointment aversion is set above a threshold. The uninsurable risks must, however, be absent in retirement to obtain non-participation.\(^2\) In contrast, CRRA, the standard EU model, does not generate non-participation in retirement for any level of risk aversion.

In this article I also solve the life-cycle problem in presence of uncertain out-of-pocket medical expenses. Although these uncertain expenses lower the risky investments under the standard EU model (Pang and Warshawsky (2010)), they result in a mild increase in risky portfolio share under the DA model. Such behavior is a result of the positive relation between risky investment and the uninsurable risk in the later phase of life-cycle. The pattern does not constitute a case against the DA model as the net impact of an unfavorable health shock can lead to an increase or a decrease in risky share under the standard model (Edwards (2008, 2010), Love and Perozek (2007)). In addition the empirical evidence about the causal impact of such shocks is mixed (Love and Smith (2010)).

DA preferences were proposed with the intent to accommodate Allais paradox-type behavior (Gul (1991)).\(^3\) The positive relation generated between the income risk and risky allocation under the DA preferences can be understood through this paradox. The paradox can be summarized as an affinity for riskless alternative if such an option is available and a preference for riskier choice if the alternate choice is risky as well. In other words the alternatives affect the risk appetite.\(^4\) Since labor income is non-tradable it can set or limit the available alternate choices and vary the risk appetite in a manner similar to that in Allais paradox. Thus if income is riskless then a sure or riskless gamble is one of the alternatives in the agent’s choice set. The result is that the agent, subject to Allais paradox-type behavior, is drawn toward conservative choices which produce lower risky shares or complete withdrawal from the risky asset market. If on the other hand the income is risky then the alternate choice is set to be risky. This is because the agent is left with the risky income even if he decides not to invest in

\(^2\)The government policy can be a possible reason for absent uninsurable risks in retirement in some economies.

\(^3\)Choi, Fisman, Gale, and Kariv (2007) and others find experimental evidence for DA preferences.

\(^4\)Consider the following version of the paradox. The first set of choices are Lottery 1A: guaranteed $200 and Lottery 1B: 80% chance of $300 and 20% chance of $0. The second set of choices are Lottery 2A: equal chances of winning $200 and $0 and Lottery 2B: 40% chance of $300 and 60% chance of $0. Most people choose Lottery 1A from the first set of choices and Lottery 2B from the second set. These choices violate the independence axiom.
the risky asset. Now given the risky alternative the agent prefers an aggressive strategy and
invests more in the risky asset. The result is that as the agent shifts from sure income to risky
income the risky share of investment increases and hence the positive relation between the
two. DA preferences also include the curvature component similar to that in EU preferences.
This EU-type feature helps revert to the standard decline in risky share with income risk
once the income uncertainty crosses a threshold. The result is an inverted U-shaped pattern
between risky share and income risk. The risky shares rise with income risk until the income
uncertainty crosses a threshold. Beyond the threshold, risky shares decline with income risk
which is the standard pattern.

The hump shaped variation in risky allocation with income risk interacts with the changes
in uninsurable risk through life-cycle to produce age dependent effects. The uninsurable
income risk is higher in the early life-cycle due to permanent shocks that affect income. These
shocks affect the labor income in the period they are realized and also affect the labor income
draws in all the remaining periods of life-cycle. Thus with many spells of income left the
uninsurable risk is higher in the early phase of life-cycle. Consequently the standard relation
between income risk and the risky share applies. In the later phase of life-cycle, however, the
permanent shocks are more akin to idiosyncratic shocks as fewer spells of income remain. Thus
the uninsurable risk is lower in this period. Consequently the positive relation between income
risk and risky allocation applies in this later phase. However note that when retirement income
is tethered to permanent shocks a sharp change occurs around retirement as the retirement
event resolves a large amount of uncertainty.

Another way to understand the increase in risky allocation with income risk is through
the extra component of marginal utility in the DA model that overweights disappointing
states. This component applies the extra weight evenly to all the disappointing outcomes.
Such a scheme increases the aversion to risky assets if income is not uncertain or is absent
altogether. In such cases the adverse asset returns strictly associate with low wealth and high
marginal utility states. However a contrasting pattern appears if income is risky. In such cases
the favorable(adverse) returns also associate with the low(high) wealth states. This change
coupled with the overweighting applied evenly to all the disappointing states increases the
appetite for risky asset. The result is an increase in the risky share of portfolio as safe source
of income turns risky. This trend is in contrast with the pattern that occurs with the EU
models. The marginal utility in EU models is convex and strictly increases with the decline in
wealth level. The spread in the marginal utility spread also increases with the income risk. In
addition the favorable asset returns are increasingly sparse as the wealth level declines. These
features together drive risky share lower with the increase in income risk. This pattern also
extends to the DA model through the EU component of the marginal utility once income risk

crosses a threshold.

Cocco, Gomes, and Maenhout (2005) and Gomes and Michaelides (2005) study the life-cycle portfolio allocation problem using CRRA and EZ preferences. In this article I approach the problem using DA and EZ-DA preference models and focus on the unique implications. Ang, Bekaert, and Liu (2005) also study the portfolio problem faced by the DA agents. However their setup does not include uninsurable risks and includes only terminal consumption. Their focus is on asset market non-participation in such setting. In contrast my setup includes uninsurable risks and allows for intermediate consumption as well. I focus on the relation between risky investment and uninsurable risk. I also prove that the non-participation outcome studied in Ang, Bekaert, and Liu (2005) is not possible if income is risky and independent.

Barsky, Juster, Kimball, and Shapiro (1997) provide survey evidence indicating variation in risk tolerance with age. In addition the experimental evidence in Albert and Duffy (2012) indicates increasing pattern in risk aversion with age. Such pattern produces an additional motivation to lower risky allocation with age, the same effect as that of the declining uninsurable risk in the DA model. In case of DA preferences however the decline in risky shares due to declining uninsurable risk is limited to the later phase of working life. Thus to the extent the effect of age on risky shares is concerned a setting with standard preferences and increasing risk aversion with age can be indistinguishable from a DA preference model. However the cross-sectional variation in uninsurable risk lead to different effects in the two models. The effect of an increase in uninsurable risk is always to lower risky investment in the case of standard preferences. The impact in the case of DA preferences is however more nuanced and dependent on age.

The habits model is another appealing model of preferences (Gomes and Michaelides (2003), Polkovnichenko (2007)). The additive habit model generates conservative investments in old age to ensure that the habit level is sustained, but the accompanying wealth accumulation is also high. The agents may also exhibit aversion to risky investments in the later phase of life-cycle due to inflexibility in the labor supply. However, the relation between income risk and risky allocation is negative in such models unlike the more complex pattern under the DA preferences.

The rest of the article is organized as follows. I present empirical evidence on the pattern in equity investment and its relation with income risk in Section 2. I show the effects of background risk on risky allocation under the DA model in a one-period setting in Section 3. I set up and calibrate the life-cycle problem in Sections 4 and 5. I present the results for the

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benchmark case and contrast them with the results from CRRA model in Section 6. I follow up with various extensions in Section 7. I match the conditional portfolio shares, participation rates and wealth accumulation pattern implied by the EZ-DA model to that from Survey of Consumer Finances in Section 8. I conclude in Section 9.

2 Patterns in portfolio shares

2.1 Age effects

The age, time and cohort are linearly dependent and thus the estimation of their effects on portfolio shares is difficult. The estimates of age effect are sensitive to assumptions about the cohort and time effects. Ameriks and Zeldes (2002) use Survey of Consumer Finances (SCF) data to measure age effects on stock market participation and the equity fraction of financial assets among equity owners. They show that the market participation rates increase during working life irrespective of whether the cohort or the time controls are used. However, the pattern in participation starting around the age of 60 depends on the type of control used. They find that the time controls yield declining age pattern. However, as the authors note, the pattern could be attributed to the missing cohort controls. Similarly, the missing time effects such as growing popularity of defined-contribution (DC) plans and Individual Retirement Accounts (IRAs) or other time trends could be the reason for the rising age pattern observed when only the cohort controls are used (Weisbenner (2002), Campbell and Viceira (2002), Ameriks and Zeldes (2002)). In a similar vein the time controls yield declining\(^6\) while the cohort controls produce increasing effects of age on equity shares of market participants’ financial assets. Each of these patterns could possibly be due to the other missing control.

I use a proxy for the time effects that is linearly independent of age and cohort to better ascertain the effect of age on equity investment. In particular I use the fraction of households with the DC retirement plan and/or IRA account in each survey year as a proxy for the time effect. The implied age effects indicate a hump shaped pattern in stock market participation rate and a declining pattern in equity investment among the participants (Fig. 1). Further, I obtain similar patterns with other alternative proxies for the time effect.\(^7\)

These results indicate that the age effects, after fully accounting for time and cohort effects,

\(^6\)The declining age effect pattern on the equity fraction among the participants is stronger in the full 1989-2010 SCF sample compared to 1989-1998 sample in Ameriks and Zeldes (2002).

\(^7\)Specifically I consider, (1) the proportion of households participating in equity market, (2) the proportion of households among the stock market participants with the DC retirement plan and/or IRA account in each survey year, and (3) the average proportion of financial assets held in retirement account in the form of equity across the participating households over time. The second and the third proxies are conditional on equity investment.
are likely to generate declining equity allocation and a hump shaped pattern in the propensity to participate in the equity market. The policy of declining allocation to equity with age at target-date funds provides another estimate of age effects. These funds accounted for $503 billion at the end of March, 2013 (Charlson and Lutton (2013)) and are one of the mandated default investment options in retirement accounts.

2.2 Relation with non-financial income risk

The income risk has a substantial influence on portfolio choice. Some studies indicate a negative relationship between risky investment and income risk while the results in other studies are suggestive of a relation that depends on the investor’s age and wealth. Angerer and Lam (2009), Betermier, Jansson, Parlour and Walden (2012), Heaton and Lucas (2000a), indicate a negative relationship between income risk and risky investment. Whereas Massa and Simonov (2006) find a positive relation between the two among high wealth households and a negative relationship among low wealth households. They use Swedish household data and control for correlation between financial and non-financial income. Bonaparte, Korniotis and Kumar (2013) use Dutch household data to study the relation between equity investment and income-return correlation. The median age of the respondents in this dataset is 54.

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Figure 1: The implied age effects for stock market participation rate from a probit regression and that for the fraction of financial wealth invested in equity from an OLS regression. The specifications include age dummies, cohort dummies and a proxy for the time effect. I use the fraction of population with defined contribution plans or IRA accounts in a given year as the time proxy. The data is from all available samples of Survey of Consumer Finances from 1989 to 2010 and include home equity in computing financial wealth. See Internet Appendix for more details.
Table 1: The coefficients in column (i) are from probit regression of stock market participation, and in column (ii) the coefficients are from OLS regression of equity fraction of financial assets for market participants. The independent variables include dummy variables ‘near-retirement’, ‘non-public sector’ and their interaction ‘near-retirement’ × ‘non-public sector’ along with other controls. The data is for head of households (HoH) from all SCF samples from 1989 to 2010 with ages between 55 to 60 and 30 to 34. near-retirement = 1 if 55 \leq \text{HoH age} \leq 60 and near-retirement = 0 if 30 \leq \text{HoH age} \leq 34. ‘non-public sector’ = 0 if HoH works in public sector and ‘non-public sector’ = 1 otherwise. The controls are for income, liquid assets, household characteristics, expected returns, along with time and cohort dummies. Internet Appendix describes the controls. Observations are weighted with SCF sample weights. The t-statistics in parenthesis are robust and adjusted for multiple imputation. ** significant at 5%; * significant at 10%.

Although the equity investments decline with rising income-return correlation the relation between equity investment and income risk (after accounting for income-return correlation) is not unambiguous.\textsuperscript{9} They also analyze data from National Longitudinal Survey of Youth 1979 Cohort (NLSY79) which has a much younger sample with a median age of 31. The relation between equity investment and income risk is unambiguously negative in this sample. Angerer and Lam (2009) also use NLSY79 to find a negative relationship between permanent income risk and risky investment and little or no impact of transitory income risk. These patterns point to a more complex relationship between income risk and risky investment.

Table 1 presents evidence suggesting that the equity investment increases with income risk in later stage of working life. I use data from SCF and compare equity investment near retirement between the ages of 55 to 60 to that between 30 to 34. I use household head’s industry to proxy for income risk. I classify public sector workers as low income risk group and, absent detailed industry classification, the rest non-public sector workers into high income risk group.\textsuperscript{10} The positive and significant interaction term ‘near retirement’ × ‘non-public sector’

\textsuperscript{9}The relation is unambiguously negative if the income risk is very high. See coefficients in Table 5 for the dummy variable named ‘High Inc Risk.’ It identifies whether variance for income growth is in the top quartile.

\textsuperscript{10}See Campbell, Cocco, Gomes and Maenhout (1999) for estimates of income risk by industry. The public version of the SCF dataset classifies those in non-public sector industries into six groups without regard for

<table>
<thead>
<tr>
<th>Dependent variable</th>
<th>Stock market participation</th>
<th>(Equity/Financial Assets) among stock market participants</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>(ii)</td>
<td></td>
</tr>
<tr>
<td>near retirement × non-public sector</td>
<td>0.444**</td>
<td>0.0155</td>
</tr>
<tr>
<td></td>
<td>(2.097)</td>
<td>(0.349)</td>
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<tr>
<td>near retirement</td>
<td>-0.575**</td>
<td>-0.0441</td>
</tr>
<tr>
<td></td>
<td>(-2.609)</td>
<td>(-0.967)</td>
</tr>
<tr>
<td>non-public sector</td>
<td>0.178</td>
<td>-0.0321</td>
</tr>
<tr>
<td></td>
<td>(1.216)</td>
<td>(-0.948)</td>
</tr>
<tr>
<td>Controls</td>
<td>Yes</td>
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<tr>
<td># Obs.</td>
<td>4,188</td>
<td>2,414</td>
</tr>
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</table>
suggests higher propensity to participate in stock market among higher income-risk workers than that among lower income-risk workers near retirement compared to no such significant differences in early working life. The effect of the interaction term is positive but insignificant on equity investment among the market participants. The investment in equities is imputed based on categorical responses in SCF.\textsuperscript{11} The resulting poor measurement could be one reason for these insignificant estimates. I include time dummies, cohort dummies, income controls, liquid asset controls, household characteristics, expected returns based on experienced stock market returns as computed in Malmendier and Nagel (2011) as controls in regressions in Table 1. I exclude retirees and part or sole owners of businesses. The Internet Appendix further describes these controls.

3 One-period portfolio problem with labor income

Gollier and Pratt (1996) show that the agents with HARA class utility functions subject to mean-preserving rise in uninsurable risk lower the demand for risky assets. In contrast, the DA preferences can yield an increase in risky investment with the rise in uninsurable risk. This counterintuitive pattern applies when background risk is low enough. However, once the background risk crosses a threshold the risky asset demand declines with the background risk. I illustrate this phenomenon for DA preferences in a one-period terminal consumption problem and contrast it with the implications from a CRRA model. The features of the one-period problem extend to the life-cycle setup. The one-period setting provides a clean way to isolate and explain different drivers of the risky asset demand.

The one-period problem is as follows. The investor has savings worth $A_t$ dollars that he optimally splits between a risky asset and a risk-free asset at time $t$. The investor also receives non-tradable positive labor income $Y_{t+1}$ at time $t+1$. The investor consumes all of the terminal wealth $W_{t+1}$ at time $t+1$:

\[
W_{t+1} = A_t R_{p,t+1} + Y_{t+1}
\]

\[
R_{p,t+1} = R_{e,t+1} x_t + R_f
\]

\[
\ln(R_{t+1}) = \ln(\bar{R}) + \epsilon_{r,t+1}
\]

\[
\epsilon_{r,t+1} \sim N(-\sigma_r^2/2, \sigma_r^2).
\]

the riskiness of their income stream.

\textsuperscript{11}Starting in 2004 SCF collects actual percentage of account balance invested in equity for most of the accounts. However, the equity portion of the combination and other mutual funds is still imputed. The half of the value of the combination funds and the complete value of other mutual funds is assigned to equity holdings.
The risky return $R_{t+1}$ has a log-normal distribution with mean $\bar{R}$. The risk-free rate is $R_f$. I set $\bar{R} - R_f = 6\%$, $R_f = 2\%$ and $\sigma_r = 18\%$. The portfolio weight is constrained such that $0 \leq x_t \leq 1$. I present the impact of mean-preserving spread in income risk by setting the expected labor income at $\bar{Y}$ and varying the standard deviation, $\sigma_y$, of the log-labor income. Further, I set return and income shocks independent of each other.

$$Y_{t+1} = \bar{Y}\exp(\epsilon_{y,t+1})$$
$$\epsilon_{y,t+1} \sim N(-\sigma_y^2/2, \sigma_y^2).$$

$\mu_{da,t}$ and $\mu_{crra,t}$ in Eq. (1) and (2) are the certainty equivalents under the DA and CRRA preferences, respectively. The outcomes below $\mu_{da,t}$ are disappointing and the outcomes above $\mu_{da,t}$ are elating.

$$\mu_{da,t}^{1-\gamma} = E_t \left[ W_{t+1}^{1-\gamma} \right] - \theta E_t \left[ (\mu_{da,t}^{1-\gamma} - W_{t+1}^{1-\gamma}) I(W_{t+1} < \mu_{da,t}) \right] \tag{1}$$
$$\mu_{crra,t}^{1-\gamma} = E_t \left[ W_{t+1}^{1-\gamma} \right]. \tag{2}$$

The two parameters $\gamma$ and $\theta$ determine the risk-reward attitudes under the DA preferences. I refer to $\gamma$ as the curvature parameter and $\theta$ as the disappointment-aversion parameter hereafter. Further, I refer to $W_{t+1}^{1-\gamma}/(1 - \gamma)$, the valuation of wealth in a specific state, as utility. I set $\gamma = 5$ and $\theta = 1$. I set the CRRA model’s curvature parameter equal to that of the DA model. The disappointment probability, $\Phi$, for DA preferences is the probability that the gamble outcome will be less than the certainty equivalent. Hence, $\Phi \equiv E_t [I(W_{t+1} \leq \mu_{da,t})]$ for the one-period problem.

The agent’s objective is to maximize the certainty equivalent (CE) by choosing the optimal portfolio weight, $x_t$. I compute the optimal $x_t$ as a function of $\sigma_y$ and plot it in Fig. 2. The optimal risky asset demand increases with the background risk under the DA model. This trend, however, flips once the income risk crosses a threshold and the risky allocation turns conservative with the increasing background risk. The portfolio weight, on the contrary, is decreasing over the entire range for the CRRA preferences.\footnote{The constraint $x_t \leq 1$ is binding at low income risk and high $\bar{Y}/A_t$ for both the DA and CRRA models.}

**Proposition 3.1.** If DA agent is sufficiently disappointment averse such that $\theta > \theta^*$ and earns a risk free income then he (1) does not invest in the risky asset. If the DA agent earns risky income which is independent of asset returns then the (2) optimal risky investment is positive for all $\theta \geq 0$. However, the requirement is that $\bar{R}_{e,t+1} > 0$. The second result applies even if the asset returns are conditionally independent of income.
The effects of age on risky allocation in the life-cycle problem relate to the patterns in risky share in Fig. 2. This figure presents the case applicable to the life-cycle setting as data points to weak or no correlation between income and returns. The patterns in Fig. 2 also help
point to the relation between income risk and risky investment at various stages of life-cycle. The mapping exercise requires knowledge of the changes in uninsurable risk with age and the changes in the human capital to savings ratio with age. In fact both of these changes follow a declining pattern with age in the life-cycle setting. The agent has a long stream of non-tradable income yet to be cashed and hence a high level of human capital in the early stages of life-cycle. However, fewer spells of income remain as the agent progresses through the life-cycle and consequently a declining trend in the human capital. The agent also cumulates savings with age to save for retirement and bequest. Thus the result is that the human capital to savings ratio declines with age. The decline in uninsurable risk is due to the permanent shocks affecting the income. The permanent shock realized at a given stage affects the income at that age and also the income for all the remaining periods. Thus in the early stages of life-cycle the uncertainty in permanent income represents a large quantity of uninsurable risk. This risk declines with age as the number of income spells to be realized declines.

Fig. 2 indicates that the effect of declining human capital to savings ratio in the life-cycle setting is to lower the risky shares. This pattern applies for both the DA and CRRA preferences. However, the decline in uninsurable risk has an opposite effect at all ages in the context of CRRA preferences. The net effect is dominated by the declining human capital to savings ratio. The result is a drop in the risky shares with age. In the case of DA preferences however the drop in uninsurable risk also aids the decline in risky shares once the level of uninsurable risk is low enough. This pattern occurs in the later phase of life-cycle. The uninsurable risk reinforces the pattern generated by the ratio and the risky allocation has a positive relation with uninsurable risk over this phase. In contrast the relation between risky shares and uninsurable risk is always negative in the case of CRRA preferences. Below I illustrate the drivers of risky allocation in the one-period problem. Same mechanism also applies in the life-cycle setting.

3.1 Drivers of non-monotonic asset demand

The first order conditions (FOCs) in Eq. (3) and (4) for CRRA and DA preferences are one of the ways to understand the phenomenon in Fig. 2.\textsuperscript{13}

\begin{align}
0 &= E_t [u'(W_{t+1})R_{e,t+1}] \\
0 &= E_t [u'(W_{t+1})D_{t+1}R_{e,t+1}] \\
\text{Where } D_{t+1} &= \frac{1 + \theta I(W_{t+1} < \mu_{da,t})}{1 + \theta \Phi(W_{t+1} < \mu_{da,t})} \text{ and } u'(W_{t+1}) = W_{t+1}^{-\gamma}. \tag{5}
\end{align}

\textsuperscript{13}I thank the anonymous referee for suggesting this approach.
Figure 3: The average excess returns, $R_{a,e,t}^{+1}$, with increasing mean-preserving spread in labor income, $\sigma_y$, that corresponds to $(a - 10)\%$ to $a\%$ percentile range of the terminal wealth, $W_{t+1}$. $a$ takes values 10, 20, ..., 100. The curves at $\sigma_y = 0$ are in increasing order of the value of “a.” The income and return shocks are uncorrelated. The portfolio weight is set to optimal value at every income risk level for the DA preferences. Similar patterns emerge if the portfolio weight is held constant or optimal weights under CRRA preferences are used.

The concave utility, $u(W_{t+1})$, and convex marginal utility, $u'(W_{t+1})$, in Eq. 3 increase the weights on adverse return outcomes as the wealth outcomes disperse with the increasing income spread. The result is a negative value for the expectation in Eq. 3 unless the asset allocation is re-optimized. The agent can achieve optimality only by reducing the risky allocation and hence the pattern for CRRA preferences in Fig. 2. Hereafter I refer to the negative impact of background risk on risky allocation as the standard effect. It is driven by the on average association of unfavorable return outcomes with the low wealth percentiles and the nature of the marginal utility, $u'(W_{t+1})$.

Fig. 3 shows the average return outcome associated with each of the ten wealth percentile bins. For instance the lowest curve graphs the average excess return corresponding to wealth in $0 - 10^{th}$ percentile range. Fig. 3 shows that the favorable return outcomes increasingly associate with the lower wealth percentiles until the income risk crosses a threshold. The trend, however, inverts for income risk higher than this threshold. Such patterns apply if the correlation between income and returns is zero or positive and is a characteristic of the non-tradable nature of labor income.

The association and the subsequent retraction of favorable return outcomes from the disappointing states and the extra emphasis on these states coded in $D_{t+1}$ (Eq. 5) are consistent
with the non-monotonic asset demand. I refer to this effect as the DA effect. \( D_{t+1} \) is the additional component of marginal utility unique to DA preferences in Eq. 5 that overweights the disappointing states. I rewrite the FOC in Eq. 4 in terms of twisted distribution in Eq. 6 that overweights disappointing outcomes and in terms of twisted return premium \( \hat{R}_{e,t+1} \) in Eq. 7. \( \hat{p}(W_{t+1}) \) is the twisted version of the data generating distribution \( p(W_{t+1}) \). The hat (\(^\wedge\)) represents quantities under the twisted distribution.

\[
0 = \hat{E}_t [u'(W_{t+1})R_{e,t+1}] \quad \text{Where } \hat{p}(W_{t+1}) = D_{t+1} \times p(W_{t+1}). \quad (6)
\]
\[
\hat{R}_{e,t+1} = -\frac{\hat{Cov}_t[u'(W_{t+1}), R_{e,t+1}]}{\hat{E}_t[u'(W_{t+1})]} \quad \text{Where } \hat{R}_{e,t+1} = \hat{E}_t[R_{e,t+1}]. \quad (7)
\]
\[
\hat{R}_{e,t+1} = \frac{R_{e,t+1} + \theta \Phi R_{e,d}}{1 + \theta \Phi} = \text{Cov}_t[R_{e,t+1}, D_{t+1}] + \hat{R}_{e,t+1} \quad (8)
\]
\[
\text{Where, } \hat{R}_{e,d} = E_t[R_{e,t+1}|W_{t+1} < \mu_{da,t}]. \quad (9)
\]

Eq. 6 and 7 focus on the interaction between \( D_{t+1} \) and returns. In fact \( \hat{R}_{e,t+1} \) captures the covariance between returns and the asymmetric weighting scheme, \( D_{t+1} \). \( \hat{R}_{e,t+1} \) is also a weighted sum of \( R_{e,t+1} \) and \( R_{e,d} \) in 1 to \((\theta \Phi)\) proportion. Eq. 8 presents both of these interpretations. The disappointment return premium, \( \hat{R}_{e,d} \), and disappointment probability, \( \Phi \), the two drivers of \( \hat{R}_{e,t+1} \) represent two separate components of the DA effect. The variation in \( \hat{R}_{e,d} \) even if \( \Phi \) and the asset’s return distribution are fixed is unique to the setup that includes background risk. In absence of such non-insurable risk the wealth and the return percentiles align and \( \hat{R}_{e,d} \) is fixed if \( \Phi \) and the return distribution are fixed.

The log-linear approximation of Eq. 6 in Eq. 10 separates the terms that drive the standard and the DA effects. The term \( \rho \) approximates the standard effect and represents the elasticity or sensitivity of terminal consumption with respect to financial wealth. The twisted log return premium, \( \tilde{r}_{t+1} - \tilde{r}_f + \tilde{\sigma}_{r,t+1}^2 / 2 \), summarizes the DA effect. The replacement of the quantities under the twisted distribution in Eq. 10 with those under the data generating distribution yields the approximate formula for CRRA preferences. The result is a constant return premium and only the standard effect applies as the income risk varies. Campbell and Viceira (2002) use such approximation for CRRA preferences to show that the condition \( \gamma > 1/\rho \) is sufficient for a negative impact of the rising mean-preserving income spread on risky investment sourced through rising \( \rho \).\(^{15}\) The same condition also approximately ensures similar trend for \( \rho \) and the

\(^{14}\)The appropriate notation, in line with the rest of the article, for the premium in disappointing states for a given \( \Phi \) is \( \hat{R}_{e,d} | \Phi_{t+1} \). However, I use \( \hat{R}_{e,d} \) for easy reading. When necessary I emphasize the dependence on \( \Phi \) and do not stress that \( \hat{R}_{e,d} \) is a time \( t \) measurable statistic of a time \( t+1 \) measurable variable.

\(^{15}\)This is only a sufficient condition and based on the approximate formula for risky asset demand under CRRA preferences. Gollier and Pratt (1996) show that a mean-preserving increase in background risk yields
standard effect for DA preferences (Fig. 4). I include derivation of Eq. 10 and provide details in the Internet Appendix.

\[ x_t \approx \frac{1}{\rho} \left( \frac{\hat{r}_{t+1} - \hat{r}_f + \hat{\sigma}_{r,t}^2/2}{\gamma \hat{\sigma}_{r,t}^2} \right); \text{ if } \hat{\sigma}_{y,r,t} \approx 0. \]  
\[ \frac{1}{\rho} = 1 + \frac{\exp \left( E_t \left[ \ln(Y_{t+1}) \right] \right)}{\exp \left( E_t \left[ \ln(A_t R_{p,t+1}) \right] \right)}; \quad 0 < \rho < 1. \]  

I graph the DA effect and its components in Fig. 4. The hump shaped pattern in \( \hat{R}_{e,t+1} \) is consistent with the pattern in the risky portfolio weight. In addition to the dominant standard effect the DA effect also helps drive the decline in portfolio weight at high income risk. The patterns in \( \hat{R}_{e,d} \) and \( \Phi \), the DA effect’s components, are also consistent with the variation in \( \hat{R}_{e,t+1} \).

The pattern in \( \hat{R}_{e,d} \) is hump shaped (Fig. 4) and similar to the pattern in \( \hat{R}_{e,t+1} \). The two are positively related for a fixed \( \Phi \). When the income is certain, returns in disappointing states are strictly lower than those in elating states as the rankings of wealth and return align when a decline in risky asset demand for all HARA class utility models.
the background risk is absent. This pattern, in fact, generates the lowest possible value of $R_{e,d}$ for a given $\Phi$. Once the income turns risky it drives a wedge between the rankings of wealth and return outcomes. The effect is that not only the low(high) returns but also the high(low) returns increasingly associate with the lower(higher) wealth percentiles. Fig. 3 reflects this pattern as all curves are drawn toward the median return outcome. The increasing mix of higher returns in disappointing states or low-wealth percentiles increases $R_{e,d}$. The pattern, however, flips at high income risk. The retraction of advantageous returns from the low wealth percentiles at high income risk is an artifact of the lower bound on income as the income must be non-negative. I illustrate this later effect and provide details in the Internet Appendix.

The $\Phi$, the other component of the DA effect, modifies the weight on $R_{e,d}$ in Eq. 8. The $\Phi$ has a U-shaped variation (Fig. 4) and has two opposing effects on $\hat{R}_{e,t+1}$. On one hand it attenuates the hump shaped variation induced in $\hat{R}_{e,t+1}$ due to a similar variation in $R_{e,d}$ but on the other hand it also aids in the generation of hump shaped variation in $\hat{R}_{e,t+1}$ as $R_{e,d} < \hat{R}_{e,t+1}$. The second effect is the negative relation between $\hat{R}_{e,t+1}$ and $\Phi$ for a fixed $R_{e,d}$ and it holds if the income and return shocks are independent or positively correlated. The variation in $R_{e,d}$ is, however, larger compared to the variation in $\Phi$ and is the driving force behind the variation in $\hat{R}_{e,t+1}$. The same applies in the life-cycle setting where the dominant DA effect is the variation in $R_{e,d}$. In effect the appetite for risky investment rises with income risk due to the association of advantageous returns with low wealth states and also the emphasis on such states built into the DA model. The usual pattern reverts as the standard effect soon dominates and the DA effect also contributes in the same direction at high enough income risk.

\[
\Lambda_t = E_t [\Omega_{t+1}] - \Phi \times \theta [\Lambda_t - E_t [\Omega_{t+1} | \Omega_{t+1} < \Lambda_t]]
\]

(12)

Where $\Lambda_t = u(\mu_{da,t})$ and $\Omega_{t+1} = u(W_{t+1})$.

(13)

I rewrite Eq. 1, the formula for the DA welfare, in Eq. 12. It shows that besides $\theta$ the value of $\Phi$ depends only on the distribution of $u(W_{t+1})$. In particular $\Phi$ is negatively related to the left skewness of $u(W_{t+1})$ due to the asymmetric emphasis on adverse outcomes. The inclusion of risky income that is independent of returns lowers the skewness of wealth and turns $u(W_{t+1})$ more left-skewed. The result is a decline in $\Phi$ until the income risk crosses a threshold. Thereafter, the right skewness of income distribution ultimately dominates as the income risk increases and results in the rising portion of the pattern in $\Phi$. I provide more

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16The bulk of income distribution increasingly concentrates near zero for high $\sigma_y$, an artifact of mean-preserving spread on income that is bounded below at zero. Thus the return and wealth rankings increasingly align at low wealth percentiles producing a retraction of advantageous returns from the low wealth percentiles.
details about the effect of income risk on the skewness of $u(W_{t+1})$ and the effect of skewness of $u(W_{t+1})$ on $\Phi$ in the Internet Appendix.

The decline in wealth skewness with the introduction of risky income and the concurrent rise in DA investor’s risky investment are suggestive of diversifying impact of the income risk. However, the welfare, $\mu_{da,t}$, strictly declines with the rising mean-preserving spread in income. The rising riskiness of income does not have any benefit for the DA agent and he is strictly worse off. I include the proof in Appendix A.

In the following sections I set up and solve the full-fledged life-cycle problem for the DA and CRRA preferences and use the twisted premium as a summary of the DA effect to explain contrasting features.

4 Life-cycle problem and solution method

The life-cycle problem is as follows. The agent at time $t$ has savings $A_t$ that are transformed into future tradable wealth according to Eq. (14). The benefit of saving is the portfolio return $R_{p,t+1}$, which depends on the chosen portfolio weight $x_t$ and the stochastic excess return $R_{e,t+1}$ on the risky asset, and a known risk-free return $R_f$. The term $W_t$ that I refer to as wealth or tradable wealth is also known as cash-on-hand in the life-cycle literature (following Deaton (1991)). $C_t$ is consumption at time $t$. $A_t$ is the difference between wealth, $W_t$, and consumption, $C_t$. In addition to the return on savings, the agent also receives exogenous non-tradable labor income $Y_{t+1}$ and is left $(1 - h_{t+1})Y_{t+1}$ after covering housing-related expenditures:

$$W_{t+1} = A_t R_{p,t+1} + (1 - h_{t+1})Y_{t+1}, \text{ where, } A_t = W_t - C_t \tag{14}$$

$$R_{p,t+1} = R_{e,t+1} x_t + R_f$$

$$R_{e,t+1} = R_{t+1} - R_f.$$

The agent cannot borrow at the risk-free rate or short the risky asset. Thus, the portfolio weight lies between 0 and 1. In addition, current consumption cannot exceed current wealth.

$$C_t \le W_t, \quad x_t \ge 0, \quad x_t \le 1. \tag{15}$$

The risky asset at the agent’s disposal is a value-weighted market index (henceforth also referred to as stock). The raw returns on the market index follow a log-normal distribution with an expected rate of return $\overline{R}$ per period (Eq. (16)). The standard deviation of log returns
is $\sigma_r$. I assume that the investor faces a constant investment opportunity set:

$$
\ln(R_t) = \ln(R) + \eta_t, \text{ where, } \eta_t \sim N(-\sigma_r^2/2, \sigma_r^2). \tag{16}
$$

The real labor income has a deterministic component $l_t \equiv l(t, Z_t)$ that depends on age and other personal characteristics $Z_t$. Eq. (17) describes the labor income process until retirement.

$$
Y_t = \exp(l_t + \nu_t + \epsilon_t) \quad \forall t \leq K
\nu_t = \nu_{t-1} + u_t \quad \text{Where, } u_t \sim N(0, \sigma_u^2), \quad \epsilon_t \sim N(0, \sigma_e^2). \tag{17}
$$

In addition to the deterministic trend $l_t$, labor income is also determined by a permanent component $\nu_t$ driven by shocks $u_t$ and an idiosyncratic component $\epsilon_t$. The two shocks distributed as $N(0, \sigma_u^2)$ and $N(0, \sigma_e^2)$ are uncorrelated. The transitory shock $\epsilon_t$ is also uncorrelated with the stock return shock $\eta_t$. I allow for correlation between $u_t$ and $\eta_t$ in the extension of the benchmark model.

The retirement income is a fixed fraction of permanent income in the last working period before retirement.

$$
\ln(Y_t) = \ln(\lambda) + l_K + \nu_K; \quad \forall K + 1 \leq t \leq T \tag{18}
\Rightarrow Y_t = \lambda \exp(l_K + \nu_K), \quad K = \text{the last period agent works}
$$

The preference specification follows Epstein and Zin (2001). I set the reciprocal of the curvature parameter, $\gamma$, equal to the elasticity of intertemporal substitution (EIS), $\psi$. I consider the case where the two differ in Section 8. The value function $J_t$ in Eq. 19a summarizes the preference over gambles and static choices. $\beta$ captures the time rate of preference and $p_t$ represents probability of surviving to the next period, $t + 1$, conditional on having survived up to period $t$.

$$
\frac{J_t(W_t, \nu_t)^{1-\gamma}}{1-\gamma} = \max_{C_t, x_t} \frac{C_t^{1-\gamma} + p_t \beta \mu_t (J_{t+1}(W_{t+1}, \nu_{t+1}))^{1-\gamma}}{1-\gamma} \tag{19a}
\mu_t^{1-\gamma} = E_t \left[ J_{t+1}(W_{t+1}, \nu_{t+1})^{1-\gamma} \right] \tag{19b}
- \theta E_t \left[ \mu_t^{1-\gamma} - J_{t+1}(W_{t+1}, \nu_{t+1})^{1-\gamma} I(J_{t+1} < \mu_t) \right]
$$

The welfare $J_t(W_t, \nu_t)^{1-\gamma}/(1-\gamma)$ is concave in wealth $W_t$ if $\gamma > 0$ and $\theta \geq 0$. The proof
is in the Internet Appendix and applies to the general case involving a bequest motive. The
proof involves Proposition 4.1, a non-trivial component. The complexity arises due to the
fact that the disappointing and elating outcomes of two gambles need not occur in the same
states of the world. This is the first article to show concavity of welfare under the DA model
in presence of non-tradable income.

**Proposition 4.1.** If \( g_{t+1}(W_{t+1}, \nu_{t+1})^{1-\gamma}/(1 - \gamma) \) is concave (quasi-concave) in wealth \( W_{t+1} \)
and \( \theta \geq 0 \), then agent’s welfare, \( \mu_t(\cdot)^{1-\gamma}/(1 - \gamma) \) satisfies following inequality for any two
distinct wealth gambles \( W_{1,t+1} \) and \( W_{2,t+1} \) and for all \( 0 < \lambda < 1 \). Where, \( W_{\lambda,t+1} = \lambda W_{1,t+1} + (1 - \lambda)W_{2,t+1} \).

\[
\lambda \mu_t(g_{t+1}(W_{1,t+1}, \nu_{t+1}))^{1-\gamma}/(1 - \gamma) + (1 - \lambda) \mu_t(g_{t+1}(W_{2,t+1}, \nu_{t+1}))^{1-\gamma}/(1 - \gamma) < (\leq) \mu_t(g_{t+1}(W_{\lambda,t+1}, \nu_{t+1}))^{1-\gamma}/(1 - \gamma)
\]

**Proof.** See Appendix A for a sketch and Internet Appendix for complete proof.

The optimization program ends at age 100 as \( p_{100} = 0 \). I numerically solve for the optimal
policy rules for all other periods by backward iteration. The value function scales with the
permanent income \( e^{\nu_t} \). The normalized value function \( J_t^\nu(W_t^\nu) \equiv J_t e^{-\nu_t} \) depends only on
one state variable \( W_t^\nu \equiv W_t e^{-\nu_t} \), the normalized wealth. I use simulated draws of shocks to
compute integrals in Eq. 19b. This step along with the fixed point problem in Eq. 19b severely
slow the solution speed. I describe the numerical method in more detail in Appendix B.

## 5 Calibration

### 5.1 Preference parameters and mortality

I follow Cocco, Gomes, and Maenhout (2005) and set the relative risk aversion \( \gamma_{crra} \) for the
CRRA preference model at 10. The DA and CRRA preferences are not equivalent, as the
CRRA model implies little risk aversion over small gambles unless \( \gamma_{crra} \) is extremely high.
However, a high \( \gamma_{crra} \) implies unrealistic risk aversion over large gambles. Consequently I
match the DA preference parameters at only one age. I choose age 64, just before retirement,
to perform the match. This choice is motivated by the fact that the portfolio implications
of the two preferences are in stark contrast right after retirement. I choose \( \gamma_{DA} = 6 \) and
\( \theta = 0.65^{17} \) so that the average savings and risk allocations are almost the same under the two

\[^{17}\text{The estimates by Choi, Fisman, Gale, and Kariv (2007) for } \theta \text{ range from 0 to 1.9. These are the } 5^{th} \text{ and}
the } 95^{th} \text{ percentiles in Table 1 for the symmetric case.} \]
The agent faces mortality risk from age 66 until age 100 and dies with probability 1 at age 100. I obtain estimates for $p_t$ from Arias (2006). The adult age starts at 20 for agents without college degrees and at age 22 for those with college degrees. All agents retire at age 65 irrespective of their educational attainment.

## 5.2 Labor income, housing expenditure and asset returns

I follow Cocco, Gomes, and Maenhout (2005) and estimate the labor income process in Eq. (17) and (18) using Panel Study of Income Dynamics (PSID) data from 1970 to 2009. I provide details of the estimation procedure in the Internet Appendix. I fit a third-order polynomial in age to the labor income profile implied by the age dummies for each of the three education groups. The polynomial coefficients and the estimates of the other parameters of the income process are in Table 2. I set the correlation between income and returns to zero for the benchmark case. \(^{19}\) I consider the case with positive correlation between permanent income and returns in Section 7.1.

I follow Gomes and Michaelides (2005) and estimate the fraction of annual income used for mortgage and rent payments, $h_t$, as a function of age, $t$, using the updated PSID sample from 1970 to 2009. I use a truncated version of this polynomial in Eq. 20 for $h_t$. The function truncates to zero for all ages 81 and above. The details about estimation are in the Internet Appendix.

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\(^{18}\)I thank an anonymous referee for suggesting this simpler method of matching the CRRA and DA preference parameters.

\(^{19}\)The estimates of the correlation between permanent income and returns are positive but small.
γ 6 (DA), 10 (CRRA)  
θ 0.65 (DA)  
β 0.96  
Age at the last non-retirement income, K 65  
Correlation of labor income and stock return shocks, 0  
Standard deviation of idiosyncratic labor income shocks, σ_ε 0.31  
Standard deviation of permanent labor income shocks, σ_u 0.09  
Standard deviation of log stock return shocks, σ_r 0.18  
Average excess risky return, (\bar{R} - R_f) 4%  
Riskless rate, (R_f - 1) 2%  

Table 3: Parameter values for the benchmark case.

\[ h_t = \max \left( 0.534 - 2.384 \times 10^{-2} \cdot t + 4.539 \times 10^{-4} \cdot t^2 - 2.990 \times 10^{-6} \cdot t^3, 0 \right) \]  

I follow the life-cycle portfolio allocation literature and set the equity premium at 4%.\textsuperscript{20} I set the risk-free rate at 2%, and the standard deviation of stock returns at 18%.

### 6 Benchmark case

In the following subsections, I analyze the case for high school graduates. Similar results follow for the college and no high school groups. I set the correlation between return shocks and income shocks to zero for the benchmark case. Table 3 lists all parameters for this case.

#### 6.1 Policy rules

The retirement policy rules for the optimal risky asset weight are shown in Fig. 5. These are functions of (tradable) wealth, \( W_t \), scaled by the permanent component of income, \( e^{\nu_6} \). They feature conservative allocation with (1) rising tradable wealth and (2) age. The drastic conservative attitude, the third feature of the DA preference model, beginning right at retirement, differentiates the DA model from the CRRA model. This conservative attitude overlays the rest of the retirement period under the DA preference model.

The retirement income is risk free and is a multiple of the permanent income \( e^{\nu_{65}} \) in the last working period. These two aspects are responsible for both the common as well as the differentiating features of the investment rules. The risk free retirement income creates a risk

\textsuperscript{20}The real returns on the assets are lower after considering holding costs and taxes. Jagannathan, McGrattan, and Scherbina (2000) argue that the equity premium is much lower than 6% due to diversification costs, taxes, and liquidity premium for bills. Claus and Thomas (2001) and Fama and French (2002) also argue for a low value of the equity premium.
Figure 5: The optimal portfolio (left panel) and consumption (right panel) policy rules around retirement and late old age for DA and CRRA preferences. These are graphed as functions of wealth scaled by permanent income.

free but non-tradable bond holding in the agent’s portfolio and turns the investment rules for the tradable wealth, \( W_t \), more aggressive. In effect the aggressive tilt attempts to offset the non-tradable bond holding. However, in the case of a large value for the tradable wealth the relative importance of the non-tradable bond is diminished and yields risky allocations that are more conservative. The result is a declining pattern in investment rules for a given age. Further, the stock of the riskless asset created by non-random retirement income also declines with age. This decline generates the second pattern, an increasingly conservative allocation with age for a given level of scaled wealth, \( W^\nu_t \). I illustrate the determinants of the third feature, unique to the DA preferences, with the help of simulations in the next subsection.

The risky allocation rules for the DA and CRRA preferences share similar features over the agent’s working life. The policy rules are decreasing in scaled wealth and grow conservative as the middle-aged agent grows older. In addition, the policy rules for the DA model grow aggressive with age in early life, similar to that for the CRRA preference model. This trend, as Cocco, Gomes, and Maenhout (2005) note, is due to the rise in human capital stock in early life as income is lower but income growth is steeper over this phase of life.

The consumption policy rules in terms of scaled consumption as a function of scaled wealth are similar for the DA and CRRA preferences. The consumption rules are concave in tradable wealth and also indicate that the agent decumulates savings in retirement. This pattern is due to increasing mortality-induced impatience. The chances of surviving to the next year decline with increasing age and the mortality is certain at age 100. Thus the benefits to savings depreciate with age and consequently accelerate the decumulation. However, the decumulation
motive is stronger for the DA than for the CRRA agent. The difference is driven by the more conservative investments of the DA agent through retirement that yield lower average benefits to savings. The rules are similar over working life under both preferences except that the DA agent accumulates savings at a more aggressive rate than does the CRRA agent.

6.2 Simulations

I simulate 10,000 independent income paths and obtain consumption and investment decisions along these paths using optimal policy rules at every age. I contrast the DA and CRRA agents’ behavior using average stock allocation, consumption and savings over life.

The risky allocations decline with age for both the DA and CRRA agents and the DA agent’s drastic conservative approach to investment in retirement is noticeable starting at age 65 (Fig. 6, top-left panel). This attitude is due to the resolution of permanent income uncertainty in the last working period that sets the retirement income for remaining life periods. The result is a resolution of uncertainty in a large stock of human capital around retirement as observed in the drop in $\sigma_t(\ln(J_{t+1}))$. In addition to the sizeable stock of retirement income the agent’s savings are also the highest around the retirement period. Thus the phenomenon around retirement is similar to that associated with the passage from modest income risk to no income risk in the one-period problem with moderate values of income to savings ratio.

Fig. 6 shows that the twisted premium, $\hat{R}_{e,t+1}$, the summary of the DA effect, drops by 2.2% as income transitions from uncertainty to certainty with the drop of 8.6% in disappointing return premium, $\hat{R}_{e,d}$, being the dominant driver for the change. The rise in disappointment probability, $\Phi$, also contributes to the decline in $\hat{R}_{e,t+1}$ to a limited extent as the change in $\Phi$ is small (increment of 0.08). The $\Phi$ is related to the skewness of utility $u(J_{t+1}) \equiv J_{t+1}^1/(1-\gamma)$.

In effect the extra emphasis on disappointing states built into the DA model coupled with the increasing association of adverse returns with the low wealth realizations (reflected in $\hat{R}_{e,d}$) dampens the appetite for the risky asset. The drastic decline in background risk creates sharp changes in the return and wealth associations and consequently the sharp change in risky allocation. The decline also highlights the positive relationship between risky investment and uninsurable risk in late working life that is unique to DA preferences. These preferences can generate such drop into retirement as long as the total uninsurable or background risk through income and other possible sources declines into retirement. The post-retirement phase

\footnote{\textsuperscript{21}Similar to the one-period problem the right-skewness of $u(J_{t+1})$ is positively related to $\Phi$. See Internet Appendix and Section 3 for a detailed treatment of one-period problem.}
Figure 6: The cross-sectional average of the fraction of savings invested in risky assets (top-left panel) and the standard deviation of the log of the gamble over next period value function (bottom-left panel) at all ages over the life of DA and CRRA agents. These averages are conditioned on positive savings. The cross-sectional average of the twisted return premium and disappointment return premium conditional on positive savings for the DA agent are in top-right panel. The cross-sectional averages of the skewness of utility and the disappointment probability for the DA agent are in bottom-right panel.
is associated with many risks and most notably the health risk.\footnote{The changes in family composition can also act as additional sources of risk (Love (2010)).} I explore its impact on stock allocation in Section 7.6.

The risky allocations rise in retirement in absence of a bequest motive. As Cocco, Gomes, and Maenhout (2005) note, the rising mortality-induced impatience implies that the agent runs through wealth rapidly. As a result, the portfolio rules are evaluated at lower tradable wealth. This interaction between portfolio and consumption rules yields rising average optimal risky allocations and hence the pattern in retirement. I incorporate bequest in the DA preference model in Section 7.4.

The DA and CRRA agents save little over early working life (Fig. 7, left panel) and hold a large stock of human capital that they are yet to encash. Consequently, the stock allocations are high but limited by the borrowing constraint over this phase of life. The risky investment behavior is as if the human capital is more bond-like than stock-like. Both agents accumulate savings throughout working life and reduce risky investment as human capital declines with age.

The overall consumption patterns are similar for both the agents. However, the relatively uneven consumption profile for the DA model is a result of over-dependence, especially in retirement periods, on risk-free asset for consumption smoothing. The aversion to attractive but risky stock imposes a higher cost to achieve a given level of smoothing and hence the more uneven pattern.

Figure 7: The cross-sectional mean savings (left panel) and consumption (right panel) at every age over the life of the CRRA and DA agents.
Figure 8: Left panel: cross-sectional average stock allocation conditional on positive savings for the CRRA and DA preferences for the case of correlated income and return. Right panel: cross-sectional average twisted return premium for the benchmark and correlated case for DA preferences.

7 Model extensions

7.1 Labor income correlated with stock market returns

I consider the effect of correlation on the DA agent’s risky allocations. Benzoni, Collin-Dufresne, and Goldstein (2007) use co-integration between income and dividends to obtain aversion to stocks in early-working life. Cocco, Gomes, and Maenhout (2005) find that correlated labor income helps explain lower stock investments in the early stages of working life. I follow Gomes and Michaelides (2005) and set the correlation between the permanent component of labor income and stock returns at 0.15 and idiosyncratic income shocks uncorrelated with the returns.

The correlation dilutes the semblance of human capital to a bond and makes it more stock-like. The correlation yields hedging demands that diminish stock investment through working life for both the DA and CRRA preferences (Fig. 8, left panel). The retirement income, however, is riskless which removes any hedging motive in retirement. These changes reflect in the rising stock investments as the CRRA agent transitions into retirement. This pattern also applies in presence of the bequest motive. Such pattern is contrary to the observed reduction in risky allocation with age. In contrast, the DA model continues to generate a drop even in the presence of correlation. Similar to the benchmark case, the DA agent’s stock allocation drops with the fall in the twisted premium (Fig. 8, right panel). However, hedging demands prior to retirement reduce the size of the drop at the onset of retirement (Fig. 8, left panel).
Figure 9: Average stock allocation (left panel) and twisted return premium (right panel) for DA preferences with different $\theta$. The twisted premium is not defined for $\theta = 1.3$ for age 65 and onwards.

7.2 Heterogeneity in disappointment aversion

Households exhibit considerable heterogeneity in their portfolio allocation (Curcuru, Heaton, Lucas, and Moore, 2004). The most notable variation is in their choice to participate in the stock market. The various samples of Survey of Consumer Finances find that only approximately half of the U.S. households participate in the stock market either directly or indirectly. Further, the age effects also indicate a decline in the propensity to participate in the equity market in retirement (Section 2). The international evidence also points to a significant heterogeneity in portfolio choice (Guiso, Haliassos, and Jappelli (2002)). I show that heterogeneity in disappointment aversion can generate substantial variation in portfolio choice and in particular the participation choice in retirement. I graph risky allocations and twisted premium for three different values of disappointment aversion parameter $\theta = \{0.3, 0.65, 1.3\}$ in Fig. 9.\(^\text{23}\) The rest of the parameters and the setup are the same as that in the benchmark case.

The increment in the weight on disappointing returns with the increase in $\theta$ lowers the twisted premium (Fig. 9, right panel). The increase in $\theta$ from 0.3 to 1.3 lowers the stock investments by about a half over mid-working life through retirement (Fig. 9, left panel). Thus the variation in $\theta$ alone generates substantial variation in investment behavior. In addition if $\theta$ is high enough then the agent does not invest in the risky stock at all in retirement. In contrast, the CRRA agent invests at least some amount in the risky stock in retirement for

\(^\text{23}\)These values are well within the experimental estimates of disappointment aversion parameter Choi, Fisman, Gale, and Kariv (2007).
all levels of risk aversion. The approximate value of the threshold disappointment aversion, \( \theta^* \), depends on the distribution of returns alone (Eq. 21). I include the proof in the Internet Appendix. The formula also applies in presence of the bequest motive described in Section 7.4. Ang, Bekaert, and Liu (2005) show that Eq. 21 provides the exact value of the threshold in the case of one-period problem without any non-tradable income. The twisted premium is non-positive and the welfare declines with an infinitesimal investment in the risky asset if \( \theta > \theta^* \). In addition the agent optimally chooses not to short the stock. Consequently non-participation is the optimal policy.

\[
\hat{\theta}^* = -\frac{E_t [R_{e,t+1}]}{E_t [R_{e,t+1} | R_{e,t+1} < 0] \times \Phi(R_{e,t+1} < 0)}
\]  

Similar to the influence of income risk on risky asset demand in Proposition 3.1 the uninsurable risks in retirement can drive the agent toward risky investment for the cases with \( \theta > \theta^* \). The uninsurable risks such as out-of-pocket medical expenses in Section 7.6 are either unrelated or positively related to asset returns and can engender expenses in excess of retirement income. Thus these risks increase the dependence on financial wealth and may imply only negligible risky allocations. In addition, the loss in welfare from complete non-participation in retirement under the DA model may as well be limited given the nature of these risks. Further, the universal health plans in some countries limit the exposure to out-of-pocket medical expense risks, an important uninsurable risk in retirement (Morgan, Mueller, and Diener (2013), Peterson and Burton (2007)). The DA model can help explain non-participation in retirement in such environments.

The changes in \( \theta \) also affect consumption-savings decisions. The average consumption through life is lower for higher \( \theta \) and the reliance on less rewarding risk-free asset implies that more savings are needed to last through the post-retirement period. Thus the savings for retirement increase with \( \theta \). Further, the agent with higher \( \theta \) also runs through the savings faster in retirement.

### 7.3 Defined Contribution plan

The employers are increasingly moving away from Defined Benefit (DB) to Defined Contribution (DC) plans. I consider a simplified model of DC benefits that excludes all retirement income including social security transfers and treat DC investment plans as a part of the

---

24If the agent has CRRA preferences then an infinitesimal investment in risky assets leaves marginal utility independent of returns. Thus the welfare improves if the agent invests at least some amount in risky assets.

25\( \theta^* = 0.72 \) for the asset return distribution in the benchmark case.
complete portfolio.  

I use the income process from the benchmark case and set retirement income at age 66 and onwards to zero. The uninsurable risk dwindles throughout working life without a sizeable drop at any age given the DC plan. This pattern in uninsurable risk generates risky allocations and twisted return premium devoid of a large drop (Fig. 10) unlike that in the benchmark case. The small drop in stock investment at retirement is the result of a transition from uncertain income in the last working period to no income thereafter. The absence of non-financial income also avoids the increasing pattern in risky allocations in retirement for both DA and CRRA preferences. However, the gradual declining pattern in Fig. 10 for the DA model is not unique to DC retirement plans. I obtain a gradual decline in risky allocations with DB retirement plans if the agent is subject to heteroscedastic income process described in Section 7.5.

### 7.4 Bequest

I add the bequest motive to the benchmark case in Eq. (22a) and (22b). $\mu_t$ is the certainty equivalent over the future welfare as defined earlier, whereas, $\mu_{b,t}$ is the certainty equivalent of the gamble over bequeathed wealth modified by parameter b. The welfare, $J_t^{1-\gamma}/(1 - \gamma)$,

---

26 The agents can borrow against the savings in DC accounts or liquidate their savings from these accounts after paying a penalty.

The twisted premium in Fig. 11 right panel is little future welfare and the gamble over bequeathed wealth in the event of demise. However, alternate formulations may not be concave. This is the first article to formulate bequest motive with DA preferences and show that the proposed formulation yields concave preferences.

\[
\frac{J_t(W_t, \nu_t)^{1-\gamma}}{1-\gamma} = \max_{C_{t+1}, \nu_{t+1}} \frac{C_{t+1}^{1-\gamma} + p_t \beta \mu_t (J_{t+1}(W_{t+1}, \nu_{t+1}))^{1-\gamma}}{1-\gamma} + (1-p_t) \beta b \mu_{b,t} (W_{t+1}/b)^{1-\gamma} \tag{22a}
\]

\[
\mu_{b,t}^{1-\gamma} = \mathbb{E}_t \left[ (W_{t+1}/b)^{1-\gamma} \right] - \theta \mathbb{E}_t \left[ (\mu_{b,t}^{1-\gamma} - (W_{t+1}/b)^{1-\gamma}) I (W_{t+1} < b\mu_{b,t}) \right] \tag{22b}
\]

The agent lives until age \( T = 100 \), and the terminal valuation \( J_{T+1}^{1-\gamma}/(1-\gamma) \) due solely to the bequest is encoded as \( b(W_{T+1}/b)^{1-\gamma}/(1-\gamma) \). The parameter \( b \) captures the intensity of the bequest motive. I follow Gomes and Michaelides (2005) and set \( b = 2.5 \) for DA model and retain all other parameter values from the benchmark case.\(^{28}\) The twisted premium in the first-order condition\(^{29}\) in Eq. (23) is a weighted sum of that pertaining to the gamble over future welfare and the gamble over bequeathed wealth in the event of demise.\(^{30}\)

\[
\hat{E}_t [R_{e,t+1}] = -\frac{p_t}{a_t} Cov_t \left[ C_{t+1}^{-\gamma}, R_{e,t+1} \chi_{1,t+1} \right] - \frac{1-p_t}{a_t} Cov_t \left[ (W_{t+1}/b)^{-\gamma}, R_{e,t+1} \chi_{2,t+1} \right] \tag{23}
\]

\[
\hat{E}_t [R_{e,t+1}] = \left[ a_{1,t} \hat{E}_{1,t} [R_{e,t+1}] + a_{2,t} \hat{E}_{2,t} [R_{e,t+1}] \right]/a_t
\]

\[
\chi_{1,t+1} = \frac{1 + \theta I (J_{t+1} < \mu_t)}{1 + \theta \Phi (J_{t+1} < \mu_t)}; \ \chi_{2,t+1} = \frac{1 + \theta I (W_{t+1} < b\mu_{b,t})}{1 + \theta \Phi (W_{t+1} < b\mu_{b,t})}
\]

\[
a_{1,t} = p_t E_t \left[ C_{t+1}^{-\gamma} \right]; \ a_{2,t} = (1-p_t) E_t \left[ (W_{t+1}/b)^{-\gamma} \right]; \ a_t = a_{1,t} + a_{2,t}
\]

\[
\hat{E}_{i,t} [\cdot] = E_t [\cdot] \times \chi_{i,t+1} \text{ for } i = 1, 2
\]

\(^{28}\) I use a higher value \( b = 4 \) to avoid increasing risky allocation pattern in retirement for the CRRA model.\(^{29}\) See Internet Appendix for derivation.\(^{30}\) The first-order condition can also be written as \( \hat{E}_t \left[ R_{e,t+1} \right] = -\left( p_t/k_t \right) Cov_{1,t} \left[ C_{t+1}^{-\gamma}, R_{e,t+1} \right] - (1 - p_t/k_t) Cov_{2,t} \left[ W_{t+1}^{-\gamma}/b^{-\gamma}, R_{e,t+1} \right] \). The twisted premium \( \hat{E}_t [R_{e,t+1}] \) is a weighted sum of \( \hat{E}_{1,t} [R_{e,t+1}] \) and \( \hat{E}_{2,t} [R_{e,t+1}] \) with weights \( k_{1,t} \) and \( k_{2,t} \) respectively, where \( k_{1,t} = p_t \hat{E}_{1,t} [C_{t+1}^{-\gamma}], k_{2,t} = (1-p_t) \hat{E}_{2,t} [W_{t+1}^{-\gamma}/b^{-\gamma}] \) and \( k_t = k_{1,t} + k_{2,t} \).
changed compared with the benchmark case. Similar to the benchmark case, the decline with age in the capitalized value of retirement income generates increasingly conservative investment rules. The motive to bequeath wealth slows the fast pace of savings decumulation observed in the benchmark case. The slower decumulation implies that the age pattern in investment rules is reflected in the average allocations. Thus, the bequest motive is the driver behind the increasingly conservative allocations in retirement. However, the twisted premium decline around retirement sets a far more conservative baseline during retirement for the DA model in comparison with that for the CRRA model. The bequest motive adds to this baseline and turns it more conservative as the mortality chance increases. In addition, the intent to bequeath wealth also provides an added motive to save, which yields higher savings over working life.

### 7.5 Heteroscedastic income risk

The labor income is homoscedastic in the benchmark case and in related extensions. Both the permanent, $u_t$, and the idiosyncratic, $\epsilon_t$, shocks add to the uncertainty in log-labor income at age $t$. Fig. 12 graphs the estimates of spread in log-labor income due to these two shocks, $\sqrt{\sigma_u^2 + \sigma_e^2}$, with age $t$.\(^{31}\) The shocks are heteroscedastic and their size varies with age. The per-period income is more uncertain over young working life than when the agent is closer

---

\(^{31}\)See Internet Appendix for the estimation procedure.
Figure 12: The graphs of the point estimates of the spread in per-period shocks to log labor income till retirement. Left-panel: the estimates for each of the three education groups. Right-panel: 95% bootstrap confidence intervals and point estimates for the high school educated group.

to retirement.\textsuperscript{32} Hence the homoscedastic model is a simplified representation of income uncertainty. This model also implies a large decline in uninsurable risk around retirement and results in an abrupt drop in risky allocation. However, the data indicates a gradual drop in portfolio share of stocks (Section 2). The differences in retirement age across agents or a progressive decline in per-period income risk or the two together can help match the observed profile. I show the effects of tapered decline in per-period income risk that highlights the positive relation between risky investment and income risk in the later phase of working life. This is a feature that is unique to DA preferences.

I use a parsimonious heteroscedastic income process, Eq. (24a)-(24c), that accounts for both riskless income after retirement at age 65 and a declining per-period income uncertainty. $l_t$ in Eq. (24a) is the same deterministic component of income described in the benchmark case. I set $t_{\text{stop}} = 66$ and $t_{\text{start}} = 20$. This sets $h_\epsilon = h_u = 1$ at age 20 and $h_\epsilon = h_u = 0$ right after retirement at age 66. The per-period spread in log-income, $\sqrt{\sigma^2 h_\epsilon^2(t) + \sigma^2 h_u^2(t)}$, drops linearly over the working life. The retirement income process after age $K = 65$ and other parameters are the same as that in the benchmark case. The bequest intensity $b$ is set at 2.5, the same as that in Section 7.4 for the DA model.\textsuperscript{33} The estimates of $\sigma_\epsilon$ and $\sigma_u$ for this heteroscedastic process are in Table 4. I provide details of the estimation procedure in the Internet Appendix.

\textsuperscript{32}See Baker and Solon (2003) for a study of Canadian income data; they find an age-related U-shaped pattern in idiosyncratic income uncertainty.

\textsuperscript{33}The CRRA model requires a higher value $b = 4$ to avoid increasing risky allocation pattern in retirement.
Table 4: Estimates of $\sigma_u$ and $\sigma_\epsilon$ for heteroscedastic income process in Eq. (24a)-(24c). $t_{\text{stop}} = 66$ and $t_{\text{start}} = 20$ for all education groups except $t_{\text{start}} = 22$ for college-educated group.

<table>
<thead>
<tr>
<th></th>
<th>No High School</th>
<th>High School</th>
<th>College</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_\epsilon$</td>
<td>0.4097</td>
<td>0.3206</td>
<td>0.2840</td>
</tr>
<tr>
<td>$\sigma_u$</td>
<td>0.1188</td>
<td>0.1338</td>
<td>0.1566</td>
</tr>
</tbody>
</table>

The gradual decline in total uninsurable risk implied by the heteroscedastic income process generates a phased drop in stock investment along with the steadily dwindling twisted return premium (Fig. 13). The changes in the disappointing return premium, $\bar{R}_{e,d}$, and the disappointment probability, $\Phi$, the two components of the DA effect, are also graded and not abrupt (not included). The stock allocations are similar in early working life under the DA and CRRA preferences, but only the DA model implies lower allocations during late working life and in retirement on account of the decline in income risk.

$$\ln(Y_t) = l_t + \nu_t + h_t(t)\epsilon_t \quad \forall t \leq K$$

(24a)

$$\nu_t = \nu_{t-1} + h_u(t)u_t \quad \text{Where } u_t \sim N(0, \sigma_u^2) \quad \epsilon_t \sim N(0, \sigma_\epsilon^2)$$

(24b)

$$h_t(t) = h_u(t) = (t_{\text{stop}} - t)/(t_{\text{stop}} - t_{\text{start}})$$

(24c)
7.6 Health risk

The declining human capital is the dominant force that turns investments conservative under the CRRA model. In addition the drop in income risk in late working life also contributes to this pattern under the DA model. However, the late phase of life also coincides with the rise in a notable source of risk, the health risk. Its impact through uncertain out-of-pocket medical expenses is a mild rise in risky allocation in retirement under the DA model. This pattern is neither a case in favor nor a case against the DA model as the empirical evidence is mixed. In addition the net effect of health risk under the standard expected utility model is also ambiguous.

Edwards (2008, 2010) notes that health shocks can increase or decrease the marginal utility of consumption. The sign of the effect on portfolio choice depends on the complementarity of health with consumption and leisure. Some studies find that the decline in health increases the marginal utility (Lillard and Weiss (1997), Edwards (2008)) while others conclude the opposite (Finkelstein, Luttmer, and Notowidigdo (2013), Viscusi and Evans (1990)). Further, Love and Perozek (2007) show that if the survival risk and the expected survival horizon depend on health shocks then, all else equal, an increase in survival risk increases the risky share of the portfolio in some cases. The health shocks are also associated with significant out-of-pocket medical expenses (De Nardi, French, and Jones (2006)). Pang and Warshawsky (2010) show that such uncertain expenses act as a background risk and imply safer investment portfolios. Thus as a net effect an adverse health shock may yield a rise or a drop in risky share under the standard model.

The empirical evidence on the relation between health status and risky allocation also is mixed. Rosen and Wu (2004) and Berkowitz and Qiu (2006) find positive relation between poor health and lower risky allocation. Edwards (2008) finds that the self-perceived health risk may explain 20% of age-related decline in risky investment after retirement. However, Love and Smith (2010) find that an agent’s health status accounts for only a small causal effect on portfolio choice. They find that after controlling for unobserved heterogeneity, health does not significantly affect portfolio choice among single households and report only a small (2-3 percentage points) effect for married households and for those in the lowest health categories.

I consider the impact of out-of-pocket (OOP) medical expenses under the DA model. The parsimonious process for OOP expenses, $M_t$, in Eq. 25 avoids additional state variables and contains the computational cost. I limit the maximum value of $M_t$ at $\phi Y_t$. I subtract this

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35 Yogo (2009) notes that some of the medical spending is in fact an endogenous investment in health which can lower health related background risk.
36 Fan and Zhao (2009) and Coile and Milligan (2009) also report similar relation between health and risky allocation.
<table>
<thead>
<tr>
<th></th>
<th>College</th>
<th>High School</th>
<th>No High School</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>8.0951</td>
<td>3.7047</td>
<td>6.6583</td>
</tr>
<tr>
<td>Age</td>
<td>-0.3064</td>
<td>-0.1939</td>
<td>-0.2657</td>
</tr>
<tr>
<td>Age^2/100</td>
<td>0.2268</td>
<td>0.1573</td>
<td>0.2001</td>
</tr>
<tr>
<td>σ_t</td>
<td>0.71544</td>
<td>0.80816</td>
<td>0.7836</td>
</tr>
</tbody>
</table>

Table 5: The Constant, Age, Age^2/100 rows provide coefficients for the 2nd-order polynomial in age for numerical implementation of the deterministic part of out-of-pocket medical expense profile, q_t, for each of the three education groups. The σ_t row includes the estimates for the standard deviations of the process.

censored medical expense, \( \tilde{M}_t \), from available resources in each retirement period in Eq. 26. I estimate the process in Eq. 25 separately for each of the three education groups using households from 1992 to 2010 waves of Health and Retirement Study (HRS) and restrict to those households with the head older than 65. I provide details in the Internet Appendix. \( q_t \) in Eq. 25 captures the deterministic age effect. Table 5 lists coefficients for the 2nd-order polynomials used to fit the regression estimates of the age effects and also the spread of \( \iota_t \) shocks. I set \( \varphi = 1.24 \) and find only 3% of the observations with the ratio \( M_t/Y_t \) above this value in the uncensored raw data. I retain the heteroscedastic income process from Section 7.5.

\[
\begin{align*}
M_t &= Y_t \exp(q_t + \iota_t) \quad \forall t > 65, \text{ where } \iota_t \sim N(0, \sigma_t^2). \\
W_t &= A_{t-1} R_{p,t} + (1 - h_t) Y_t - \tilde{M}_t \quad \forall t > 65, \text{ where } \tilde{M}_t = \min\{\varphi Y_t, M_t\}.
\end{align*}
\] (25) (26)

The medical expense shocks can be severely disastrous as they can be higher than the retirement income. The effect is to increase the dependence on financial wealth for consumption. The result is a portfolio shares pattern rising in wealth at low values of wealth (Fig. 14, top-right panel). In addition the agent saves and does not invest the present value of the stream of highest future medical expenses in excess of retirement income, \( (\tilde{M}_t - Y_t)_{max} \), in risky asset in any period. The result is a higher wealth threshold for stock participation as indicated in the policy rules in the top-right panel of Fig. 14. This behavior is also the reason for using \( \tilde{M}_t \), the bounded version of \( M_t \), in Eq. 26. The bound on \( M_t \) is consistent with other possible essential expenses and a consumption floor insured through government policy.

The increment in portfolio shares in retirement till the agent is in early 80s is due to the rising twisted premium over this phase (Fig. 14). The rise in the variance of \( M_t/Y_t \) due to the rise in the mean value of \( \ln(M_t/Y_t) \) with age is the source behind this pattern. The increase in the expected value of OOP expenses with age also turns expense shocks more severe and raises the curvature component of the marginal utility, the standard effect. It tempers the
impact of rising twisted premium. The twisted premium, however, drops\textsuperscript{37} later in retirement and both the effects induce safer portfolio choices over the terminal phase.

The average risky allocation in retirement is 6.7\% in Fig. 14, up from 3.4\% in absence of uncertain OOP expenses. These two cases help segregate the impact of uncertain OOP expenses. In contrast to the simplifying assumption separating income and OOP expense uncertainty in the current setup, the agent is likely to be subject to significant uncertain OOP expenses while still earning uncertain income. The delay in retirement might be one reason for such pattern and a more complex setup can capture this feature. The overlap between the two uncertainties can avoid the mildly increasing age pattern in stock allocation in retirement observed under the current setup.

8 Epstein-Zin preferences with DA aggregator

Gomes and Michaelides (2005) show that separating risk aversion from the elasticity of intertemporal substitution (EIS) and adding a one-time entry cost to participate in the equity market creates a recipe to match the equity participation rates. The agents with high risk aversion and high EIS turn out to be the dominant participants in the equity market whereas the agents with low risk aversion and low EIS mostly abstain from the equity market. The low

\textsuperscript{37}This drop is akin to a similar drop in one period model at high income risk.
risk averse and low EIS agents save a small buffer stock and lack a stronger drive to save for the retirement. The result is that such agents are unwilling to pay the small one-time entry cost. The small fraction of these agents that do pay the entry cost invest almost all their savings in the risky asset. I show that adding disappointment aversion helps lower the risky investments of such agents to some extent. Further disappointment aversion also increases wealth accumulation and lowers the risky investment over the lifetime (Section 7.2). These trends have opposing effects on participation decision. The higher savings imply that the agent is more likely to pay the market entry cost. Whereas, holding the wealth accumulation constant, the agents with lower optimal investment in stocks are less willing to pay the participation cost. However, the net effect is an increase in the participation rate with the increasing disappointment aversion parameter.

I follow Gomes and Michaelides (2005) and consider a model with preference heterogeneity to match participation rate, conditional equity allocation and the distribution of financial wealth to income ratio. I consider a population of investors equally split between two sets of preference parameters. The 50% of investors have low EIS, a low value for the curvature component of risk aversion and low disappointment aversion parameter ($\psi = 0.18, \gamma = 1.1, \theta = 0.35$), while the other 50% have moderate values for these parameters ($\psi = 0.4, \gamma = 4, \theta = 0.6$). The preference formulation in Eq. (27a)-(27c) follows Epstein and Zin (2001) and includes the bequest motive. I use heteroscedastic income process from Section 7.5 with a correlation of 0.15 between permanent income and return shocks and set bequest intensity, $b$. 

Figure 15: The participation rates and conditional equity allocations over life for two different sets of preference parameters. The left panel corresponds to an agent with low EIS, low curvature component of risk aversion and low disappointment aversion. The right panel corresponds to an agent with moderate values for the same set of parameters.
The agent pays a fixed cost equaling fraction $F$ of the permanent income $Y_{p,t+1}$ to enter the equity market (Eq. 27d). $I_p$ is the indicator variable set to one if the agent decides to pay the cost or otherwise the indicator is set to zero. I set entry cost at 4% of the permanent income. Eq. 27e is the budget equation after the agent enters the equity market. This setup, except for the slightly higher participation cost, follows Gomes and Michaelides (2005).

\[
J_t = \max_{C_t,x_t} \left[ (1 - \beta) C_t^{1-1/\psi} + \beta \left[ p_t \mu_t^{1-\gamma} + (1 - p_t) b \mu_{b,t}^{1-\gamma} \right]^{1-1/\psi} \right]^{1/1-\psi} \quad (27a)
\]

\[
\mu_t^{1-\gamma} = E_t \left[ J_{t+1}^{1-\gamma} \right] - \theta E_t \left[ \mu_t^{1-\gamma} - J_{t+1}^{1-\gamma} I(J_{t+1} < \mu_t) \right] \quad (27b)
\]

\[
\mu_{b,t}^{1-\gamma} = E_t \left[ (W_{t+1}/b)^{1-\gamma} \right] - \theta E_t \left[ (\mu_{b,t}^{1-\gamma} - (W_{t+1}/b)^{1-\gamma}) I(W_{t+1} < b \mu_{b,t}) \right] \quad (27c)
\]

\[
J_T = \max_{C_T,x_T} \left[ (1 - \beta) C_T^{1-1/\psi} + \beta \left[ b \mu_{b,T}^{1-\gamma} \right]^{1-1/\psi} \right]^{1/1-\psi} \quad (27c)
\]

\[
W_{t+1} = (W_t - C_t) R_f + (1 - h_{t+1}) Y_{t+1} - F I_p Y_{p,t+1}, \text{ where } Y_{p,t+1} = e^{\eta_{t+1} + \nu_{t+1}}. \quad (27d)
\]

\[
W_{t+1} = (W_t - C_t) R_{p,t+1} + (1 - h_{t+1}) Y_{t+1} \quad (27e)
\]

The low EIS and low risk-aversion (low curvature and disappointment aversion component) group saves little and only a small fraction of the agents from this group enter the equity market (Fig. 15 left panel). In contrast to the agents with CRRA aggregator and similar parameters (but $\theta = 0$) the equity market participants from this group reduce their allocation to stocks as they progress through their working life. The declining twisted premium lowers the appetite for risky assets. The higher savings accompanying higher $\gamma$ and $\theta$ imply that the agent eagerly pays the entry cost despite the lower risky weight over the lifetime.

I compare the model implied average participation rates and average conditional allocations for different age groups with the age effects implied by Survey of Consumer Finances (SCF) data in Fig. 16. I use the age effects from Fig. 1 which control for cohort effects and use the fraction of households with DC and/or IRA accounts in each survey year as the time control. The model and the data differ in participation rates in later phase of life. One way to improve the match over this phase is to incorporate a small group of investors with disappointment aversion parameter above the critical threshold. Further, the conditional allocations have sizeable differences in the early phase of life. The model implies substantially higher allocations over this phase.

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38 The SCF data includes cohorts born in 1910 through 1985 and the samples are drawn from triennial surveys from 1989 to 2010 with an additional survey in 2009. I use cohort effect for those born in 1950 and time effect for the survey year 2001, the respective midpoints, in Fig. 16.
Figure 16: The participation rates (left panel) and conditional equity allocations (right panel) implied by the model and SCF data for different age groups. The model involves 50% of investors with low EIS and low risk-aversion (curvature component and disappointment aversion) and the other 50% with moderate values for the same preference parameters. The data points from SCF are the age effects in Fig. 1 that control for cohort effect and use fraction of households with DC/IRA accounts in a survey year as the time control. I fix time effect to that of year 2001 and cohort effect to that of 1950.

The inclusion of low probability disastrous zero income shocks as that in Carroll (1997) helps lower the conditional allocations. However, this feature also implies higher participation rates due to higher savings by the agents to insure against such shocks.

I compare the wealth accumulation pattern implied by the model with that from SCF in Table 6. I follow Gomes and Michaelides (2005) and compare the distributions over three age groups: buffer stock savers (20-35), retirement savers (36-65) and retirees (66 and higher). I use all survey samples to compute the distribution in the data. Similar to the model in Gomes and Michaelides (2005) the DA aggregator model also comes close to matching the wealth accumulation pattern of poorer households except in the retirement phase. However, the extent to which the EZ-DA model overshoots the wealth to income ratio for the median household is substantially higher. The DA aggregator model also yields larger wealth accumulation at the extreme but the quantity is not as high as that in the retirement phase in the data.

9 Conclusion

I study the life-cycle portfolio problem of a disappointment averse (DA) agent and find that the declining income risk in late working life provides an additional motivation to reduce risky allocation. The DA model generates an age-dependent relationship between income
Table 6: The distribution of wealth-to-labor income ratios for different age groups. Top panel: the distribution from SCF data using all survey samples from 1989 to 2010. Bottom panel: the distribution generated from the model using initial wealth distribution from SCF data.

<table>
<thead>
<tr>
<th>Age groups</th>
<th>SCF data</th>
<th>36-65</th>
<th>≥ 66</th>
</tr>
</thead>
<tbody>
<tr>
<td>20-35</td>
<td>0.001</td>
<td>0.033</td>
<td>0.187</td>
</tr>
<tr>
<td>10(^{th}) percentile</td>
<td>0.008</td>
<td>0.008</td>
<td>0.010</td>
</tr>
<tr>
<td>Median</td>
<td>0.331</td>
<td>1.790</td>
<td>6.365</td>
</tr>
<tr>
<td>90(^{th}) percentile</td>
<td>2.496</td>
<td>8.555</td>
<td>29.334</td>
</tr>
<tr>
<td>90(^{th}) percentile</td>
<td>1.201</td>
<td>8.941</td>
<td>18.470</td>
</tr>
</tbody>
</table>

risk and risky portfolio shares in congruence with the suggestive empirical evidence. I prove that the welfare under DA preferences with a positive curvature parameter and non-negative disappointment aversion parameter is concave in wealth. The addition of the DA aggregator to a heterogeneous Epstein-Zin model with a one-time equity market participation cost helps lower the conditional equity allocations of agents with the low elasticity of intertemporal substitution and low risk aversion. The DA model also suggests that a heteroscedastic income process better matches the observed equity allocations. The DA preferences also generate non-participation in equity markets in retirement if the agent’s aversion to disappointment is strong enough. Such non-participation occurs despite the fact that the equity premium is positive and no other frictions are present. I also relate the life-cycle results to a one-period investment problem with uncertain labor income. I show that the DA preferences generate a hump-shaped pattern in risky portfolio share with income risk. I prove that the risky portfolio share must increase for a sufficiently disappointment averse agent if the income and returns are independent.

This article suggests that background risk is a useful lens through which to probe unique features of competing preference models and a useful tool to limit the set of preferences to those that are consistent with the observations.
A Proofs

Proposition A.1. The optimal welfare of a DA agent, $\mu_{da,t}(\sigma_Y)$, endowed with savings, $A_t$, and facing a one-period investment problem while earning a non-tradable income, $Y_{t+1}$, with standard deviation $\sigma_Y$ declines with the mean-preserving increase in the income risk if $\theta \geq 0$ and $\gamma > 0$. $\mu_{da,t}(\sigma_Y)$ represents the maximum welfare corresponding to income with a standard deviation of $\sigma_Y$ at the optimal portfolio weights:

$$\mu_{da,t}(\sigma_Y) \equiv \mu_{da,t}(\sigma_Y, x^*_t) \equiv \max_{x_t} \mu_{da,t}(\sigma_Y, x_t)$$

Where, $x^*_t = \arg\max_{x_t} \mu_{da,t}(\sigma_Y, x_t)$.

Proof. Let the income with a standard deviation of $\sigma_{Y,2}$ represent a mean-preserving spread over income with the standard deviation equal to $\sigma_{Y,1}$ that is; $\sigma_{Y,1} < \sigma_{Y,2}$. Let $x^*_t,1$ and $x^*_t,2$ be the optimal portfolio weights corresponding to standard deviation of income $\sigma_{Y,1}$ and $\sigma_{Y,2}$, respectively. Thus, $\mu_{da,t}^*(\sigma_{Y,1}) = \mu_{da,t}(\sigma_{Y,1}, x^*_t,1)$ and $\mu_{da,t}^*(\sigma_{Y,2}) = \mu_{da,t}(\sigma_{Y,2}, x^*_t,2)$.

Note that the definition of optimal welfare $\mu_{da,t}^*(\sigma_{Y,1})$ implies the following:

$$\mu_{da,t}(\sigma_{Y,1}, x^*_t,1) \geq \mu_{da,t}(\sigma_{Y,1}, x^*_t,2) \Rightarrow \mu_{da,t}^*(\sigma_{Y,1}) \geq \mu_{da,t}(\sigma_{Y,1}, x^*_t,2). \quad (A.1)$$

Theorem 3 in Gul (1991) implies the conditions $\gamma > 0$ and $\theta \geq 0$ are sufficient for the second-order stochastic dominance for DA preferences. This property yields the first inequality below, and combining this inequality with Eq. A.1 yields the final result:

$$\mu_{da,t}(\sigma_{Y,1}, x^*_t,2) > \mu_{da,t}(\sigma_{Y,2}, x^*_t,2) \Rightarrow \mu_{da,t}(\sigma_{Y,1}, x^*_t,2) > \mu_{da,t}^*(\sigma_{Y,2}) \Rightarrow \mu_{da,t}(\sigma_{Y,1}, x^*_t,2) \geq \mu_{da,t}(\sigma_{Y,1}, x^*_t,2).$$

Thus, the optimal welfare declines with the increasing mean-preserving spread in income.

Proof of proposition 3.1. The DA agent’s welfare $\mu_t$ and terminal wealth $W_{t+1}$ given the savings $A_t$ and portfolio weight $x_t$ in presence of income $Y_{t+1}$ are as follows:
\[ W_{t+1} = A_t[R_{e,t+1} x_t + R_f] + Y_{t+1} \]
\[ \mu_{da,t}^{1-\gamma} = E_t \left[ W_{t+1}^{1-\gamma} \right] - \theta E_t \left[ \left( \mu_{da,t}^{1-\gamma} - W_{t+1}^{1-\gamma} \right) I(W_{t+1} < \mu_{da,t}) \right]. \]

\( R_{e,t+1} \) is the risky excess return and \( R_f \) is the risk-free rate. The wealth and the welfare limits as the portfolio weight \( x_t \) tends to zero are \( W^0_{t+1} \) and \( \mu^0_{da,t} \), respectively.

\[ \lim_{x_t \to 0^+} W_{t+1} = W^0_{t+1} = A_t R_f + Y_{t+1} \]
\[ \lim_{x_t \to 0^+} \mu_{da,t} = \mu^0_{da,t}. \]

Thus, if income is risk-free, \( \mu^0_{da,t} = W^0_{t+1} = A_t R_f + Y_{t+1} \). Further, the differential of \( \mu_{da,t} \) with respect to the portfolio weight \( x_t \) is

\[ \frac{d\mu_{da,t}}{dx_t} = \frac{E_t \left[ W_{t+1}^{-\gamma} [R_{e,t+1}] (1 + \theta I(W_{t+1} \leq \mu_{da,t})) \right]}{1 + \theta \Phi(W_{t+1} \leq \mu_{da,t})}. \]

**Case 1: Risk-free income**

I consider the case of risk-free income and simplify the inequality \( W_{t+1} < \mu_{da,t} \) as \( x_t \to 0^+ \) and as \( x_t \to 0^- \).

\[ \lim_{x_t \to 0^+} I(W_{t+1} < \mu_{da,t}) = \lim_{x_t \to 0^+} I(A_t R_{e,t+1} x_t < 0) \]
\[ \Rightarrow \lim_{x_t \to 0^+} I(W_{t+1} < \mu_{da,t}) = I(R_{e,t+1} < 0) \]

Similarly, \( \lim_{x_t \to 0^-} I(W_{t+1} < \mu_{da,t}) = I(R_{e,t+1} > 0). \)

I apply the above results to compute the differential of \( \mu_{da,t} \) with respect to \( x_t \) as \( x_t \to 0^+ \):

\[ \lim_{x_t \to 0^+} \frac{\mu_{da,t} \gamma}{dx_t} = (W_{t+1}^{0\gamma} E_t [R_{e,t+1} (1 + \theta I(R_{e,t+1} > 0))] / (1 + \theta \Phi(R_{e,t+1} > 0)) \]
\[ \Rightarrow \lim_{x_t \to 0^-} \frac{\mu_{da,t} \gamma}{dx_t} > 0 \quad \forall \theta > 0 \quad \therefore E_t[R_{e,t+1}] > 0 \]

But, \( \lim_{x_t \to 0^+} \frac{\mu_{da,t} \gamma}{dx_t} = (W_{t+1}^{0\gamma} E_t [R_{e,t+1} (1 + \theta I(R_{e,t+1} < 0))] / (1 + \theta \Phi(R_{e,t+1} < 0)) \]
\[ \Rightarrow \lim_{x_t \to 0^+} \frac{\mu_{da,t} \gamma}{dx_t} < 0 \quad \forall \theta > \theta^*. \]
The above results imply that the welfare of the agent improves if he chooses not to short the risky asset, and his welfare declines if he goes long the asset. Thus, the agent optimally chooses not to invest in the risky asset when income is risk free and $\theta > \theta^*$. 

**Case 2: Risky income**

I again simplify the inequality $W_{t+1} < \mu_{da,t}$ as $x_t \to 0$ when income is risky:

$$\lim_{x_t \to 0} I(W_{t+1} < \mu_{da,t}) = I(A_t R_f + Y_{t+1} < \mu_{da,t}^0)$$

I use the above inequality to compute the differential of $\mu_{da,t}$ with respect to $x_t$ as $x_t \to 0$:

$$\lim_{x_t \to 0} \mu_{da,t}^{-\gamma} \frac{d\mu_{da,t}}{dx_t} = E_t[f(Y_{t+1})g(Y_{t+1})R_{e,t+1}]$$

Where, $f(Y_{t+1}) = (A_t R_f + Y_{t+1})^{-\gamma}$, and $g(Y_{t+1}) = \frac{1 + \theta I(A_t R_f + Y_{t+1} < \mu_{da,t}^0)}{1 + \theta \Phi(A_t R_f + Y_{t+1} < \mu_{da,t}^0)}$.

Note, however, that $R_{e,t+1}$ and $Y_{t+1}$ are independent or $R_{e,t+1}$ is conditionally independent of $Y_{t+1}$ \(^{39}\). This relation between the returns and the income implies the following:

$$\lim_{x_t \to 0} \mu_{da,t}^{-\gamma} \frac{d\mu_{da,t}}{dx_t} > 0 \quad : f > 0, \quad g > 0, \quad \overline{R}_{e,t+1} > 0.$$ 

Thus, the DA agent’s welfare increases if he invests some positive amount of the savings in the risky asset. This result applies for all $\theta > 0$.

**Proof of proposition 3.2.** Let $K_{\theta,t} = E_t [R_{e,t+1}] + \theta E_t [I(R_{e,t+1} < 0)]$. The inequality $\theta < \theta^*$ implies $K_{\theta,t} > 0$.

\(^{39}\)If asset returns $R_{e,t+1}$ are conditionally independent of labor income $Y_{t+1}$ then the final result still applies. Let $h(Y_{t+1}) = f(Y_{t+1})g(Y_{t+1})$. Then the definition of conditional independence in Ingersoll (1987) on page 12 implies that

$$E_t[R_{e,t+1}|Y_{t+1}] = E_t[R_{e,t+1}].$$

Hence, $E_t[h(Y_{t+1})R_{e,t+1}] = E_t[h(Y_{t+1})]E_t[R_{e,t+1}]$ and the final result still obtains.
\[ E_t[R_{e,t+1}] \]

If \( \theta^* > \theta \)

\[ \Rightarrow -\theta E_t[R_{e,t+1} I(R_{e,t+1} < 0)] < E_t[R_{e,t+1}] \]

\[ \Rightarrow K_{\theta,t} > 0. \] (A.2)

Let \( \mu_{da,t} \) be the welfare for an arbitrary portfolio weight \( x_t \) and income \( Y_{t+1} \) that yields a wealth gamble \( W_{t+1} \). This set-up implies, \( E_t[R_{e,t+1} I(W_{t+1} < \mu_{da,t})] \geq E_t[R_{e,t+1} I(R_{e,t+1} < 0)] \).

The equality applies only in the case of risk-free income. This inequality implies the inequality in Eq. A.3:

\[ E_t[R_{e,t+1} I(W_{t+1} < \mu_{da,t})] \geq E_t[R_{e,t+1} I(R_{e,t+1} < 0)] \]

\[ E_t[R_{e,t+1} + \theta E_t[R_{e,t+1} I(W_{t+1} < \mu_{da,t})]] \geq K_{\theta,t} \quad \therefore \theta \geq 0. \] (A.3)

I combine inequalities in Eq. A.2 and Eq. A.3 to obtain the following inequality:

\[ E_t[R_{e,t+1} + \theta E_t[R_{e,t+1} I(W_{t+1} < \mu_{da,t})]] > 0 \text{ if } 0 \leq \theta < \theta^*. \] (A.4)

The welfare \( \mu_{da,t} \) for \( \gamma = 0 \) is given by

\[ \mu_{da,t} = E_t[W_{t+1}] - \theta E_t[(\mu_{da,t} - W_{t+1}) I(W_{t+1} < \mu_{da,t})]. \]

The differential of \( \mu_{da,t} \) with respect to the portfolio weight \( x_t \) is

\[ \frac{d\mu_{da,t}}{dx_t} = \frac{E_t[R_{e,t+1} (1 + \theta I(W_{t+1} < \mu_{da,t}))]}{1 + \theta \Phi(W_{t+1} < \mu_{da,t})}. \]

I rewrite the above differential and use Eq. A.4 to obtain the inequality in Eq. A.5, which implies welfare improves with the increase in portfolio weight \( x_t \), where \( x_t \leq 1 \).

\[ \frac{d\mu_{da,t}}{dx_t} = \frac{E_t[R_{e,t+1}] + \theta E_t[R_{e,t+1} I(W_{t+1} < \mu_{da,t})]}{1 + \theta \Phi(W_{t+1} < \mu_{da,t})} \]

\[ \Rightarrow \frac{d\mu_{da,t}}{dx_t} > 0 \quad \therefore 0 \leq \theta < \theta^* \text{ and } \Phi \geq 0. \] (A.5)

Because the welfare improves with increasing portfolio weight, the agent invests all of the savings in the risky asset without violating any constraints.
Sketch of the proof for proposition 4.1.

Case: $\theta = 0$
If $\theta = 0$, the agent’s welfare folds back to expected utility case. The properties of expected utility are sufficient to prove the proposition.

Case: $\theta > 0$ and both $W_{1,t+1}$ and $W_{2,t+1}$ are sure gambles
If $W_{1,t+1}$ and $W_{2,t+1}$ are sure gambles then the welfare again folds back to expected utility case and the properties of expected utility are sufficient to prove the proposition.

Case: $\theta > 0$ and at least one of $W_{1,t+1}$ and $W_{2,t+1}$ gambles is an uncertain gamble
I rewrite the terms in the proposition to economize on the notation.

Let $\psi_t(W_{t+1}, \nu_{t+1}) \equiv \mu_t(g_{t+1}(W_{t+1}, \nu_{t+1}))^{1-\gamma} / (1-\gamma)$ and $G_{t+1}(W_{t+1}, \nu_{t+1}) \equiv g_{t+1}(W_{t+1}, \nu_{t+1})^{1-\gamma} / (1-\gamma)$.

The inequalities in the proposition with the new notation are:

$$0 < (\leq) \psi_{\lambda,t} - E_t [\lambda G_{t+1}(W_{1,t+1}, \nu_{t+1}) + (1-\lambda)G_{t+1}(W_{2,t+1}, \nu_{t+1})]$$

$$+ \theta E_t [(\psi_{\lambda,t} - \lambda G_{t+1}(W_{1,t+1}, \nu_{t+1}) - (1-\lambda)G_{t+1}(W_{2,t+1}, \nu_{t+1}))$$

$$\times I (\lambda G_{t+1}(W_{1,t+1}, \nu_{t+1}) + (1-\lambda)G_{t+1}(W_{2,t+1}, \nu_{t+1}) < \psi_{\lambda,t})].$$

I use the definition/notation $\psi_{\lambda,t} \equiv \psi_t(W_{\lambda,t+1}, \nu_{t+1})$ in the inequality above. I further economize on notation by defining $G_{1,t+1} \equiv G_{t+1}(W_{1,t+1}, \nu_{t+1})$, $G_{2,t+1} \equiv G_{t+1}(W_{2,t+1}, \nu_{t+1})$, $\psi_{1,t} \equiv \psi_t(W_{1,t+1}, \nu_{t+1})$ and $\psi_{2,t} \equiv \psi_t(W_{2,t+1}, \nu_{t+1})$. I use the definitions of welfares $\psi_{1,t}$ and $\psi_{2,t}$ along with the inequality above to obtain the following inequality written in the new notation.
\[ \Delta_{1,2,\lambda} + \lambda \theta E_t[(\psi_{1,t} - G_{1,t+1}) I (G_{1,t+1} < \psi_{1,t})] \]
\[ + (1 - \lambda) \theta E_t[(\psi_{2,t} - G_{2,t+1}) I (G_{2,t+1} < \psi_{2,t})] < (\leq) \theta E_t[(\psi_{\lambda,t} - \lambda G_{1,t+1} - (1 - \lambda) G_{2,t+1}) \times I (\lambda G_{1,t+1} + (1 - \lambda) G_{2,t+1} < \psi_{\lambda,t})] \]  

(A.6)

\( \Delta_{1,2,\lambda} \) is defined as \( \Delta_{1,2,\lambda} \equiv \lambda \psi_{1,t} + (1 - \lambda) \psi_{2,t} - \psi_{\lambda,t} \). The above inequality is independent of the relation between \( \psi_{\lambda,t}, \psi_{1,t} \) and \( \psi_{2,t} \). I consider three possible cases. These are (1) \( \psi_{\lambda,t} < \lambda \psi_{1,t} + (1 - \lambda) \psi_{2,t} \), (2) \( \psi_{\lambda,t} = \lambda \psi_{1,t} + (1 - \lambda) \psi_{2,t} \) and (3) \( \psi_{\lambda,t} > \lambda \psi_{1,t} + (1 - \lambda) \psi_{2,t} \). These three cases translate into (1) \( \Delta_{1,2,\lambda} > 0 \), (2) \( \Delta_{1,2,\lambda} = 0 \) and (3) \( \Delta_{1,2,\lambda} < 0 \) respectively.

**Case (1):** \( \Delta_{1,2,\lambda} > 0 \)

If \( \Delta_{1,2,\lambda} > 0 \) then this assumption combined with Eq. A.6 yields the following inequality.

\[ \Delta_{1,2,\lambda} < \theta \lambda H_1 + \theta (1 - \lambda) H_2 \]  

(A.7)

Where, \( H_1 \) and \( H_2 \) are as follows.

\[ H_1 = E_t[(\psi_{1,t} - G_{1,t+1}) I (G_{1,t+1} + (1 - \lambda) G_{2,t+1} < \psi_{\lambda,t})] \]
\[ - E_t[(\psi_{1,t} - G_{1,t+1}) I (G_{1,t+1} < \psi_{1,t})] \]

\[ H_2 = E_t[(\psi_{2,t} - G_{2,t+1}) I (\lambda G_{1,t+1} + (1 - \lambda) G_{2,t+1} < \psi_{\lambda,t})] \]
\[ - E_t[(\psi_{2,t} - G_{2,t+1}) I (G_{2,t+1} < \psi_{2,t})] \]

Consider \( H_1 \). It is made up of two integrals, \( K_{H_1} \) and \( Q_{H_1} \) as defined below.

\[ K_{H_1} = E_t[(\psi_{1,t} - G_{1,t+1}) I (G_{1,t+1} < \psi_{1,t})] \]
\[ Q_{H_1} = E_t[(\psi_{1,t} - G_{1,t+1}) I (\lambda G_{1,t+1} + (1 - \lambda) G_{2,t+1} < \psi_{\lambda,t})] \]

\( Q_{H_1} \) is bounded above at \( K_{H_1} \), which translates to \( H_1 \leq 0 \). Similarly, \( H_2 \leq 0 \). Since, \( \theta > 0 \) and \( 0 < \lambda < 1 \), I have,

\[ \theta \lambda H_1 + \theta (1 - \lambda) H_2 \leq 0. \]

But, given the assumption that, \( \Delta_{1,2,\lambda} > 0 \), Eq. A.7 indicates,
Thus the assumption of $\Delta_{1,2,\lambda} > 0$ yields an absurd result indicating incorrect assumption.

**Case (2):** $\Delta_{1,2,\lambda} = 0$

Same arguments and steps in “Case (1)” indicate that $H_1 \leq 0$ and $H_2 \leq 0$ and unless $G_{t+1}(\cdot, \nu_{t+1})$ is quasi-concave the result is a pair of equations contradicting each other.

**Combined:**

Thus, if $G_{t+1}(\cdot, \nu_{t+1})$ is concave (quasi-concave) then $\lambda\psi_{1,t} + (1 - \lambda)\psi_{2,t} < (\leq)\psi_{\lambda,t}$.

## B Numerical method

I use endogenous grid method to save on computational cost (Carroll (2006)). The integrals involved in computing expectations are not amenable to the quadrature method. I use simulated draws of shocks to compute these integrals. I use equi-distributed Sobol sequences to generate these draws.\(^{40}\) The computation of integrals combined with the search for a DA fixed point impedes the solution speed. I use the golden section search with parabolic interpolation for solving for the optimal portfolio weight and optimal consumption in the benchmark case. I use the same methods in the extensions except for the case in which I model the health risk. I use a grid search method for both the portfolio and consumption optimization problems when the curvatures of intertemporal and static choice are different, and for the extension of the benchmark case with health risk.

The DA fixed-point problems reduce to a generic problem of finding $\mu$ such that $f(\mu) = 0$, where $f(\mu) \equiv \mu^{1-\gamma} + \theta E [ (\mu^{1-\gamma} - G^{1-\gamma}) I(G < \mu)] - E [G^{1-\gamma}]$. The function, $f$, is monotonic in $\mu$ as $f'(1 - \gamma) > 0$. Thus, the problem, $f(\mu) = 0$, is amenable to a bisection algorithm.

The two end points of the search are $[G_{\text{min}}, G_{\text{max}}]$. Note that $f(G_{\text{min}}) \times f(G_{\text{max}}) < 0$; thus the solution to $f(\mu) = 0$ lies in $[G_{\text{min}}, G_{\text{max}}]$.

\(^{40}\)See Judd (1998) for the benefits of equi-distributed sequences over pseudo-random number generator sequences. I also performed optimization using pseudo-random number generator sequences and found no difference.
References


