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An Optimal Mortgage Refinancing Strategy with Stochastic Interest Rate

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Abstract This article puts forward a framework for assessing the optimal refinancing strategy in continuous time when the interest rate is stochastic and follows a Vasicek model. The optimal refinancing time is obtained by minimizing the conditional expectation of the discounted total payment. A moment generating function is used to derive a closed-form approximation to the refinancing function with infinite maturity under the Vasicek model. The approximation is studied both analytically and numerically. The results indicate three different types of behaviour in the refinancing function, depending on the underlying parameters in the model. Two types indicate optimal refinancing in finite time. We outline a strategy by which a borrower can continually evaluate whether to refinance. By providing a systematic way to evaluate the likelihood of refinancing, these results should be of interest to those trading mortgage-backed securities.

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1 Introduction

A fixed-rate mortgage contract is a financial product that requires the contract holder (the mortgage borrower) to make a periodic repayment of the loan to the contract issuer (the mortgage lender) until the end of the contract. Matching the payment of principal and interest method is a widely used scheme, *i.e.*, the borrower regularly makes the same amount of repayment before its maturity.

The residential mortgage market has seen a slow recovery since the 2008 financial crisis and the recent low level of mortgage rates has caused mortgage borrowers' great interest in seeking refinancing opportunities. For instance, the borrower can prepay the outstanding balance to terminate the existing mortgage contract by entering a new mortgage contract with a lower mortgage rate based on the current spot rate. Consequently, the total interest payment is reduced.

However, refinancing costs (attorney, document, title fees, etc.) and transaction costs are considerably large. Frequent refinancing can quickly cancel out the potential gain. In this work, we concentrate on the optimal refinancing problem where only one refinancing opportunity is allowed under the stochastic interest rate environment. One then can relax this restriction and allow more than one refinancing through suitable generalizations of the model.

In a deterministic interest rate environment, it is always optimal to refinance the mortgage whenever the value of gain from refinancing is positive (Siegel 1984). However, in the stochastic interest rate environment, a potential benefit may exist if one waits for the rate to decline further. In order to see how the stochastic factor influences the optimal refinancing time, we assume no refinancing costs, transaction costs or prepayment penalty.

The one-opportunity no-cost refinancing problem can be considered as an optimal early exercise decision for American options, in which a possible solution is known as the option-based approach. That is, given a lower prevailing mortgage rate, at any time one can exercise the option to refinance the mortgage debt. Kalotay et al. (2004) worked backwards through the lattice interest rate model and compared the value with no refinancing and the value of a newly refinanced mortgage. Stanton (1995), Dunn and Spatt (1999) and Longstaff (2004) studied discrete-time models with a finite horizon that allowed computation of the endogenous values of the fixed rate mortgages. Agarwal et al. (2002) and Agarwal et al. (2013) considered minimizing the net present value of the interest payments with fixed discount rate. Chen and Ling (1989) built a backward-solving model and calculated the minimum differential between the contract rate and the current interest rates under the stochastic environment. Gan et al. (2012) focused on using Monte Carlo simulation in finding a desirable refinancing time that minimizes the total payment under the stochastic interest rate of the Vasicek model. Such a simulation approach has been recently extended by Xie et al. (2017) to solve for the two-dimensional refinancing problem.

In this paper, we provide analytical modeling for refinancing decisions from a mortgage borrower's point of view and construct the mortgage repayment and refinancing in a continuous-time setting. We use a Vasicek model for the stochastic interest rate due to its analytical tractability and statistical flexibility; however, this framework can be extended to other affine models. We then derive a closed form of the net present value (NPV) function, and define the optimal refinancing time to be the time at which its expected value is minimized. By introducing the simplifications of an infinite-time horizon of the mortgage and the Vasicek interest rate model, we obtain an expression which can be studied both numerically and analytically. We shall verify that these simplifying assumptions do not materially affect the results of our model. Our results demonstrate three separate behaviours, depending on the parameters in the model. Two indicate optimal refinancing in finite time.

Our analytical results illustrate parameter regimes where decisions can be made simply, and others where numerical calculations must be made. We outline a strategy whereby a borrower can continually incorporate new data into the calculations to determine whether or not to refinance.

The paper proceeds as follows. Section 2 contains a few assumptions and notations of the optimal mortgage refinancing. Section 3 gives a general solution to the refinancing problem, an outline of the refinancing strategy, and a closed form of the approximation to the refinancing function with infinite maturity under the Vasicek model. Section 4 presents the results of numerical simulations illustrating the effect of parameters in the Vasicek model on the refinancing decision. An analytical discussion of our results is given in Sect. 5, and we conclude in Sect. 6. The Appendix presents simulation results that verify that the Vasicek model is appropriate for our system.

2 Model Setup

To model the mortgage refinancing problem and find the optimal refinancing time that benefits the borrower most, we make the following assumptions:

1. The contract follows the repayment of principal and interest method; that is, a fixed amount m_1 is paid continuously at the beginning of the contract, based on the initial mortgage rate. This payment stream will repay both the principal and any interest due by the end of the contract period.
2. We assume a fixed positive difference κ between the rate of the newly-issued mortgage contract c_t and the current risk-free interest rate r_t.
3. One opportunity of adjustment to the payment is under consideration. The borrower desires an optimal refinancing strategy that minimizes the expectation of the net present value of the future payment.
4. The one-opportunity refinancing does not involve transaction cost, refinancing cost, taxation, prepayment, or default risk.

Notation

r_t Stochastic risk-free interest rate at time t, measured in terms of percentage per year.

c_t Stochastic mortgage rate at time t, *i.e.*, c_0 is the initial mortgage rate.

κ A fixed positive difference between the rate of the newly-issued mortgage contract c_t and the current risk-free interest rate r_t, *i.e.*, $c_t = \kappa + r_t$.
p_0 Initial principal of the mortgage.
m_1 Fixed amount paid continuously for the initial contract with the mortgage rate c_0.
m_2 Fixed amount paid continuously after refinancing.
T Mortgage maturity date.
$P(t)$ Principal balance at time t; the initial principal is $P(0) = p_0$ and $P(T) = 0$.
t^* Potential optimal refinancing time.

The model is defined in the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$, where $(\mathcal{F}_t)_{t\geq 0}$ represents the information flow of the market and $\mathbb{Q}$ is the risk-neutral probability measure. For the purposes of this manuscript, the interest rate $\{r_t, t \geq 0\}$ follows a Vasicek model with initial value r_0, *i.e.*,

$$dr_t = \alpha(\mu - r_t)\, dt + \sigma\, dW_t, \quad \alpha > 0, \quad \mu > 0, \quad \sigma > 0, \tag{1}$$

where $\{W_t, t \geq 0\}$ is a Wiener process under the measure $\mathbb{Q}$. It is possible for interest rates to become negative under such a model; we shall address that shortcoming in more detail below. Moreover, we note that the types of analysis we present will still apply for general affine interest rate models for which unique solutions exist.

3 Mortgage Refinancing

The value of the mortgage payment satisfies the ordinary differential equation (ODE):

$$\begin{aligned} dP(t) &= -m_1\, dt + c_0 P(t)\, dt, \quad t \in [0, t^*], \\ P(0) &= p_0, \end{aligned}$$

so its solution is

$$P(t) = \frac{m_1}{c_0}\left[1 - e^{-c_0(T-t)}\right], \quad m_1 = \frac{c_0 p_0}{1 - e^{-c_0 T}}. \tag{2}$$

Here m_1 has been chosen so that $P(T) = 0$. Suppose at time t^*, the adjusted mortgage payment gives the following ODE:

$$\begin{aligned} dP(t) &= -m_2\, dt + c_{t^*} P(t)\, dt, \quad t \in [t^*, T], \\ P(t^*) &= \frac{m_1}{c_0}\left[1 - e^{-c_0(T-t^*)}\right], \end{aligned} \tag{3}$$

where the second condition ensures continuity with (2) at $t = t^*$. Note that instead we could have imposed $P(t^{*+}) = P(t^{*-}) + C$, where C would represent one-time refinancing costs.

The solution of (3) is given by

$$P(t) = \frac{m_2}{c_{t^*}}\left[1 - e^{-c_{t^*}(T-t)}\right], \quad t \in [t^*, T],$$
$$m_2(c_{t^*}, t^*) = \frac{c_{t^*} P(t^*)}{1 - e^{-c_{t^*}(T-t^*)}}, \tag{4}$$

where m_2 has been chosen to ensure that $P(T) = 0$. The intrinsic value of the refinancing is defined as the extra benefit to the mortgage borrower due to the "refinancing option"—that is, the difference between the value in holding the contract until its maturity and the value of the mortgage that refinancing is occurred. Our objective is to obtain the time t^* that maximizes this difference. Since the value of the contract is a constant given at the beginning of the contract, our objective is equivalent to minimizing the value of the adjusted total payments, which is the sum of the net present value of the continuous cash flows before and after the refinancing decision, *i.e.*,

$$\begin{aligned} v(t^*) &= \int_0^{t^*} m_1 e^{-\int_0^t r_s\,ds}\,dt + \int_{t^*}^{T} m_2(c_{t^*}, t^*) e^{-\int_0^t r_s\,ds}\,dt \\ &= m_1 \int_0^{t^*} e^{-\int_0^t r_s\,ds}\,dt \\ &\quad + \frac{m_1}{c_0}\left(1 - e^{-c_0(T-t^*)}\right)\int_{t^*}^{T} \frac{c_{t^*}}{1 - e^{-c_{t^*}(T-t^*)}} e^{-\int_0^t r_s\,ds}\,dt, \end{aligned} \tag{5}$$

where we have used (4). The above equation implies that two factors influence $v(t^*)$: the refinancing time and the refinancing mortgage rate. The dependence of the two factors and the randomness of the interest rate increase the difficulties in solving for the minimum value of $v(t^*)$.

Our approach is to firstly eliminate the randomness of the interest rate by taking the conditional expectation of $v(t^*)$ given r_0 and then to look for the optimal refinancing time t^*, denoted as t^{**}, that leads to the minimum values of $E[v(t^*)|r_0]$, *i.e.*,

$$t^{**} = \arg\min_{t^* \in [0,T]} E[v(t^*)|r_0]. \tag{6}$$

From the borrowers' perspective, the optimization strategy proceeds as follows. At any time (which can be shifted to correspond to $t = 0$), the borrower uses information from the behaviour of the interest rates for previous times to estimate the parameters α, μ, and σ in the Vasicek model (1). Then the borrower will calculate t^{**} using (6).

If $t^{**} = 0$, the borrower refinances immediately. If $t^{**} \neq 0$, the borrower does not refinance, and continues to gather information about interest rates to refine the estimates of the parameters. This iterative process continues iteratively until $t^{**} = 0$. Hence the situation can be interpreted as a stopping-time problem.

In the discrete-time case, the problem of calculating t^{**} can be solved by Monte Carlo simulation (see Gan et al. 2012). However, in continuous time, difficulties in solving (6) arise due to the complex form of the second term on the right hand side of (5). In the following section, we simplify the problem by assuming an infinite time horizon for the mortgage contract.

3.1 Mortgage Refinancing with Infinite Time Horizon

In order to justify using the infinite time horizon approximation, we must estimate the errors we make in introducing it. Letting $T \to \infty$ while assuming $t^* < \infty$ and $E[e^{t^*}] < \infty$, we may make the following replacements (with the listed errors):

$$\begin{aligned} m_1 &\to c_0 p_0 + O(e^{-c_0 T}), \\ 1 - e^{-c_0(T-t^*)} &\to 1 + O(e^{-c_0 T}), \\ 1 - e^{-c_{t^*}(T-t^*)} &\to 1 + O(e^{-c_{t^*} T}). \end{aligned} \tag{7}$$

Letting $X_t = \int_0^t r_s \, ds$ and substituting the above into (5), we obtain

$$v(t^*) = c_0 p_0 \int_0^{t^*} e^{-X_t} \, dt + p_0 \int_{t^*}^{\infty} c_{t^*} e^{-X_t} \, dt - p_0 \int_T^{\infty} c_{t^*} e^{-X_t} \, dt, \tag{8}$$

where we have broken the second integral in (5) into two pieces to isolate the one remaining error involved in the approximation.

To bound the last integral above, let

$$r_{\min} = \inf_{t \in [0,\infty)} r_t, \quad c_{\max} = \sup_{t \in [0,\infty)} c_t.$$

Then we have

$$\int_T^{\infty} c_{t^*} e^{-X_t} \, dt \leq \int_T^{\infty} c_{\max} e^{-r_{\min} t} \, dt = \frac{c_{\max}}{r_{\min}} e^{-r_{\min} T}.$$

Therefore, ignoring the final integral in (8) introduces an $O(e^{-r_{\min} T})$ error, which is larger than the errors in (7) by the definition of c_t. But this error becomes transcendentally small as T increases, and so our infinite time horizon approximation is reasonable. Moreover, we shall show in Sect. 4 that changing the value of T does not affect the overall shape of the graph. In particular, varying T does not change the sign of $F'(0)$, which is the key parameter the borrower uses when estimating t^{**} and hence deciding whether or not to refinance.

Therefore, it is safe to omit the last integral in (8) to yield

$$v(t^*) = c_0 p_0 \int_0^{t^*} e^{-X_t} \, dt + p_0 \int_{t^*}^{\infty} r_{t^*} e^{-X_t} \, dt + \kappa p_0 \int_{t^*}^{\infty} e^{-X_t} \, dt, \tag{9}$$

where we have used the fact that $c_{t^*} = r_{t^*} + \kappa$. Note that (9) is similar to the expression for a perpetuity, an annuity for which the payments continue forever. That is, the amount of the periodic payment equals the present value of the perpetuity multiplied by the interest rate.

Under the stochastic processes r and X, the conditional expectation of $v(t^*)$ given r_0 is

$$E[v(t^*)|r_0] = c_0 p_0 E\left[\int_0^{t^*} e^{-X_t}\,dt \middle| r_0\right] + p_0 E\left[\int_{t^*}^{\infty} r_{t^*} e^{-X_t}\,dt \middle| r_0\right] + \kappa p_0 E\left[\int_{t^*}^{\infty} e^{-X_t}\,dt \middle| r_0\right]. \tag{10}$$

As expected, $v(t^*)$ is proportional to the initial principal p_0, and hence we can scale by p_0 and minimize the normalized *refinancing function* $F(t^*)$ instead:

$$\begin{aligned} F(t^*) &= c_0 E\left[\int_0^{t^*} e^{-X_t}\,dt \middle| r_0\right] + E\left[\int_{t^*}^{\infty} r_{t^*} e^{-X_t}\,dt \middle| r_0\right] \\ &\quad + \kappa E\left[\int_{t^*}^{\infty} e^{-X_t}\,dt \middle| r_0\right] \\ &= c_0 \int_0^{t^*} E\left[e^{-X_t}|r_0\right] dt + \int_{t^*}^{\infty} E[r_{t^*} e^{-X_t}|r_0]\,dt + \kappa \int_{t^*}^{\infty} E\left[e^{-X_t}|r_0\right] dt, \end{aligned} \tag{11}$$

where the last equality holds by Fubini's Theorem (Klebaner 2012, p. 53).

To minimize $F(t^*)$, we must compute $E\left[e^{-X_t}|r_0\right]$ and $E[r_{t^*} e^{-X_t}|r_0]$, which can be done by Laplace transforms (Bladt and Nielsen 2010). After the elimination of the random factor, one can find the optimal refinancing time by calculating $F(t^*)$ for $t^* \in [0, +\infty)$.

For later algebraic convenience, we define the following expression:

$$I(\lambda) = E\left[e^{-X_t + \lambda r_{t^*}}|r_0\right]. \tag{12}$$

Substituting (12) into (11), we obtain

$$F(t^*) = c_0 \int_0^{t^*} I(0)\,dt + \int_{t^*}^{\infty} I'(0)\,dt + \kappa \int_{t^*}^{\infty} I(0)\,dt. \tag{13}$$

3.1.1 Vasicek Model

Some of our results vitally depend on the Vasicek model, which as noted above, can yield negative interest rates. Obviously, this is a significant problem from the practical standpoint, if it is likely to occur. Nevertheless, the Vasicek model is commonly used for interest rates (Vasicek 1977; Cairns 2004) due to the fact that the probability of negative r is very small as a result of the short time scale and the low volatility σ (we verify exactly how small for our particular problem in the Appendix).

The solution of Eq. (1) is

$$r_t = \mu + (r_0 - \mu)e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)}\, dW_s.$$

With this result, it can be shown that $X_t = \int_0^t r_s\, ds$, the variable of interest in our analysis, is given by

$$X_t = \int_0^t r_s\, ds = \mu t + (r_0 - \mu)\frac{1 - e^{-\alpha t}}{\alpha} + \int_0^t \sigma \int_0^s e^{-\alpha(s-r)}\, dW_r\, ds. \tag{14}$$

Let $\mu_1(t^*)$ and $\mu_2(t)$ be the conditional expectations of r_{t^*} and X_t given r_0, respectively. They have the forms

$$\mu_1(t^*) = E[r_{t^*}|r_0] = \mu + (r_0 - \mu)e^{-\alpha t^*}, \tag{15}$$

$$\mu_2(t) = E[X_t|r_0] = \mu t + (r_0 - \mu)\frac{1 - e^{-\alpha t}}{\alpha}. \tag{16}$$

Correspondingly, let $\sigma_1^2(t^*)$ and $\sigma_2^2(t)$ be the variances of r_{t^*} and X_t, and $\text{Cov}[r_{t^*}, X_t|r_0]$ be the covariance of r_{t^*} and X_t. Using It ô's Isometry, we have

$$\begin{aligned}
\sigma_1^2(t^*) &= \sigma^2 \frac{1 - e^{-2\alpha t^*}}{2\alpha}, \\
\sigma_2^2(t) &= \frac{\sigma^2}{\alpha^2}\left[t - \frac{2(1 - e^{-\alpha t})}{\alpha} + \frac{1 - e^{-2\alpha t}}{2\alpha}\right], \\
\text{Cov}[r_{t^*}, X_t|r_0] &= \frac{\sigma^2}{\alpha}\left[\frac{1 - e^{-\alpha t^*}}{\alpha} - \frac{e^{-\alpha(t-t^*)}(1 - e^{-2\alpha t^*})}{2\alpha}\right].
\end{aligned} \tag{17}$$

3.1.2 The Closed Form of the Refinancing Function

In this subsection, the aim is to compute $I(\lambda) = E\left[e^{-X_t + \lambda r_{t^*}}|r_0\right]$ and then the refinancing function $F(t^*)$. We require the following lemma (Wackerly et al. 2007):

Lemma 3.1 *Given a random vector* $\mathbf{X} \in \mathcal{R}^n$ *that follows an n-variate normal distribution, i.e.,* $\mathbf{X} \sim N_n(\mu, \Sigma)$*, and a vector of real numbers* $\mathbf{t} = (t_1, t_2, \cdots, t_n)'$*, the moment generating function of* $\mathbf{X}$ *is given by*

$$M_{\mathbf{X}}(\mathbf{t}) = e^{\mathbf{t}'\mu + \frac{1}{2}\mathbf{t}'\Sigma\mathbf{t}}.$$

Note that for a fixed t, r_t and X_t are normally distributed and (r_t, X_t) is a bivariate normal distribution. Let $\mathbf{t} = (-1, \lambda)^T$ and compute the moment generating functions for the bivariate vector (X_t, r_t). By Lemma 3.1, we get

$$I(\lambda) = E\left[e^{-X_t + \lambda r_{t^*}}|r_0\right] = e^{-\mu_2(t) + \sigma_2^2(t)/2 + \lambda\mu_1(t^*) + \lambda^2\sigma_1^2(t^*)/2 - \lambda \text{Cov}[r_{t^*}, X_t|r_0]}. \tag{18}$$

Using this expression in (13), we have the exact expression

$$F(t^*) = c_0 \int_0^{t^*} e^{-\mu_2(t)+\sigma_2^2(t)/2}\, dt \\ + \int_{t^*}^{\infty} \left\{\mu_1(t^*) - \mathrm{Cov}[r_{t^*}, X_t|r_0]\right\} e^{-\mu_2(t)+\sigma_2^2(t)/2}\, dt \\ + \kappa \int_{t^*}^{\infty} e^{-\mu_2(t)+\sigma_2^2(t)/2}\, dt. \quad (19)$$

Our objective is to find the t^* that minimizes the function $F(t^*)$, which can be done using MATLAB. However, we should choose the right parameters that guarantee the convergence of the refinancing function $F(t^*)$. The following claim indicates the condition for the convergence of the function $F(t^*)$.

Lemma 3.2 *For any $t^* \in [0, \infty)$, $F(t^*)$ as defined in Eq. (19) converges if and only if*

$$\sigma^2 < 2\alpha^2\mu.$$

Proof As the exponential is bounded for all finite t, the first term in (19) is bounded. The remaining two terms converge if the exponents of the integrands go to $-\infty$ as $t \to \infty$. For large t, we have

$$-\mu_2(t) + \frac{1}{2}\sigma_2^2(t) \sim \left(-\mu + \frac{\sigma^2}{2\alpha^2}\right) t = \frac{(\sigma^2 - 2\alpha^2\mu)t}{2\alpha^2}. \quad (20)$$

Hence $F(t^*)$ converges if and only if the numerator of (20) is negative, which occurs when $\sigma^2 < 2\alpha^2\mu$. □

We now compute an actual value using the following theorem:

Theorem 3.3 *Given $\sigma^2 < 2\alpha^2\mu$,*

$$\lim_{t^* \to +\infty} F(t^*) = \frac{c_0}{\alpha} e^{A_1 - A_2^2/4A_3} G_1(1),$$

where

$$G_1(x) = \int_0^x u^{-\left(\frac{A}{\alpha}+1\right)} e^{A_3\left(u+\frac{A_2}{2A_3}\right)^2} du,$$

$$A = -\mu + \frac{\sigma^2}{2\alpha^2},$$

$$A_1 = -\left(r_0 - \mu + \frac{3\sigma^2}{4\alpha^2}\right)\frac{1}{\alpha},$$

$$A_2 = \frac{1}{\alpha}\left(\frac{\sigma^2}{\alpha^2} + r_0 - \mu\right),$$
$$A_3 = -\frac{\sigma^2}{4\alpha^3}.$$

Proof Since $\sigma^2 < 2\alpha^2\mu$, the integrands in (19) are well-behaved and hence we have that

$$F(\infty) = c_0 \int_0^\infty e^{-\mu_2(t)+\sigma_2^2(t)/2}\, dt. \tag{21}$$

Fully expanding the exponent in the above, we obtain

$$\begin{aligned} -\mu_2(t) + \frac{1}{2}\sigma_2^2(t) &= -\left(r_0 - \mu + \frac{3\sigma^2}{4\alpha^2}\right)\frac{1}{\alpha} + \left(\frac{\sigma^2}{2\alpha^2} - \mu\right)t \\ &\quad + \frac{1}{\alpha}\left(\frac{\sigma^2}{\alpha^2} + r_0 - \mu\right)e^{-\alpha t} - \frac{\sigma^2}{4\alpha^3}e^{-2\alpha t} \\ &= A_1 + At + A_2 e^{-\alpha t} + A_3 e^{-2\alpha t}. \end{aligned} \tag{22}$$

We change variables by letting $u = e^{-\alpha t}$, which gives

$$\begin{aligned} \int_0^\infty e^{-\mu_2(t)+\frac{1}{2}\sigma_2^2(t)}\, dt &= \frac{1}{\alpha} e^{A_1 - A_2^2/(4A_3)} \int_0^1 u^{-(\frac{A}{\alpha}+1)} e^{A_3(u+\frac{A_2}{2A_3})^2}\, du \\ &= \frac{1}{\alpha} e^{A_1 - A_2^2/(4A_3)} G_1(1). \end{aligned}$$

We note that the integrand of $G_1(x)$ has a singularity at the origin. Hence the integral converges when $1 + A/\alpha < 1$, which is equivalent to $\sigma^2 < 2\alpha^2\mu$. □

Before presenting our numerical results, we note that

$$\begin{aligned} F(0) &= \int_0^\infty \{\mu_1(0) - \mathrm{Cov}[r_0, X_t|r_0]\}\, e^{-\mu_2(t)+\sigma_2^2(t)/2}\, dt \\ &\quad + \kappa \int_0^\infty e^{-\mu_2(t)+\sigma_2^2(t)/2}\, dt \\ &= (r_0 + \kappa)\int_0^\infty e^{-\mu_2(t)+\sigma_2^2(t)/2}\, dt = F(\infty), \end{aligned} \tag{23}$$

where we have used (21). The fact that $F(0) = F(\infty)$ is just a restatement of the no-arbitrage principle: the expected NPV of the contract at the beginning and end of the contract must be the same. Otherwise, one could initially make a risk-free profit by buying or selling the contract.

Table 1 Values of the parameters for the refinancing function $F(t^*)$

Parameters	c_0	κ	r_0	α	μ	σ
Values	0.035	0.005	0.03	0.1	0.06	0.03

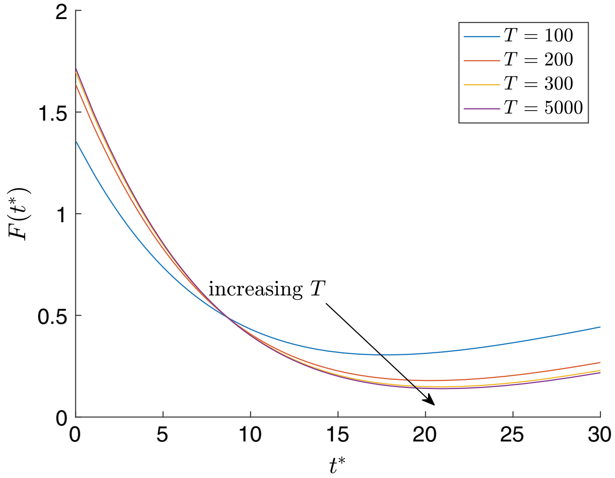

Fig. 1 Convergence of numerical results for various values of T

4 Numerical Results

There are five independent parameters in the refinancing function (19), namely κ, α, μ, σ, r_0, playing important roles in shaping the curve $y = F(t^*)$. Initially, we give each parameter a value shown in Table 1.

We begin by verifying the assumption of the infinite time horizon in Sect. 3.1. Figure 1 shows how the behaviour of $F(t)$ varies for the parameters in Table 1 and various values of T. We see that for relatively large values of T, $F(t)$ converges to the behaviour when $T \to \infty$. Moreover, even for smaller values of T, the behaviour of the graph most relevant to the borrower [*i.e.*, the sign of $F'(0)$, which determines whether the borrower should immediately refinance] remains the same.

Depending on the choice of the values of the parameters, the shapes of the curve can be very different. Our experiments show that there are basically three types of shapes for $y = F(t^*)$. Listed below are the descriptions of each type together with a typical set of parameters that will yield the specific shape. These typical sets of parameters and the corresponding plots of $y = F(t^*)$ are illustrated in Fig. 2. Note that in contrast to the other figures in this manuscript, we use an extremely long time horizon for this graph to illustrate the asymptotic behaviour.

1. The curve declines quickly at the beginning and reaches the lowest point in a short time. After a concave increase for a considerable period, the upward trend slows down and the curve asymptotes to an upper bound. Thus, there is an optimal refinancing time, but it is in the future, so the borrower should wait to refinance. Note from the curves that despite the infinite-time horizon for the problem, the

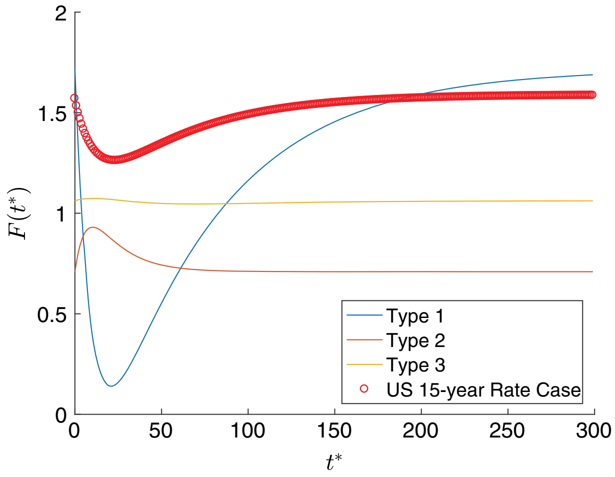

Fig. 2 Three types of the curve for the refinancing function $F(t^*)$, along with data using historical mortgage rate data

optimal refinancing time is well within the duration of a 15- or 30-year mortgage. An example set of parameters of this type is $r_0 = 0.03, \sigma = 0.03, \alpha = 0.1, \mu = 0.06, \kappa = 0.005$.

2. The curve increases quickly at the beginning and reaches the highest point in a short time. After a convex increase for a considerable period, the downward trend slows down and the curve asymptotes to a lower bound. Thus, the optimal refinancing time is now, and the borrower should refinance immediately. An example set of parameters of this type is $r_0 = 0.03, \sigma = 0.003, \alpha = 0.1, \mu = 0.06, \kappa = 0.005$.
3. The curve increases quickly at the beginning, reaching a maximum before decreasing below the long-term asymptote. Then the curve reaches a minimum before increasing with lower speed until the curve asymptotes to an upper bound. Thus, there is a finite optimal refinancing time. An example set of parameters of this type is $r_0 = 0.03, \sigma = 0.003, \alpha = 0.001, \mu = 0.06, \kappa = 0.005$. The variance in this case is quite small, and is barely noticeable in Fig. 2. With this choice of parameters, the optimal refinancing time is around $t^* = 70$, which is outside the realm of a typical contract. Therefore, in this case the borrower should refinance immediately at the local minimum $t^* = 0$. However, one could imagine other parameter sets that would drive the optimal refinancing time smaller.

In order to see the effects of the recent interest rate environment, we examined monthly data on 15-year fixed mortgage rates in monthly basis from Freddie Mac (2016) for the period from Jan. 1992 to Feb. 2016. Using maximum likelihood calibration, we found the parameters in the Vasicek model to be $\mu = 0.0241, \sigma = 0.0066$, and $\alpha = 0.0641$. Hence the current interest rate environment yields Type 1 behaviour, as seen in Fig. 2.

In order to see how the optimal refinancing time changes with respect to the parameters, we run the algorithm using the values in Table 1 as an initial reference, changing the value of one parameter while fixing the value of others. Table 2 provides the

Table 2 Curve type with different parameters

μ	$\alpha = 0.1$ $\sigma = 0.03$	σ	$\alpha = 0.1$ $\mu = 0.06$	α	$\sigma = 0.03$ $\mu = 0.06$
0.05	Type 1	0.001	Type 2	0.1	Type 1
0.07	Type 1	0.01	Type 2	0.15	Type 1
0.09	Type 1	0.015	Type 2	0.2	Type 2
0.11	Type 2	0.02	Type 3	0.25	Type 2
0.13	Type 2	0.025	Type 1	0.3	Type 2
0.15	Type 2	0.03	Type 1	0.35	Type 2

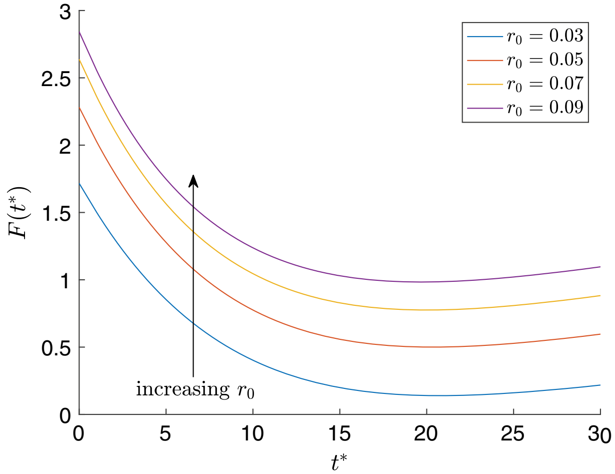

Fig. 3 Curves for $y = F(t^*)$ with different initial rates r_0 and so c_0

comparison of optimal refinancing times as one of the key parameters changes systematically.

In Fig. 3, we plot the influence of r_0 on the shape of $y = F(t^*)$. We note that with the parameters chosen, if the curve is of Type 1, it stays of the same type for any choice of r_0. Figure 4 shows the influence of κ on the shape of $y = F(t^*)$: *e.g.*, if the curve is of Type 1, it stays of the same type for any choice of κ. Note that as κ decreases, the optimal refinancing time slightly decreases. This makes financial sense, since with $r_0 < \mu$ (as in the graph), we would expect to refinance more quickly when we pay a smaller interest rate premium.

In contrast, the parameters α, μ and σ play more significant roles. In Fig. 5, we see that by increasing the value of μ from 0.05 to 0.15, the shape of the curve changes from Type 1 to Type 2 and so the optimal time for refinancing shifts from some time in the future to now. This makes financial sense, since for fixed r_0, increasing μ makes immediate refinancing more attractive, since it is unlikely that rates will remain as low as r_0 in the future.

We see similar behaviour in Fig. 6 when we decrease σ from 0.03 to 0.001. When $r_0 < \mu$ (as shown in Table 1), a smaller variance indicates that it is less likely for rates

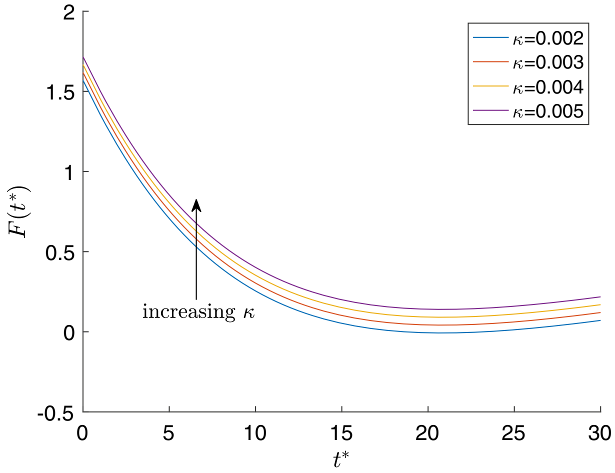

Fig. 4 Curves for $y = F(t^*)$ with different κ and so c_0

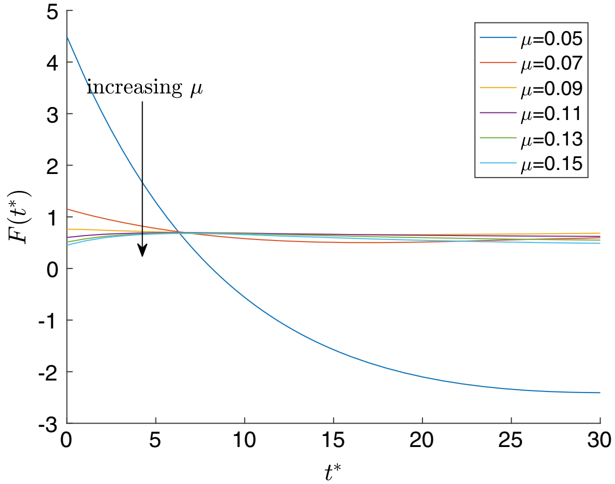

Fig. 5 Curves for $y = F(t^*)$ with different long term rates μ

to remain as low as r_0 in the future; it is more likely that they will remain in a tight band around μ. Hence it is more favourable to refinance immediately, and the graphs with lower σ show this Type 2 behaviour. Note that $\sigma = 0.02$ corresponds to the Type 3 case, which is a transitional case between Types 1 and 2.

In Fig. 7 we see that when we increase α from 0.1 to 0.35, we also transition from Type 1 (delay) behaviour to Type 2 (refinance) behaviour. For $r_0 < \mu$, an increase in α means that one would expect rates to rise more quickly to the mean μ. Hence it is advantageous to refinance immediately, and the graphs with higher α show this Type 2 behaviour.

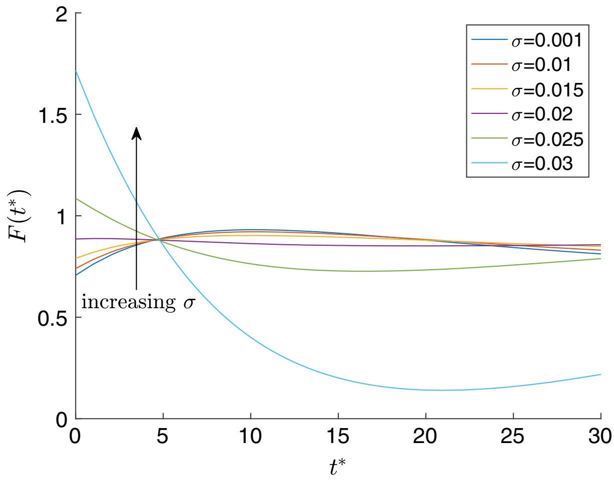

Fig. 6 Curves for $y = F(t^*)$ with different volatilities σ

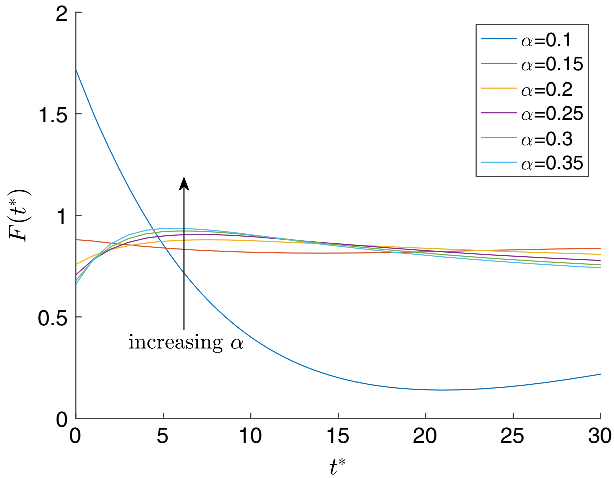

Fig. 7 Curves $y = F(t^*)$ with different reverting rates α

Lastly, in Fig. 8 we show a graph indicating whether the borrower should refinance immediately [*i.e.*, whether $F'(0) > 0$] for various values of μ and σ. The graph shows that for high μ and low σ, the borrower should refinance immediately, since it is unlikely that rates will remain as low as their current state. Then there is a region with lower μ and higher σ where the borrower should wait, since there is now a better chance that rates will decrease even further in the future.

The vertical line in Fig. 8 corresponds to the fixed value of $\mu = 0.06$ in Table 1; the curves in Fig. 6 correspond to values on this line. Similarly, curves in Fig. 5 correspond to values on the horizontal line.

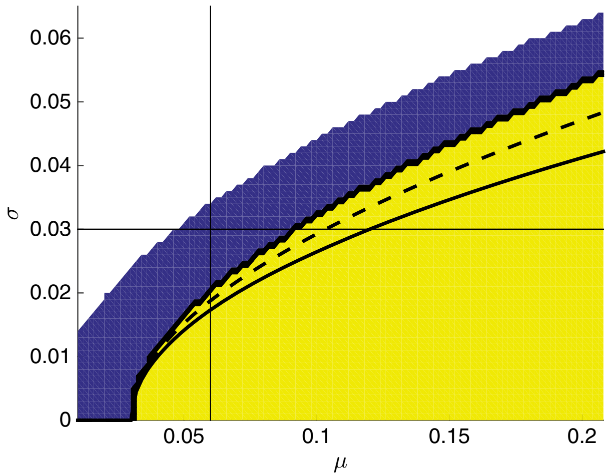

Fig. 8 Decision plot for various μ and σ. Dark region: borrower should wait to refinance. Light region: borrower should refinance immediately. Crosshairs correspond to the parameter values in Figs. 5 and 6. Solid curve: bound in (28). Dashed curve: bound in (36)

Note that the above analysis of parameter effects is based on the values given in Table 1. For other values of parameters, one might expect to have slightly different conclusions depending on the trend of the expected interest rates over time.

5 Analytical Results

In order to interpret our results from Sect. 4, we analyze the function $F(t^*)$ for small and large argument. Since $F(0) = F(\infty)$, the signs of $F'(0)$ and $F'(\infty)$ will help determine the shape of the graph. Calculating $F'(t^*)$ using Leibniz's Rule, we obtain

$$\begin{aligned} F'(t^*) &= c_0 e^{-\mu_2(t^*)+\sigma_2^2(t^*)/2} - \left\{\mu_1(t^*) - \mathrm{Cov}[r_{t^*}, X_t|r_0]\right\}_{t=t_*} e^{-\mu_2(t^*)+\sigma_2^2(t^*)/2} \\ &\quad + \int_{t^*}^{\infty} \frac{\partial}{\partial t^*} \left\{\mu_1(t^*) - \mathrm{Cov}[r_{t^*}, X_t|r_0]\right\} e^{-\mu_2(t)+\sigma_2^2(t)/2}\, dt \\ &\quad - \kappa e^{-\mu_2(t^*)+\sigma_2^2(t^*)/2} \\ &= (1 - e^{-\alpha t^*}) \left[r_0 - \mu + \frac{\sigma^2(1 - e^{-\alpha t^*})}{2\alpha^2} \right] e^{-\mu_2(t^*)+\sigma_2^2(t^*)/2} \\ &\quad + \int_{t^*}^{\infty} \left[-\alpha e^{-\alpha t^*} \left(r_0 - \mu + \frac{\sigma^2}{\alpha^2} \right) + \frac{\sigma^2}{\alpha} e^{-\alpha t} \cosh \alpha t^* \right] \\ &\quad \times e^{-\mu_2(t)+\sigma_2^2(t)/2}\, dt, \end{aligned} \tag{24}$$

where we have used (15) and (17). Note that F' is independent of κ; hence varying κ will cause simply a translation in the graph, as discussed above.

We may obtain explicit bounds for $F'(\infty)$ by noting that

$$F'(t^*) \sim \left(r_0 - \mu + \frac{\sigma^2}{2\alpha^2}\right)\exp(At^* + A_1) + \frac{\sigma^2 \cosh(\alpha t^*)}{\alpha}\int_{t^*}^{\infty} \exp((A-\alpha)t + A_1)\,dt, \quad t^* \to \infty,$$

where we have used (22). Continuing to simplify, we obtain

$$F'(t^*) \sim \left(r_0 - \mu + \frac{\sigma^2}{2\alpha^2} + \frac{\sigma^2\alpha}{2\alpha^2(\alpha+\mu)-\sigma^2}\right)\exp(A_1 t^* + A).$$

Since we are considering the case where $F(\infty)$ is finite, obviously the above expression tends to 0 by the convergence proof. However, we see that the sign of F' for large t^* is given by the sign of the parenthetical quantity. In particular:

$$\text{if } r_0 > \mu - \frac{\sigma^2}{2\alpha^2} - \frac{\sigma^2\alpha}{2\alpha^2(\alpha+\mu)-\sigma^2}, \text{ the asymptote is an upper limit (cases 1 and 3),} \tag{25}$$

$$\text{if } r_0 < \mu - \frac{\sigma^2}{2\alpha^2} - \frac{\sigma^2\alpha}{2\alpha^2(\alpha+\mu)-\sigma^2}, \text{ the asymptote is a lower limit (case 2).} \tag{26}$$

We conclude by proving two results regarding $F'(0)$, which is the key value a borrower considers when deciding whether to refinance immediately.

Theorem 5.1

$$\textit{If } r_0 > \mu, \qquad F'(0) < 0 \textit{ and the buyer should delay refinancing.} \tag{27}$$

$$\textit{If } r_0 < \mu - \frac{\sigma^2}{\alpha^2}, \quad F'(0) > 0 \textit{ and the buyer should refinance immediately.} \tag{28}$$

From a financial perspective, if $r_0 > \mu$, the interest rate today is above the expected long-term mean, so the borrower should delay refinancing. Since $\sigma^2 < 2\alpha^2\mu$ by the convergence proof, we have that the bound in (28) is more restrictive than the bound in (26). Hence we are in case 2. This makes financial sense, for if r_0, the interest rate today, is significantly less than μ, the expected long-term mean, the borrower should refinance immediately. The boundary given by (28) is the solid curve in Fig. 8.

Proof Substituting $t^* = 0$ into (24), we have

$$F'(0) = \int_0^{\infty} K(r_0)e^{-\mu_2(t)+\sigma_2^2(t)/2}\,dt, \tag{29}$$

$$K(r_0) = -\alpha(r_0 - \mu) - \frac{\sigma^2}{\alpha}(1 - e^{-\alpha t}). \tag{30}$$

The exponential in the integrand of $F'(0)$ is always positive. Hence if $K(r_0)$ has the same sign for $t > 0$, that will also be the sign of $F'(0)$. The last term in (30) is always negative, so if $r_0 > \mu$, then $K(r_0) < 0$ and hence $F'(0) < 0$.

Similarly,

$$r_0 < \mu - \frac{\sigma^2}{\alpha^2} \qquad \Longrightarrow \qquad K(r_0) > \frac{\sigma^2 e^{-\alpha t}}{\alpha} > 0,$$

so $F'(0) > 0$. □

Note also that for some choice of parameters, the bound in (28) may be negative, and hence case 2 never occurs. This is the situation that appears in Fig. 2. Essentially, since the Vasicek model allows negative interest rates, our formulation can indicate that under certain circumstances (given by the parameters in Table 1) one should refinance only if r_0 is negative. However, we note that the parameters obtained from real-world data have a smaller ratio σ/α, and hence a positive bound.

Note that the bounds in (27) and (28) provide intervals where mortgage refinancing decisions may be made easily, without calculating F. This then leaves the case

$$\mu - \frac{\sigma^2}{\alpha^2} < r_0 < \mu, \tag{31}$$

where detailed computations must be made. By continuity we know that $F'(0) = 0$ somewhere in this interval, and under certain conditions, we can establish that it is unique:

Theorem 5.2 *If $\sigma^2 < \alpha^3$, $F'(0) = 0$ for exactly one r_0 in the interval* (31).

Proof We compute

$$\frac{\partial F'(0)}{\partial r_0} = \int_0^\infty B(r_0) e^{-\mu_2(t)+\sigma_2^2(t)/2}\, dt, \tag{32}$$

$$B(r_0) = -\alpha + \left[\alpha(r_0 - \mu) + \frac{\sigma^2}{\alpha}(1 - e^{-\alpha t})\right] \frac{1 - e^{-\alpha t}}{\alpha}, \tag{33}$$

where we have used (16). The exponential in the integrand of (32) is always positive. Hence if $B(r_0)$ has the same sign for $t > 0$, that will also be the sign of $\partial F'(0)/\partial r_0$.

We then check the sign of B at the two endpoints of our interval (31):

$$B\left(\mu - \frac{\sigma^2}{\alpha^2}\right) = -\alpha - \frac{\sigma^2 e^{-\alpha t}(1 - e^{-\alpha t})}{\alpha^2} < 0, \tag{34}$$

$$B(\mu) = \frac{\sigma^2 (1 - e^{-\alpha t})^2}{\alpha^2} - \alpha < \frac{\sigma^2}{\alpha^2} - \alpha. \tag{35}$$

Therefore, if $\sigma^2 < \alpha^3$, then $B(\mu) < 0$. Since $B'(r_0) = 1 - e^{-\alpha t} > 0$, $B(r_0) < 0$ for r_0 in the interval (31), as is $\partial F'(0)/\partial r_0$. Thus $F'(0)$ is monotonic in (31), and its zero is unique. □

Our numerical evidence leads us to believe that $F'(0) > 0$ for

$$r_0 < \mu - \frac{\sigma^2}{2\alpha^2} - \frac{\sigma^2\alpha}{2\alpha^2(\alpha+\mu) - \sigma^2} + \epsilon, \tag{36}$$

where ϵ is some positive number. The combination of $F'(0) > 0$ and $F'(\infty) > 0$ leads to case 3. The boundary given by (36) (with $\epsilon = 0$) is the dashed curve in Fig. 8.

6 Conclusions and Further Research

This paper aims to find the optimal refinancing time of a mortgage loan when the mortgage interest rate in the market follows a stochastic model. We assume such a stochastic model is mathematically tractable, in the form of a Vasicek model, for instance. The net present value of the refinancing mortgage is formulated, *i.e.*, Eq. (5). Accordingly, a general approach is provided to obtain the optimal refinancing time that gives the minimum value of the expected total net present payment for the mortgage contract.

Several simplifying steps were employed to solve the complex form of Eq. (5). First, we assume the mortgage contract has an infinite maturity, which leads to the much simpler Eq. (11). We demonstrated that such an assumption introduces small errors into the value of $F(t^*)$. However, the sign of $F'(0)$ (which is the key fact needed to determine whether to immediately refinance) does not change. A full analysis of the finite time horizon is an interesting problem in its own right, and will form the basis of further research.

Second, by taking the stochastic interest rates as following a Vasicek model, the moment generating function can be used to derive a closed-form approximation to the expected total net present payment for the mortgage. It is proven that to guarantee the convergence of the refinancing function $F(t^*)$, σ^2 must be less than $2\alpha^2\mu$. Hence the values of the parameters should be checked to meet the condition of convergence before applying the method provided in this paper for solving the optimal refinancing problem.

Once the refinancing function had been computed, we performed numerical simulations to conclude that there are three types of curves for the refinancing function, though only two involve refinancing at a finite time. Despite the fact that we are assuming an infinite time horizon, the optimal refinancing time is often within the duration of a standard mortgage contract.

The reverting rate α, the long-term rate μ and the volatility σ play significant roles on shaping the refinancing curve and so the "optimal refinancing" time, while the current interest rate, r_0, and the difference between the current interest rate and mortgage rate, κ, have little influence. These results were then verified with further analytical work. This work showed parameter ranges where the refinancing decision could be made without detailed calculations, and others where detailed calculations are required.

Further research will build on the current work by relaxing many of the assumptions made in this manuscript. We have already discussed in Sect. 2 how a one-time refi-

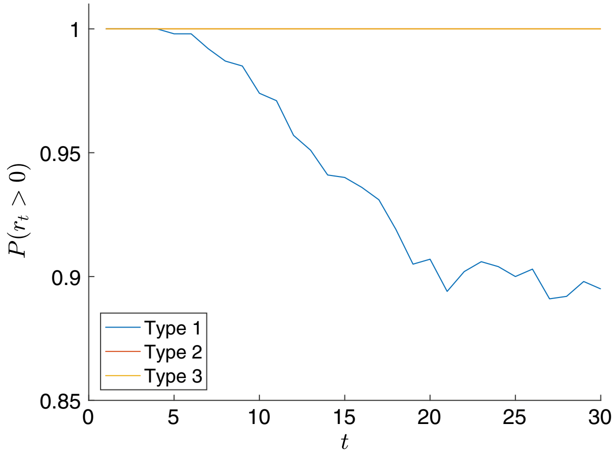

Fig. 9 Probability of positive r_t for each of the three graph types in Fig. 2

nancing cost may be implemented. Multiple refinancing opportunities would involve additional terms in the NPV function, but similar techniques to those used herein should be able to provide insight. A more significant complication would be a non-constant difference between the stochastic mortgage rates and the stochastic interest rates, though variations in κ are typically slight. Asymptotic techniques could allow further study of the equations for finite T.

A more considerable area of further research would be the use of a more realistic model than the Vasicek one. As verified in the Appendix, in our problem the Vasicek model yields positive interest rates with high probability. However, to *guarantee* positive interest rates, a more complicated model (for instance, CIR) must be used.

By defining the optimal refinancing time as that which minimizes the expected value of the NPV function (5), we have established a theoretical definition that can be analyzed in many different contexts. Using an infinite time horizon simplifies the problem enough that it can be examined analytically, not just numerically. Thus the work in this manuscript provides a firm foundation which can be expanded to handle the other topics discussed above.

Appendix

As discussed above, the Vasicek model does have the drawback that it is possible for r_t to become negative. To gain additional faith in our analysis, we ran 1000 interest-rate simulations using (1) and the parameters in Sect. 4. From those simuations, we estimated the probability that r_t would remain positive; the results are shown in Figs. 9, 10, 11, 12 and 13.

In Fig. 9, we see that only curves of Type 1 (optimal refinancing time in the future) have a non-negligible chance of having negative interest rates. Even in this case, however, over the life of a 30-year mortgage, the chance of having a negative interest rate was less than 11%. As expected, in Fig. 10, we see that the probability of having

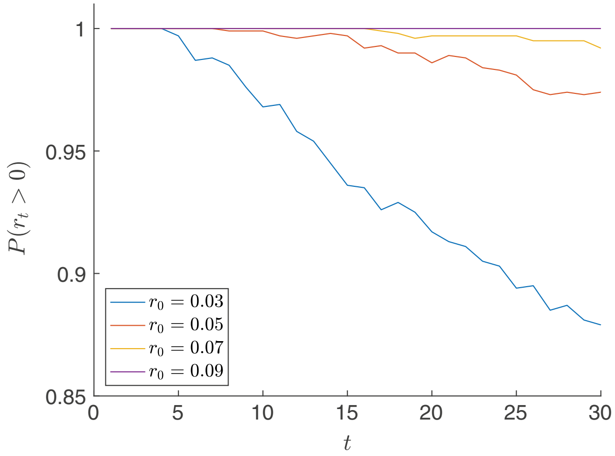

Fig. 10 Probability of positive r_t for the values of r_0 in Fig. 3

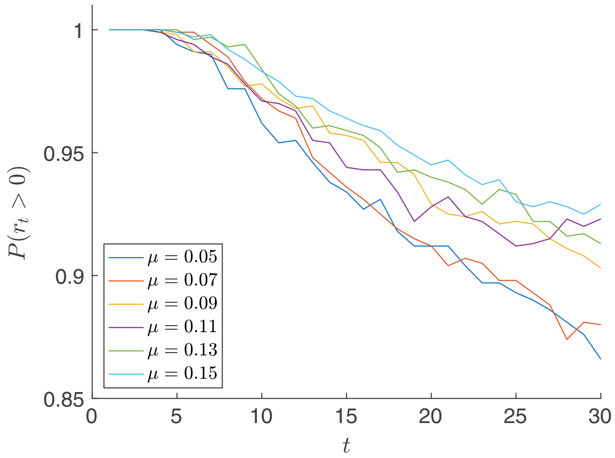

Fig. 11 Probability of positive r_t for the values of μ in Fig. 5

a negative interest rate increases as the initial rate r_0 decreases, with an extreme case still having a probability of a positive interest rate greater than 85%.

In Fig. 11, we see that as expected, the larger the long-term rate μ, the higher the probability of having a positive interest rate. However, since the transition to the long-term rate begins with $r_0 = 0.03 < \mu$, the effect of varying μ is muted. Much clearer are the results in Fig. 12, which show that higher levels of the variance lead to a higher probability of the interest rate becoming negative. But again, such probabilities are less than 15%.

In Fig. 13, we see that as the reversion rate α increases, the probability of having a negative interest rate decreases, as wandering rates are more quickly diverted to the (positive) long-term rate.

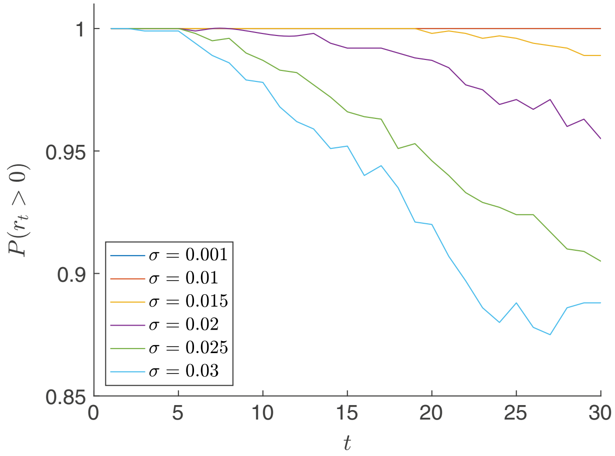

Fig. 12 Probability of positive r_t for the values of σ in Fig. 6

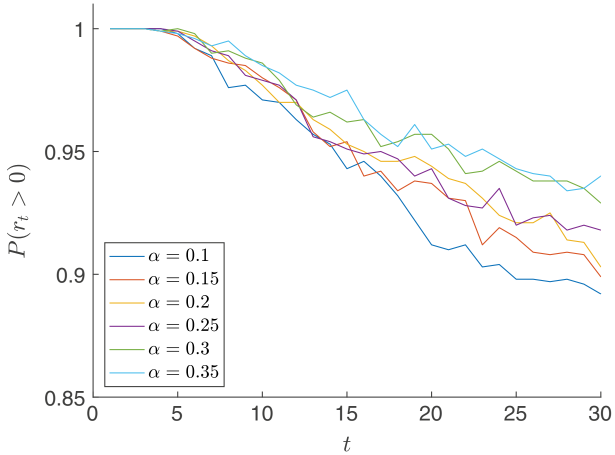

Fig. 13 Probability of positive r_t for the values of α in Fig. 7

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