On the number of k-gons in finite projective planes

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Abstract

Introduction

Over the years, many questions have surfaced regarding counting the number of certain substructures within a projective plane. In this paper we contribute to an open question in the area. We omit the standard definitions related to finite geometries and graph theory. For all undefined notions we refer the reader to Casse [5] for the notions in finite geometry, and to Bollobas [3] for all graph theoretic notions. We will also need the following definitions and notation.

Let Π denote a projective plane of order n with $N = n^2 + n + 1$ and $N_{(k)} = k!\binom{N}{k}$. Then N represents the number of points and the number of lines in Π. If A and B are points of Π, we write AB for the line containing them. We write S_k for the group of all permutations acting on $\{1, 2, \ldots, k\}$, the symmetric group.

Define a *quasi k-gon* to be a sequence $(P_1, P_2, \ldots, P_k)$ of k distinct points of Π with $k \geq 3$, together with a set $\mathcal{L}$ of all distinct lines of the form P_iP_{i+1} for $1 \leq i \leq k$. In this paper, all addition and subtraction in the indices is done modulo k. We will also allow ourselves to write $\mathcal{L} = \{P_iP_{i+1} : 1 \leq i \leq k\}$ with the understanding that different values of i can produce the same element in $\mathcal{L}$, By use of set notation in $\mathcal{L}$, we mean that $\{a, a, b\} = \{a, b\}$

It follows immediately that a quasi k-gon is uniquely determined by the sequence $(P_1, \ldots, P_k)$, so we denote it by QG_k and write $QG_k = (P_1, \ldots, P_k)$. If all lines of quasi k-gon are distinct, we call it a *k-gon* and denote it by G_k to distinguish a k-gon from a quasi k-gon. We will denote the set of points of QG_k by $\mathcal{P}_{QG_k}$ and the set of lines associated to QG_k by $\mathcal{L}_{QG_k}$.

The *point-line incidence graph* Γ_Π of Π, also known as the *Levi graph* of Π, is the bipartite graph with the set of points of Π to be one vertex part and the set of lines of Π to be the other vertex part. A point P is adjacent to a line ℓ in Γ_Π if P lies on ℓ in Π. We write $P \sim \ell$ to denote adjacency of a point and line in Γ_Π.

Let $c_k(\Pi)$ denote the number of distinct k-gons in Π and $c_{2k}(\Gamma_\Pi)$ denote the number of cycles of length $2k$ in Γ_Π. The connection between finite geometries and graph theory is of significant interest as there are many cases in which finite geometries were used to produce some of the best known results for various extremal type problems in graph theory.

Let $\mathrm{ex}(n, H, \mathcal{F})$ denote the maximum number of copies of a graph H in an n-vertex graph containing no graphs in $\mathcal{F}$ as a subgraph. When $H = K_2$(just an edge), then a simplified notation is used for $\mathrm{ex}(n, K_2, \mathcal{F})$, namely $\mathrm{ex}(n, \mathcal{F})$, and we call $\mathrm{ex}(n, \mathcal{F})$ the Turán number of $\mathcal{F}$. The problem of determining $\mathrm{ex}(n, \mathcal{F})$ is usually referred to as a *Turán type* problem. For the extensive literature related to Turán type problems, see Bollobas [4] , Füredi [9], Füredi and Simonovits [10], Verstraëte [17], Mubayi and Verstraëte [22], Lazebnik, Sun, and Wang [21].

Some of the earliest attention $\mathrm{ex}(n, H, \mathcal{F})$ received was from Erdös [7] who stated a conjecture regarding the extremal graph of $\mathrm{ex}(n, C_5, C_3)$. This conjecture was resolved by Hatami, Hladkýi, Král, Norine, and Razborov [16] and independently by Grzesik [14], building on the work of Györi [15]. The more recent wave of interest in $\mathrm{ex}(n, H, \mathcal{F})$ was initiated by Alon and Shikelman [2]. In relation to the work done in this note, we note that for a projective plane Π of order n, we have

$$c_{2k}(\Gamma_\Pi) \leq \mathrm{ex}(2N, C_{2k}, \mathcal{C}_4).$$

There have been several new results regarding the growth rate of $\mathrm{ex}(n, H, \mathcal{F})$ where $\mathcal{F} = \mathcal{C}_{2m}$ or $\mathcal{F} = C_{2m}$, with resolution up the leading term in certain cases. We refer the reader to the papers [11], [12], and [23] for the most up to date reading regarding $\mathrm{ex}(n, H, \mathcal{C}_{2m})$ and $\mathrm{ex}(n, H, C_{2m})$.

Counting k-gons in the projective plane is of interest in it's own right as well. An important consequence of answering such questions is that they may allow us to classify(up to isomorphism) and characterize projective planes as extremal objects. One can easily show that that the number of closed walks of length $2k$ in the Levi graph of any projective plane Π of order n is dependent only on n, and not on the actual plane. Let Π be a projective plane of order n and Γ_Π its Levi graph. Let A be the adjacency matrix of Γ_Π, as A is a symmetric-(0, 1) matrix, all its eigenvalues are real. Let $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_{2N}$ be the eigenvalues of A. It is easy to show, by considering eigenvalues of A^2, the eigenvalues of A are given by $\lambda_1 = n + 1$ and $\lambda_{2N} = -(n + 1)$ each with multiplicity one, and all other eigenvalues are equal to $\pm\sqrt{n}$ each with multiplicity $N - 1$. It follows, see(Biggs Algebraic graph theory), that the number of closed walks of length $2k$ in Γ_Π is given by

$$\mathrm{Trace}(A^{2k}) = \sum_{i=1}^{2N} \lambda_i^{2k} = 2(n+1)^{2k} + 2(N-1)n^k.$$

This may lead one to ask, what other structures appear in a finite projective plane Π and does the number of these structures in Π depend only the order of the plane? For example: Define a k-*arc* in a projective plane Π to be a set of k points of Π, no three of which are collinear. For $k \leq 6$, Glynn [13] showed that the number of k-arcs in a plane of order n does not depend on the plane. Furthermore, in [13], Glynn computes an exact formula for the number of 7-arcs in any finite projective plane, and using this formula deduces that there do not exist projective planes of order 6, as evaluating the formula at 6 yields a negative value. Glynn's work counting k-arcs was recently extended by Kaplan, Kimport, Lawrence, Peilen and Weinreich [18] who determined an explicit formula for the number of 9-arcs in an arbitrary projective plane. It is worth mentioning that for $k = 7, 8, 9$ the formula for the number of k-arcs depends on more than just the order of the plane.

In [19] Lazebnik, Mellinger, and Vega demonstrate that it is possible to embed a k-gon of every possible size into any affine or projective plane. This was further extended by Aceves, Heywood, Klahr, and Vega [1] who showed that one can embed a k-gon of every possible size in the projective space $PG(d, q)$. Moreover, in a different paper Lazebnik, Mellinger, and Vega [20] motivated the study of counting k-gons with the following two questions:

1. Assuming n is large compared to k, which partial planes with N points contain the largest number of k-gons? Equivalently, which C_4-free bipartite graphs with partitions of size N contain the greatest number of $2k$-cycles?

2. Do all projective planes of order n contain the same number of k-gons?

Fiorini and Lazebnik [8] show that projective planes have the largest number of triangles(3-gons) amongst all partial planes. This work is extended by De Winter, Lazebnik, and Verstraëte [6] who show that the same holds when $k = 4$. In [20], progress towards question 2 is made as the exact value of $c_k(\Pi)$ is determined for $k = 3, 4, 5, 6$ showing that in these cases $c_k(\Pi)$ is dependent only on the order of Π. This work was further extended by Voropaev[24] again demonstrating $c_k(\Pi)$ is dependent only on the order of Π up to $k = 10$. Determining explicit formulas for larger k may very well have interesting consequences just like in the example of formula for the number of 7-arcs in a projective plane.

In this paper, we make some progress towards resolving question 2 as we determine the leading term in the asymptotic of the number of k-gons in an arbitrary projective plane. The magnitude of the leading term for the number of k-gons is shown to be the same as that of the number of closed walks of length $2k$ in the Levi graph of a finite projective plane, but with different leading coefficients.

Here we list our main results.

Main Theorem. *Let Π be a projective plane of order n and Γ_Π it's point-lince incidence graph. Then for fixed k and $n \to \infty$ we have that asymptotically*

$$c_{2k}(\Gamma_\Pi) \sim \frac{1}{2k} n^{2k} = \frac{1}{2^{k+1}k} |\Gamma_\Pi|^k.$$

Corollary. Let n be a prime power and $v = 2(n^2 + n + 1)$, then as $v \to \infty$

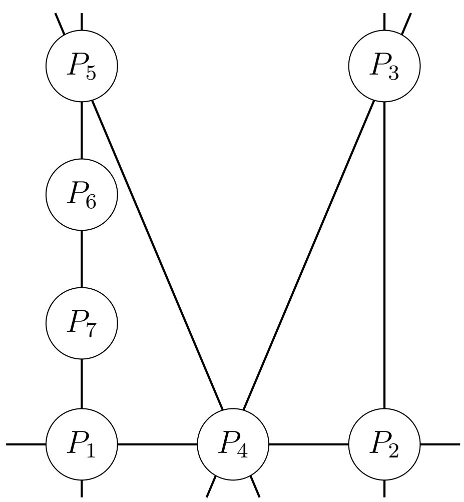
$$\text{ex}(v, C_{2k}, \mathcal{C}_4) \geq \left(\frac{1}{2^{k+1}k} - o(1) \right) v^k.$$

Proof of Main Theorem

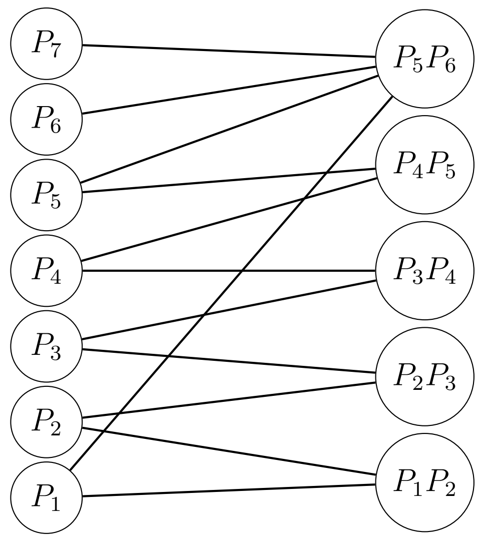
Let Π be a projective plane and Γ_Π be its Levi graph. Let $QG_k = (P_1, \ldots, P_k)$ be a quasi k-gon. Define the subgraph Γ_{QG_k} of Γ_Π corresponding to QG_k as follows: The set of vertices $V(\Gamma_{QG_k})$ is given by $\mathcal{P}_{QG_k} \cup \mathcal{L}_{QG_k}$. The edges $E(\Gamma_{QG_k})$ are obtained by joining a vertex P_i to all vertices in the set $\{P_{i-1}P_i, P_iP_{i+1}\}$ for $1 \le i \le k$. If $P_{i-1}P_i = P_iP_{i+1}$ then P_i has only one neighbor. It is clear that in the case that all lines are distinct, meaning QG_k is actually a k-gon, the corresponding graph Γ_{QG_k} is a cycle of length $2k$.

Let QG_7 be a quasi 7-gon given by the following figure. We use QG_7 to demonstrate the corresponding graph Γ_{QG_7}.

QG_7



Γ_{QG_7}



Let us take a moment to comment on the above figure. Here $QG_7 = (P_1, P_2, \ldots, P_7)$, with the corresponding set of lines $\{P_iP_{i+1} : 1 \le i \le k\}$. We assume that $P_1P_2, P_2P_3, P_3P_4, P_4P_5, P_5P_6$ are all distinct lines. Observe that P_4 lies on the line P_1P_2, however, $P_4 \not\sim P_1P_2$ in Γ_{QG_7}. By definition of Γ_{QG_7}, we have only that $P_4 \sim P_3P_4$ and $P_4 \sim P_4P_5$. Furthermore, note that $P_5P_6 = P_6P_7 = P_7P_1$ and therefore P_6 and P_7 each only have one neighbor, namely P_5P_6.

The symmetric group S_k acts on quasi k-gons in Π in the following way: If $QG_k = (P_1, \ldots, P_k)$ and $\sigma \in S_k$, then $\sigma(QG_k) := (P_{\sigma(1)}, \ldots, P_{\sigma(k)})$. Hence

$$\mathcal{P}_{\sigma(QG_k)} = \{P_{\sigma(1)}, \ldots, P_{\sigma(k)}\} = \{P_1, \ldots, P_k\} = \mathcal{P}_{QG_k}$$

and the lines of $\sigma(QG_k)$ are

$$\mathcal{L}_{\sigma(QG_k)} = \{P_{\sigma(i)}P_{\sigma(i+1)} : 1 \le i \le k\}.$$

Note that in general, $\mathcal{L}_{QG_k}$ is not necessarily equal to $\mathcal{L}_{\sigma(QG_k)}$.

We call two quasi k-gons $QG_k = (P_1, \ldots, P_k)$ and $QG'_k = (P'_1, \ldots, P'_k)$ *equivalent*, and write $QG_k \equiv QG'_k$, if $\Gamma_{QG_k} = \Gamma_{QG'_k}$, that is, they have the same vertex and edge set. It is obvious that equivalence of quasi k-gons is an equivalence relation and that

$QG_k \equiv QG'_k$ if and only if there exists $\sigma \in S_k$ such that $\sigma(QG_k) = QG'_k$. Therefore $S(QG_k) := \{\sigma \in S_k : QG_k \equiv \sigma(QG_k)\}$ forms a subgroup of S_k.

Remark: Given a quasi k-gon QG_k and permutation $\sigma \in S_k$ we stress the following points: We do not think of QG_k as a partial plane in Π defined by the points and lines of QG_k. Therefore, if $QG_k \equiv \sigma(QG_k)$, then σ should not be thought of as a collineation on this partial plane. As an example, we refer to the figure above of QG_7 and consider $\sigma = (1234567)$. In the lemma that follows, we demonstrate that $QG_7 \equiv \sigma(QG_7)$, however, observe that while P_5, P_6, P_7, P_1 lie on one line in Π, $P_{\sigma(5)} = P_6$, $P_{\sigma(6)} = P_7$, $P_{\sigma(7)} = P_1$ and $P_\sigma(1) = P_2$ are not collinear in Π.

Lemma. *Let Π be a projective plane and $QG_k = (P_1, \ldots, P_k)$ in Π. Then $S(QG_k)$ contains a quasi Dihedral group D_k as a subgroup.*

Proof. Let $\sigma = (12\ldots k)$, so that $\sigma(QG_k) = (P_{\sigma(1)}, \ldots, P_{\sigma(k)}) = (P_2, \ldots, P_k, P_1)$. We wish to show that $\sigma(QG_k) \equiv QG_k$. Note that the vertex set $V(\Gamma_{\sigma(QG_k)}) = \mathcal{P}_{\sigma(QG_k)} \cup \mathcal{L}_{\sigma(QG_k)} = \mathcal{P}_{QG_k} \cup \mathcal{L}_{\sigma(QG_k)}$. Here

$$\begin{aligned}\mathcal{L}_{\sigma(QG_k)} &= \{P_{\sigma(i)}P_{\sigma(i+1)} : 1 \le i \le k\} = \{P_{i+1}P_{i+2} : 1 \le i \le k\} \\ &= \{P_iP_{i+1} : 1 \le i \le k\} = \mathcal{L}_{QG_k}.\end{aligned}$$

Thus we have $V(\Gamma_{\sigma(QG_k)}) = V(\Gamma_{QG_k})$. The edge set $E(\Gamma_{\sigma(QG_k)})$ is given by joining $P_{\sigma(i)} = P_{i+1}$ to all distinct lines in $\{P_{\sigma(i-1)}P_{\sigma(i)}, P_{\sigma(i)}P_{\sigma(i+1)}\} = \{P_iP_{i+1}, P_{i+1}P_{i+2}\}$ where $1 \le i \le k$. These are exactly the same edges that appear in Γ_{QG_k}. Thus, $\sigma \in S(QG_k)$ and has order k.

Now consider the permutation ρ of $\{1, 2, \ldots, k\}$, such that $\rho(i) = k + 1 - i$. That is, $\rho(QG_k) = (P_k, P_{k-1}, \ldots, P_1)$. Set $j = k + 1 - i$, then $\rho(i) = j$ and $\rho(i + 1) = j - 1$. Observe that $V(\Gamma_{\rho(QG_k)}) = \mathcal{P}_{QG_k} \cup \mathcal{L}_{\rho(QG_k)}$ where

$$\begin{aligned}\mathcal{L}_{\rho(QG_k)} &= \{P_{\rho(i)}P_{\rho(i+1)} : 1 \le i \le k\} = \{P_jP_{j-1} : 1 \le j \le k\} \\ &= \{P_{j-1}P_j : 1 \le j \le k\} = \mathcal{L}_{QG_k}.\end{aligned}$$

Therefore $V(\Gamma_{\rho(QG_k)}) = V(\Gamma_{QG_k})$. The edge set $E(\Gamma_{\rho(QG_k)})$ is given by joining $P_{\rho(i)} = P_j$ to all distinct lines in $\{P_{\rho(i-1)}P_{\rho(i)}, P_{\rho(i)}P_{\rho(i+1)}\} = \{P_{j+1}P_j, P_jP_{j-1}\}$ for $1 \le j \le k$. These are exactly the same edges that appear in Γ_{QG_k}. Thus, $\rho \in S(QG_k)$ and has order 2.

Clearly, the action of ρ cannot be obtained by taking powers of σ. Therefore $\langle \sigma, \rho \rangle \cong D_k$ forms a subgroup of the group of symmetries of any quasi k-gon QG_k in Π. □

Corollary. Let Π be a projective plane, and $G_k = (P_1, \ldots, P_k)$ be a k-gon in Π. Then the group of symmetries of G_k is precisely the Dihedral group D_k.

Proof. Let D_k represent the subgroup of $S(G_k)$ described above. If $\tau \in S_k$ is a permutation satisfying $\tau(i + 1) = \tau(i) \pm 1 \pmod{k}$, for all i, $1 \le i \le k$. Then in fact, we must have that either:

1. $\tau(i + 1) = \tau(i) + 1$ for all $1 \le i \le k$.

2. $\tau(i+1) = \tau(i) - 1$ for all $1 \leq i \leq k$.

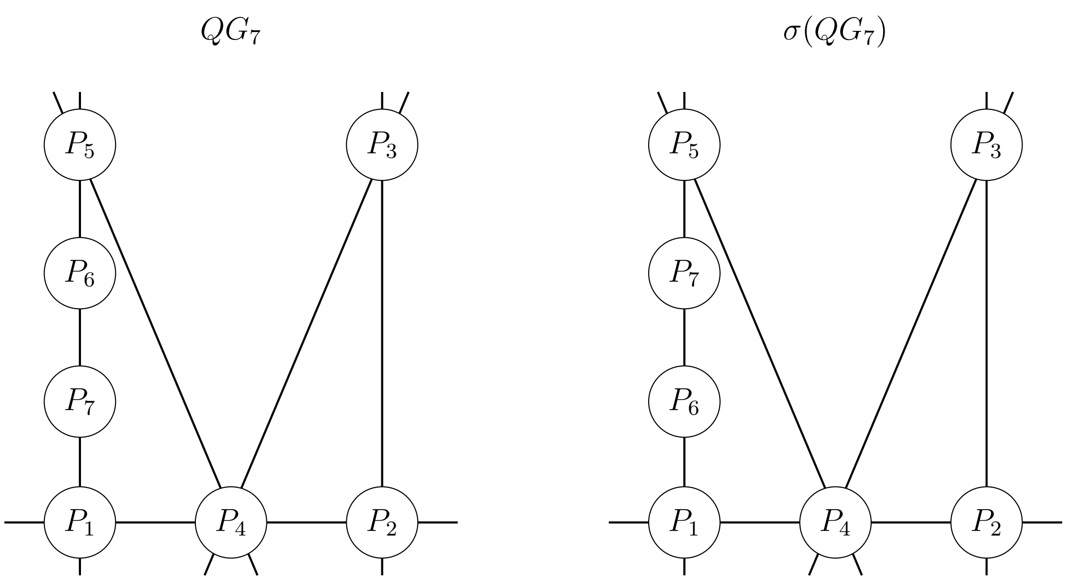
Any permutation satisfying either condition must in fact be an element of D_k.

Suppose then that $\tau \in S_k \setminus D_k$, which implies that there exists an i, $1 \leq i \leq k$ for which $\tau(i+1) \neq \tau(i) \pm 1$. For this i, let $\tau(i) = \ell$ and $\tau(i+1) = j$. We note that the line $P_\ell P_j$ is a vertex in $\Gamma_{\tau(G_k)}$. If $P_\ell P_j \neq P_i P_{i+1}$ for all $1 \leq i \leq k$, then this line corresponds to a vertex that is not in Γ_{G_k} and so $\tau \notin S(G_k)$.

So suppose that $P_\ell P_j = P_m P_{m+1}$ for some m, $1 \leq m \leq k$. If $\ell \neq m$ and $\ell \neq m+1$ then in $\Gamma_{\tau(G_k)}$ we have $P_\ell \sim P_m P_{m+1}$. This edge is not in Γ_{G_k} as the lines of G_k are distinct and $P_{\ell-1}P_\ell \sim P_\ell \sim P_\ell P_{\ell+1}$ in Γ_{G_k}. Thus $\tau \notin S(G_k)$. The same exact argument can be followed with $j \neq m$ and $j \neq m+1$.

It is easy to see that if $P_\ell P_j = P_m P_{m+1}$ for some $1 \leq m \leq k$, then we have either that $\ell \neq m$ and $\ell \neq m+1$ or $j \neq m$ and $j \neq m+1$. Suppose not, then we must have either $\ell = m$ and $j = m+1$ or $\ell = m+1$ and $j = m$. Both of these options are contradictions since we assumed $j \neq m \pm 1$. Thus $S(G_k) = D_k$. □

We now provide an example showing that it is possible to to have $S(QG_k)$ be strictly larger than the quasi dihedral group discussed above. We refer to our previous example of QG_7, and we consider the permutation $\sigma = (67)$.



The reader should convince themselves that both QG_7 and $\sigma(QG_7)$ have the same corresponding graph, namely the graph Γ_{QG_7} which we have drawn in the previous figure.

Theorem 1. *Let Π be a finite projective plane of order n and Γ_Π its Levi graph. Then*

$$c_{2k}(\Gamma_\Pi) < \frac{1}{2k} N_{(k)}$$

Proof. Recall that if G_k is a k-gon in Π, then Γ_{G_k} is a cycle of length $2k$ in Γ_Π. Then it is clear that $c_{2k}(\Gamma_\Pi)$ is given by the number of non-equivalent k-gons in Π. Obviously the number of non-equivalent k-gons is less than the number of non-equivalent quasi

k-gons. Let $Q_k = \{QG_k \in \Pi\}$ denote the set of all quasi k-gons in Π. Recall that $\equiv$ is an equivalence relation on Q_k. Then let $C_1, \ldots, C_t$ be the set of all equivalence classes of Q_k, where t is then the number of all non-equivalent quasi k-gons in Π. Clearly

$$N_{(k)} = |Q_k| = \sum_{i=1}^{t} |C_i|.$$

We wish to use the above equation to place a bound on t. Note that $|C_i| = S(QG_k)$ for any $QG_k \in C_i$. Furthermore, as a consequence of our lemma, we know that $2k$ divides $S(QG_k)$ for all QG_k. Therefore

$$c_{2k}(\Gamma_\Pi) < t = \sum_{i=1}^{t} 1 < \sum_{i=1}^{t} \frac{|C_i|}{2k} = \frac{1}{2k} N_{(k)}.$$

□

Theorem 2. *Let* Π *be a finite projective plane of order* n *and* Γ_Π *be the point-line incidence graph of* Γ_Π. *Then*

$$c_{2k}(\Gamma_\Pi) > \frac{1}{2k} N_{(k)} - \frac{1}{2k}\binom{k-1}{2} N_{(k-1)}(n-1)$$

Proof. We wish to show that the number of quasi k-gons that are not k-gons is at most

$$k!\binom{k-1}{2}\binom{n^2+n+1}{k-1}(n-1). \tag{1}$$

To do this we construct a set of quasi k-gons of the given size and show that any quasi k-gon that is not a k-gon is captured in this construction. The construction is as follows. Choose any $k-1$ points of Π, call them $\{P_1, \ldots, P_{k-1}\}$. There are $\binom{n^2+n+1}{k-1}$ ways to do this. Now consider the collection of all possible lines of the form P_iP_j from this collection of points. There are at most $\binom{k-1}{2}$ distinct lines in this collection. Note that there are at most $(n-1)$ other points that lie on any P_iP_j that are distinct from $\{P_1, \ldots, P_{k-1}\}$. Now choose P_k to be any one point on any of the lines P_iP_j that is distinct from $\{P_1, \ldots, P_{k-1}\}$. There are at most $\binom{k-1}{2}(n-1)$ choices for P_k. Thus we have constructed a set of k points, and there are at most

$$\binom{k-1}{2}\binom{n^2+n+1}{k-1}(n-1)$$

many such sets. Now every set above can be ordered $k!$ ways, and thus produce $k!$ quasi k-gons per set. Thus we have a set of quasi k-gons of size

$$k!\binom{k-1}{2}\binom{n^2+n+1}{k-1}(n-1).$$

We claim that if $QG_k = (P_1, \ldots, P_k)$ is any quasi k-gon that is not a k-gon, then QG_k is accounted for in the above construction. Since QG_k is not a k-gon, then this implies that the number of distinct lines in QG_k is at most $k-1$. This means we must have

$P_\ell P_{\ell+1} = P_m P_{m+1}$ for some $1 \leq \ell < m \leq k$. Note this implies that $P_\ell, P_{\ell+1}, P_m, P_{m+1}$ all lie on the same line. Furthermore, at least three of the points $P_\ell, P_{\ell+1}, P_m, P_{m+1}$ must be distinct since $\ell \neq m$. Now, we must have one at least one of the two following cases hold:

1. $m + 1 \neq \ell (\text{mod } k)$
2. $\ell + 1 \neq m (\text{mod } k)$

If both cases held, this would imply $k = 2$ for which we recall that a quasi k-gon had $k \geq 3$ by definition.

Regardless of whether case 1, case 2, or both cases hold, the following proof goes in exactly the same manner. Therefore, without loss of generality, suppose $m+1 \neq \ell (\text{mod } k)$ then consider the subset $\mathcal{P}' = \{P'_1, \ldots, P'_{k-1}\} \subset \{P_1, \ldots, P_k\}$ that does not include P_{m+1}. Note that this means $P_\ell, P_{\ell+1} \in \mathcal{P}'$. This means in our construction above, we begin by choosing the set $\mathcal{P}'$ of $k-1$ points and note that $P_\ell P_{\ell+1}$ is one of the distinct lines obtained in the following step of the construction. Therefore, we may choose any point on $P_\ell P_{\ell+1}$ distinct from $\mathcal{P}'$ as the kth point to add to $\mathcal{P}'$. More specifically, we may choose P_{m+1} as our kth point. Set $P'_k = P_{m+1}$ and then choose $\tau \in S_k$ that gives $P'_{\tau(i)} = P_i$ so that the quasi k-gon $(P'_{\tau(1)}, \ldots, P'_{\tau(k)}) = (P_1, \ldots, P_k)$. Thus QG_k is captured in the above construction. Furthermore, if QG_k is a quasi k-gon, captured in the above construction, then all quasi k-gons equivalent to QG_k must also be captured in the construction.

Recall that each quasi k-gon has D_k as a subgroup of the group of symmetries, and so we may divide out by all such symmetries as we did in the case of the upper bound. By deleting the quasi k-gons obtained from our construction, from the count in the upper bound, we are left with only k-gons that have each been counted at most once. Therefore

$$C_{2k}(\Gamma_\Pi) > \frac{k!}{2k}\binom{n^2+n+1}{k} - \frac{k!}{2k}\binom{k-1}{2}\binom{n^2+n+1}{k-1}(n-1).$$

□

Corollary. Let Π be a projective plane of order n and Γ_Π it's point-lince incidence graph. Then for fixed k and $n \to \infty$ we have that asymptotically

$$c_{2k}(\Gamma_\Pi) \sim \frac{1}{2k} n^{2k} = \frac{1}{2^{k+1}k} N^k.$$

Corollary. Let n be a prime power and $v = 2(n^2+n+1)$, then

$$\text{ex}(v, C_{2k}, \mathcal{C}_4) \geq \left(\frac{1}{2^{k+1}k} - o(1)\right) v^k.$$

Concluding Remarks

Acknowledgments

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