Abstract

We show that Abelian Cayley graphs contain many closed walks of even length. This implies that for each $k \geq 3$, there exists $C = C(\epsilon, k) > 0$ such that for each Abelian group $G$ and each symmetric subset $S$ of $G$ with $1 \notin S$, the number of eigenvalues $\lambda_i$ of the Cayley graph $X = X(G,S)$ such that $\lambda_i \geq k - \epsilon$ is at least $C \cdot |G|$. This can be regarded as an analogue for Abelian Cayley graphs of a theorem of Serre for regular graphs. To cite this article: S.M. Cioabă, C. R. Acad. Sci. Paris, Ser. I

1. Introduction

Let $X$ be a $k$-regular, connected graph on $n$ vertices. Denote by $t_r(u)$ the number of closed walks of length $r$ starting at a vertex $u$ of $X$ and let $\Phi_r(X) = \sum_{u \in X} t_r(u)$. Let $k = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of the adjacency matrix of $X$. The graph $X$ is called Ramanujan if $|\lambda_j| \leq 2\sqrt{k-1}$ for each $\lambda_j \neq \pm k$. One of the hardest problems in graph theory is constructing infinite families of $k$-regular Ramanujan graphs for $k \geq 3$ fixed. The only constructions known (see [4,7]) are for $k = 1$ a power of a prime and are obtained from Cayley graphs of certain matrix groups.

J.-P. Serre [4] proved the following result regarding the largest eigenvalues of regular graphs. For a simple proof and related results, see [2,3,8].

**Theorem 1.1** (Serre). For each $\epsilon > 0$ and $k$, there exists a positive constant $c = c(\epsilon, k)$ such that for any $k$-regular graph $X$, the number of eigenvalues $\lambda_i$ of $X$ with $\lambda_i \geq (2 - \epsilon)\sqrt{k-1}$ is at least $c|X|$.
In this Note, we prove that Abelian Cayley graphs have a large number of closed walks of even length. We use this fact to give a simple proof of the following Serre-type theorem for Abelian Cayley graphs.

**Theorem 1.2.** For each \( \epsilon > 0 \) and \( k \), there exists a positive constant \( C = C(\epsilon, k) \) such that for any Abelian group \( G \) and for any symmetric set \( S \) of elements of \( G \) with \( |S| = k \) and \( 1 \notin S \), the number of the eigenvalues \( \lambda_i \) of the Cayley graph \( X = X(G, S) \) such that \( \lambda_i \geq k - \epsilon \) is at least \( C \cdot |G| \).

Cayley graphs are defined as follows. Let \( G \) be a finite multiplicative group, with identity 1 and suppose \( S \) is a subset of \( G \) such that \( 1 \notin S \) and \( s \in S \) implies \( s^{-1} \in S \). The Cayley graph \( X = X(G, S) \) is the graph with vertex set \( G \) and with \( x, y \in G \) adjacent if \( xy^{-1} \in S \). Notice that adjacency is well-defined since \( S \) is symmetric. Also, \( G \) is regular with valency \( k = |S| \) and it contains no loops since \( 1 \notin S \). It is easy to see that \( X \) is connected if and only if \( S \) generates \( G \). If \( G \) is an Abelian group and \( S \) is a symmetric subset of \( k \) elements of \( G \), then the eigenvalues of \( X(G, S) \) are \( \lambda_X = \sum_{x \in S} \chi(x) \) where \( \chi \) ranges over all the characters of \( G \) (see Li [9]). This fact was used by Friedman, Murty and Tillich [6] who proved that the second largest eigenvalue of a \( k \)-regular Abelian Cayley graph with \( n \) vertices is at least \( k - O(kn^{-4/3}) \).

There are Abelian Cayley graphs that are Ramanujan (see Li [9]). The proof that these graphs are Ramanujan is often based on number theoretic estimates of character sums. For each choice of a degree of regularity, all these constructions produce only a finite number of Ramanujan graphs. Theorem 1.2 shows that it is impossible to construct an infinite family of constant degree Abelian Cayley graphs that are Ramanujan. This also follows from [1] and [6].

### 2. Closed walks in Abelian Cayley graphs

Let \( G \) be a finite Abelian group. There is a simple bijective correspondence between the closed walks of length \( r \) starting at a vertex \( u \) of a Cayley graph \( X(G, S) \) and the \( r \)-tuples \( (a_1, \ldots, a_r) \in S^r \) with \( \prod_{i=1}^r a_i = 1 \). To each closed walk \( u = u_0, u_1, \ldots, u_{r-1}, u_r = u \), we associate the \( r \)-tuple

\[
(u_0 u_1^{-1}, u_1 u_2^{-1}, \ldots, u_{r-2} u_{r-1}^{-1}, u_{r-1} u_r^{-1}) \in S^r.
\]

Suppose that

\[
S = \{x_1, x_2, \ldots, x_s, x_1^{-1}, x_2^{-1}, \ldots, x_s^{-1}, y_1, \ldots, y_t\},
\]

where each \( x_i \) has order greater than 2 for \( 1 \leq i \leq s \) and each \( y_j \) has order 2 for \( 1 \leq j \leq t \). The degree of the Cayley graph of \( G \) with respect to \( S \) is \( k = 2s + t \). Let \( W_{2r}(X) \) be the number of \( 2r \)-tuples from \( S^{2r} \) in which \( r \) number of appearances of \( x_i \) is the same as the number of appearances of \( x_i^{-1} \) for all \( 1 \leq i \leq s \) and the number of appearances of \( y_j \) is even for all \( 1 \leq j \leq t \). More precisely, \( W_{2r}(X) \) counts \( 2r \)-tuples from \( S^{2r} \) in which \( p \) positions are occupied by \( x_i \)’s, \( p \) positions are occupied by \( x_i^{-1} \)’s and the remaining \( 2r - 2p \) positions are occupied by \( y_j \)’s (each of them appearing an even number of times), where \( 0 \leq p \leq r \). These choices imply that the product of the \( 2r \) elements in this type of \( 2r \)-tuple is 1.

Thus, \( t_{2r}(u) \geq W_{2r}(X) \) for each \( u \in V(X) \). This implies

\[
\Phi_{2r}(X) = \sum_{u \in V(X)} t_{2r}(u) \geq n W_{2r}(X),
\]

for each \( r \geq 1 \).

We evaluate \( W_{2r}(X) \) by choosing first the \( 2p \) positions for the \( x_i \)’s and their inverses. This can be done in \( \binom{2r}{2p} \) ways. Then we choose \( p \) positions for the \( x_j \)’s. This is done in \( \binom{2p}{p} \) ways and the rest are left for \( x_i^{-1} \)’s. Since this happens for all \( 0 \leq p \leq r \), we get the following expression for \( W_{2r}(X) \)

### Lemma 2.1

For each \( r \geq 1 \), we have

\[
W_{2r}(X) = \sum_{p=0}^{r} \binom{2r}{2p} \binom{2p}{p} \sum_{i_1+\ldots+i_p=p} \left( \binom{p}{i_1, \ldots, i_s} \right)^2 \sum_{2j_1+\ldots+2j_l=2r-2p} \binom{2r-2p}{2j_1, \ldots, 2j_l}.
\]
We now obtain lower bounds for
\[ c(p, s) = \sum_{i_1 + \ldots + i_p = s} \left( \binom{p}{i_1, \ldots, i_p} \right)^2 \quad \text{and} \quad d(r - p, t) = \sum_{2j_1 + \ldots + 2j_r = 2r - 2p} \left( \binom{2r - 2p}{2j_1, \ldots, 2j_r} \right). \]

Obtaining a closed formula for any of these two sums seems to be an interesting and difficult combinatorial problem in itself. We use the Cauchy–Schwarz inequality to obtain a lower bound on \( c(m, l) \).

\[ c(m, l) = \sum_{i_1 + \ldots + i_m = l} \left( \binom{m}{i_1, \ldots, i_l} \right)^2 \geq \frac{(\sum_{i_1 + \ldots + i_m = l}(\binom{m}{i_1, \ldots, i_l})^2}{\binom{m+l-1}{l-1}^2} = \frac{l^{2m}}{(m+l-1)} . \]

Our lower bound for \( d(m, l) \) follows from the following result of Fixman [5].

\[ d(m, l) = 2^{-l} \sum_{j=0}^{l} \left( \binom{l}{j} \right) (l - 2j)^{2m} > 2^{1-l} l^{2m}. \]

Hence, we have

\[ c(m, l) > \frac{l^{2m}}{(m+l-1)}, \quad d(m, l) > \frac{l^{2m}}{2^{l-1}}. \tag{3} \]

These two inequalities and Lemma 2.1 easily imply the next result. Recall that \( k = 2s + t \).

**Lemma 2.2.** For each \( r \geq 1 \), we have

\[ W_{2r}(X) > \frac{k^{2r}}{2^k(2r+1)(k+r-1)}. \]

**Proof.** Using Lemma 2.1, inequalities (3) and \( \binom{2p}{p} > \binom{2^{p}}{2^{p+1}} \), we get

\[ W_{2r}(X) = \sum_{p=0}^{r} \left( \frac{2r}{2p} \right) \binom{2p}{p} c(p, s) d(r - p, t) > \sum_{p=0}^{r} \left( \frac{2r}{2p+1} \right) \binom{2^{2p}}{2^{2p+1}} \binom{2^{2p}}{2^{2p+1}} \binom{2^{2p}}{2^{2p+1}} \frac{k^{2r-2p}}{2^{2r-2p}} \]

\[ > \sum_{p=0}^{r} \left( \frac{2r}{2p+1} \right) \frac{(2s)^{2p} t^{2r-2p}}{(2r+1)(k+r-1)} > \frac{1}{(2r+1)(k+r-1)} \sum_{p=0}^{r} \left( \frac{2r}{2p+1} \right) \binom{2^{2p}}{2^{2p}} \binom{2^{2p}}{2^{2p}} \binom{2^{2p}}{2^{2p}} \binom{2^{2p}}{2^{2p}} \binom{2^{2p}}{2^{2p}} \frac{k^{2r}}{2^{2r+1}(k+r-1)}. \]

\[ \square \]

3. The proof of Theorem 1.2

We now present the proof of Theorem 1.2.

**Proof.** Let \( \epsilon > 0 \). Consider an Abelian group \( G \) and \( S \) a subset of \( G \) of size \( k \). Denote by \( n \) the order of \( G \) and by \( m \) the number of eigenvalues \( \lambda_i \) of \( X = X(G, S) \) such that \( \lambda_i \geq k - \epsilon \). Then there are exactly \( n - m \) eigenvalues of \( X \) that are less than \( k - \epsilon \). It follows that

\[ \text{tr}(kI + A)^{2l} = \sum_{i=1}^{n} (k + \lambda_i)^{2l} < (n - m)(2k - \epsilon)^{2l} + m(2k)^{2l}, \tag{4} \]

for each \( l \geq 1 \).

Using Lemma 2.2 and (2), we obtain the following
\[ \text{tr}(kI + A)^{2l} = \sum_{i=0}^{2l} \binom{2l}{i} k^i \Phi_{2l-i}(X) \geq \sum_{j=0}^{l} \binom{2l}{2j} k^{2j} \Phi_{2l-2j}(X) > n \sum_{j=0}^{l} \binom{2l}{2j} k^{2l-2j} \frac{k^{2l-2j}}{2^k(2(l-j)+1)(\binom{k+l-j-1}{k-1})} > n \frac{(2k)^{2l}}{2^{k+1}(2l+1)(\binom{k+l-1}{k-1})} \]

for each \( l \geq 1 \). Combining this inequality with (4), it follows that

\[ m/n > \frac{\frac{1}{2^{k+1}(2l+1)(\binom{k+l-1}{k-1})} (2k)^{2l} - (2k-\epsilon)^{2l}}{(2k)^{2l} - (2k-\epsilon)^{2l}}. \] (5)

for each \( l \geq 1 \). Now

\[ \lim_{l \to \infty} \frac{1}{2^{k+1}(2l+1)(\binom{k+l-1}{k-1})} (2k)^{2l} = 2k > 2k - \epsilon = \lim_{l \to \infty} \frac{2\sqrt{2k-\epsilon)^{2l}}. \]

This implies that there exists \( l_0 = l(\epsilon, k) \) such that

\[ \frac{1}{2^{k+1}(2l+1)(\binom{k+l-1}{k-1})} (2k)^{2l} - (2k-\epsilon)^{2l} > (2k-\epsilon)^{2l}, \]

for each \( l \geq l_0 \). Letting

\[ C(\epsilon, k) = \frac{(2k-\epsilon)^{2l_0}}{(2k)^{2l_0} - (2k-\epsilon)^{2l_0}} > 0 \]

it follows that

\[ m/n > C(\epsilon, k). \]

This proves the theorem. \( \square \)

References