SPANNING TREES, TOUGHNESS, AND EIGENVALUES OF REGULAR GRAPHS

by

Wiseley Wong

A dissertation submitted to the Faculty of the University of Delaware in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics

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SPANNING TREES, TOUGHNESS, AND EIGENVALUES OF REGULAR GRAPHS

by

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Spectral graph theory is a branch of graph theory which finds relationships between structural properties of graphs and eigenvalues of matrices corresponding to graphs. In this thesis, I obtain sufficient eigenvalue conditions for the existence of edge-disjoint spanning trees in regular graphs, and I show this is best possible. The vertex toughness of a graph is defined as the minimum value of $\frac{|S|}{c(G \setminus S)}$, where $S$ runs through all subsets of vertices that disconnect the graph, and $c(G \setminus S)$ denotes the number of components after deleting $S$. I obtain sufficient eigenvalue conditions for a regular graph to have toughness at least 1, and I show this is best possible. Furthermore, I determine the toughness value for many families of graphs, and I classify the subsets $S$ of each family for when this value is obtained.
Chapter 1
INTRODUCTION

1.1 Introduction and history

A graph can be regarded as a mathematical model relating various objects such as computers, molecules, or locations. This simple representation is applicable in various fields, including computer science, chemistry, biology, and sociology. The objects are vertices in a graph, and we join them together with an edge if they satisfy certain properties, such as directly transferring data, bonding molecules, or traveling routes. In some applications, it is valuable to detail the edges by putting weights (cost of travel) or a direction (one-way streets) on them.

The Seven Bridges of Königsberg is a historically well-known problem ([31], [61, Section 2.4]). It involves finding a route on four land masses that crosses seven bridges exactly once, and ends on the same land mass as the starting one (see Figure 1.1). This problem is notable for laying the foundations of graph theory. The traveling salesman problem is another famous problem ([61, Section 6.4]), often applied in optimization and computer science. Given a list of cities and the cost to travel between any two cities, it asks to find the cheapest route to visit all cities once, and return to the starting city.

Numerous extensions and variations on graph theory problems have been studied, leading to many open problems and research in other areas of mathematics. This has produced many different branches of graph theory. In this thesis, much of the work involves the branch of spectral graph theory, which finds relationships between structural properties of graphs and eigenvalues of matrices corresponding to graphs (see [2, 12] for more details).
Spectral graph theory was first applied in 1931 in chemistry [43], which related eigenvalues with stability of molecules. Moreover, physics and quantum mechanics associated eigenvalues with energy states. It was not until 1957 when the first mathematical paper in this area was written [26]. Eigenvalues have played a significant role in expander graphs (informally, an expander is a highly connected and sparse graph) [41]. Expanders have applications in computer science, error-correcting codes, and Markov chains [51].

Linear algebra has played a significant role in graph theory [2, 8, 34]. The adjacency matrix and Laplacian matrix are two common matrices that are used to represent a graph. The spectrum of a graph is the multiset of eigenvalues of a corresponding matrix. The choice of the matrix often depends on properties of interest in the graph. For example, the spectrum of the adjacency matrix determines if the graph is bipartite, but not if the graph is connected. On the other hand, the spectrum of the Laplacian matrix provides the exact opposite. Algebraic properties of these matrices are used to find connections with the structure and properties of graphs.

The most important algebraic parameters used in spectral graph theory are the eigenvalues of these matrices. Kirchhoff’s matrix tree theorem [45] is a classical result relating Laplacian eigenvalues and the number of spanning trees in a graph. Although spectral methods provide solutions to many graph theory problems, the eigenvalues of
a matrix give rise to interesting problems themselves. Given a graph, the spectrum is clearly determined. However, the converse is not necessarily true; there may be two different graphs, both of which have the same spectrum. Two graphs are said to be \textit{cospectral} if they have the same spectrum. If a graph \( G \) does not have any graph cospectral to it, \( G \) is said to be determined by its spectrum. Whether a graph is determined by its spectrum is an open problem for many graphs \([15, 33, 37]\). The smallest example of cospectral graphs is shown in Figure 1.2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{cospectral_graphs.png}
\caption{Cospectral graphs that have adjacency matrix spectrum \( 2^{(1)}, -2^{(-1)}, 0^{(3)} \), where the exponent denotes the multiplicity of the eigenvalue.}
\end{figure}

In another direction, it is of interest to determine the influence of the eigenvalues of a graph on its structural and combinatorial properties. This will be illustrated in Chapters 2 and 3 of the thesis. In Chapter 2, we obtain an upper bound on the second largest adjacency matrix eigenvalue for \( k \)-regular graphs that determines the existence of at least two or three edge-disjoint spanning trees. Furthermore, we show this bound is best possible. Chapter 3 involves a graph parameter known as graph toughness. Informally, it estimates how connected a graph is. We obtain an upper bound on the second largest adjacency matrix eigenvalue for \( k \)-regular graphs that sufficiently determines a graph to be \( 1 \)-tough, and we show this is best possible. In addition, we find a sufficient bound on the fourth largest eigenvalue for \( k \)-regular graphs for the existence of a spanning \( d \)-tree, which is a spanning tree with degree at most \( d \).
Various graph parameters can be estimated in terms of eigenvalues. A simple example is the Hoffman Ratio bound (see Theorem 1.3.4), which relates the independence number with the smallest eigenvalue of the adjacency matrix. In Chapters 4 and 5, we use this bound to help us determine the toughness value of some families of graphs, most of which are strongly regular graphs. The results in these chapters were highly motivated by Brouwer who, in 1995, found relationships between eigenvalues and graph toughness of regular graphs. In Chapter 6, we discuss some conjectures and open problems that were motivated by the work in the previous chapters.

We first provide some background in graph theory. For undefined terms, see [61]. Unless otherwise stated, we assume all graphs are undirected and simple: the graphs have no loops and no multiple edges. A graph $G$ with $n$ vertices and $m$ edges consists of a vertex set $V(G) = \{v_1, ..., v_n\}$ and an edge set $E(G) = \{e_1, ..., e_m\}$. The elements of $V(G)$ are called vertices, and the elements of $E(G)$ are called edges. We say $v_1$ and $v_2$ are adjacent if and only if $e = \{v_1, v_2\} \in E(G)$. For simplicity, an edge $\{v_1, v_2\} \in E(G)$ will be denoted $v_1v_2$, where $v_1$ and $v_2$ are the endpoints of the edge. If a vertex $v$ is an endpoint of an edge $e$, we say $v$ is incident with $e$. The degree of a vertex $v$ is the number of edges incident with $v$. A graph is $k$-regular if all vertices have degree $k$. The neighborhood of a vertex $v$, denoted $N(v)$, is the set of vertices that are adjacent to $v$. A subgraph $H$ of a graph $G$ is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph $H$ is an induced subgraph of $G$ if $E(H)$ contains all edges $v_1v_2$ of $G$ whose endpoints are in $V(H)$. If $H$ is an induced subgraph of $G$ with vertex set $S = V(H)$, the subgraph induced by $S$ is denoted $G[S]$. A spanning subgraph $H$ of $G$ is a subgraph with $V(H) = V(G)$. A tree is a connected graph that does not contain any cycles. The complement of a graph $G$, denoted $\overline{G}$, is the graph with $V(\overline{G}) = V(G)$ and $\{v_1, v_2\} \in E(\overline{G})$ if and only if $\{v_1, v_2\} \notin E(G)$. The complete graph $K_n$ is the graph on $n$ vertices such that every two vertices are joined by an edge. A clique in a graph $G$ is a complete subgraph. The complement of a clique is an independent set, or coclique, which is a subset of vertices $S$ such that $G[S]$ contains no edges. The cardinality of the largest independent set of a graph $G$ is denoted $\alpha(G)$. A walk of length $k$ between
two vertices $u$ and $v$, is a sequence of vertices $u = u_0, u_1, ..., u_k = v$, such that for $0 \leq i \leq k - 1$, $u_i$ is adjacent to $u_{i+1}$. If all these vertices are distinct, the walk is a \textit{path}. If there exists a path between any two vertices, the graph is \textit{connected}. Otherwise the graph is \textit{disconnected}. A \textit{component} of $G$ is a maximal connected subgraph. A disconnected graph contains at least two components.

### 1.2 Matrices

The \textit{adjacency matrix} $A(G)$ of a graph $G$ on $n$ vertices is an $n \times n$ matrix, with entry $a_{ij} = 1$ if vertex $i$ is adjacent to vertex $j$, and 0 otherwise. We assume all graphs are simple and undirected, so $A(G)$ is a symmetric, real matrix. By the spectral theorem ([42, Theorem 2.5.6], [61, Section 8.6]), the eigenvalues of $A(G)$ are real and $A(G)$ possesses an orthonormal basis of eigenvectors. The eigenvalues of $A(G)$ will be denoted

$$\lambda_1(G) \geq \lambda_2(G) \geq ... \geq \lambda_n(G).$$

The \textit{Laplacian matrix} of a graph $G$ on $n$ vertices is given by $L = D - A(G)$, where $D$ denotes denotes the $n \times n$ diagonal matrix whose $i$-th diagonal entry is the degree of vertex $i$, and $A(G)$ is the adjacency matrix of $G$. The Laplacian matrix is positive semidefinite since for any vector $x$, $x^tLx = \sum_{i \sim j} (x_i - x_j)^2 \geq 0$. The eigenvalues of $L$ will be denoted

$$0 = \mu_1(G) \leq \mu_2(G) \leq ... \leq \mu_n(G).$$

### 1.3 Eigenvalue interlacing

A powerful tool used in spectral graph theory is eigenvalue interlacing (for more general results and details, see [12, 34, 36, 42]). Denote the $i$-th largest eigenvalue of a matrix $M$ by $\lambda_i(M)$. Let $A$ and $B$ be square matrices of order $n$ and $m$, respectively, with $n \geq m$. If

$$\lambda_i(A) \geq \lambda_i(B) \geq \lambda_{n-m+i}(A) \quad (1.1)$$

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for $1 \leq i \leq m$, we say the eigenvalues of $B$ interlace the eigenvalues of $A$. The interlacing is tight if there exists an integer $k$, $1 \leq k \leq m$, such that

$$\lambda_i(A) = \lambda_i(B) \text{ for } 1 \leq i \leq k, \text{ and } \lambda_{n-m+i}(A) = \lambda_i(B) \text{ for } k+1 \leq i \leq m.$$

**Theorem 1.3.1** (Haemers, [35]). Let $S$ be a complex $n \times m$ matrix such that $S^*S = I_m$, where $S^*$ denotes the adjoint of $S$. Let $A$ be a Hermitian square matrix of order $n$. Let $B = S^*AS$, and $v_1, \ldots, v_m$ be an orthonormal set of eigenvectors of $B$. Then

1. the eigenvalues of $B$ interlace the eigenvalues of $A$.

2. if $\lambda_i(B) = \lambda_i(A)$ or $\lambda_i(B) = \lambda_{n-m+i}(A)$ for some $i$, $1 \leq i \leq m$, then there exists an eigenvector $v$ of $B$ for $\lambda_i(B)$, such that $Sv$ is an eigenvector of $A$ for $\lambda_i(B)$.

3. if for some integer $l$, $0 \leq l \leq m$, $\lambda_i(A) = \lambda_i(B)$ for all $i$, $1 \leq i \leq l$, then $Sv_i$ is an eigenvector of $A$ for $\lambda_i(A)$, for $1 \leq i \leq l$.

4. if the interlacing is tight, then $SB = AS$.

A direct consequence of Theorem 1.3.1 is an interlacing result accredited to Cauchy.

**Theorem 1.3.2** ([2, 12], Cauchy Interlacing). Let $\lambda_i(M)$ denote the $i$-th largest eigenvalue of a matrix $M$. If $A$ is a real symmetric $n \times n$ matrix and $B$ is a principal submatrix of $A$ with order $m \times m$, then for $1 \leq i \leq m$,

$$\lambda_i(A) \geq \lambda_i(B) \geq \lambda_{n-m+i}(A). \quad (1.2)$$

This theorem implies that if $H$ is an induced subgraph of a graph $G$, then the eigenvalues of $H$ interlace the eigenvalues of $G$.

Consider a partition $V(G) = V_1 \cup \ldots V_s$ of the vertex set of $G$ into $s$ non-empty subsets. For $1 \leq i, j \leq s$, let $b_{i,j}$ denote the average number of neighbors in $V_j$ of the vertices in $V_i$. The quotient matrix of this partition is the $s \times s$ matrix whose $(i,j)$-th entry equals $b_{i,j}$. The partition is called equitable if for each $1 \leq i, j \leq s$, any vertex
$v \in V_i$ has exactly $b_{i,j}$ neighbors in $V_j$. Another consequence of Theorem 1.3.1 is that the eigenvalues of a quotient matrix interlace the eigenvalues of the adjacency matrix of a graph.

**Theorem 1.3.3** ([12, 35, 36]). For a graph $G$, let $B$ be the quotient matrix of a partition of $V(G)$. Then the eigenvalues of $B$ interlace those of $A$. Furthermore, the interlacing is tight if and only if the partition is equitable.

A consequence of Theorem 1.3.3 is the Hoffman Ratio bound ([12, Theorem 3.5.2]). The bound is particularly applicable in Chapters 4 and 5.

**Theorem 1.3.4** (Hoffman Ratio Bound). If $G$ is a connected, $k$-regular graph on $n$ vertices, then

$$\alpha(G) \leq \frac{n}{1 - \frac{k}{\lambda_n}}.$$

**Proof.** For any independent set $A$, the partition of the vertex set into $A$ and $V(G) \setminus A$ yields the quotient matrix

$$B = \begin{bmatrix}
0 & k \\
k \frac{|A|}{n-|A|} & k - \frac{k|A|}{n-|A|}
\end{bmatrix}.$$ 

The smaller eigenvalue of $B$ is $-\frac{k|A|}{n-|A|}$. By Theorem 1.3.3, $\lambda_n(G) \leq -\frac{k|A|}{n-|A|}$, and the result follows. 

1.4 Graph eigenvalue properties

**Lemma 1.4.1** ([2, 26]). Let $\bar{d}(G)$ and $\Delta(G)$ denote the average degree and maximum degree of $G$, respectively. Then

$$\bar{d}(G) \leq \lambda_1(G) \leq \Delta(G).$$

**Proof.** The lower bound follows by Theorem 1.3.3, with a partition of $V(G)$ into one part (alternatively, one can use the Rayleigh quotient with the all-ones vector). For the
upper bound, let $x$ be an eigenvector corresponding to $\lambda_1$. Without loss of generality, let $x_l > 0$ be the largest coordinate of $x$. Then

$$\lambda_1 x_l = (Ax)_l = \sum_{i \sim l} x_i \leq \Delta(G)x_l.$$  

\[ \square \]

**Lemma 1.4.2.** If $G$ is a $k$-regular graph, then $\lambda_1 = k$ and its multiplicity equals the number of connected components.

*Proof.* If $1$ denotes the all ones vector, we have $A1 = k1$, and so $k$ is an eigenvalue. Lemma 1.4.1 implies $\lambda_1 = k$. For the final part of the statement, since the spectrum of a graph equals the union of the spectra of the components, we show that when $G$ is connected, any eigenvector corresponding to $k$ is a multiple of $1$. If $G$ is connected, let $x$ be an eigenvector of $k$ and $x_l > 0$ be the largest coordinate of $x$. Then

$$kx_l = (Ax)_l = \sum_{i \sim l} x_i \leq kx_l.$$  

For equality to occur, necessarily $x_i = x_l$ for all $i$ with $i \sim l$. We can now repeat this argument by using a vertex adjacent to vertex $l$. By continuing the argument through all vertices of $G$, we see that $x_1 = x_2 = \ldots = x_n$, and so all eigenvectors corresponding to eigenvalue $k$ are a multiple of $1$. \[ \square \]

### 1.5 Strongly regular graphs

A graph $G$ is said to be *strongly regular* graphs with parameters $(n, k, \lambda, \mu)$ if $G$ is $k$-regular with $n$ vertices, every adjacent pair of vertices have $\lambda$ common neighbors, and every non-adjacent pair of vertices have $\mu$ common neighbors. If $G$ is strongly regular with parameters $(n, k, \lambda, \mu)$, then $\overline{G}$ is also strongly regular with parameters $(n, n - k - 1, n - 2 - 2k + \mu, n - 2k + \lambda)$.

The following lemma determines when a strongly regular graph is disconnected.

**Lemma 1.5.1.** Let $G$ be a strongly regular graph with parameters $(n, k, \lambda, \mu)$ that is not $K_n$ or $K_n$. The following are equivalent:
1. $G$ is disconnected.

2. $\lambda = k - 1$.

3. $\mu = 0$.

4. $G$ is a disjoint union of $m$ copies of $K_{k+1}$, $m > 1$.

Two examples of strongly regular are the cycle $C_5$ and the Petersen graph. Their parameters are $(5, 2, 0, 1)$ and $(10, 3, 0, 1)$, respectively. We now present two families of strongly regular graphs. There are many other examples of strongly regular graphs (for instance, see [14]).

The Lattice graph $L_v$ is a strongly regular graph with parameters $(v^2, 2(v - 1), v - 2, 2)$. It is the line graph of the complete bipartite graph $K_{v,v}$. That is, the vertices of $L_v$ are the edges of $K_{v,v}$, and $ef \in E(L_v)$ if and only if $e = u_1u_2$ and $f = u_2u_3$ for some $e, f \in E(K_{v,v})$ and $u_1, u_2, u_3 \in V(K_{v,v})$. The graph can be seen as a $v \times v$ grid, with two vertices adjacent if and only if they are in the same row or column. For more properties, see Section 4.2.

![The Lattice graph $L_3$.](image)

The Triangular graph $T_v$ is a strongly regular graph with parameters $\left(\binom{v}{2}, 2v - 4, v - 2, 4\right)$. It is the line graph of the complete graph $K_v$. The vertex set of the graph are the 2-size subsets of $[v]$, with two vertices adjacent if and only if they share an element. For more properties, see Section 4.4.
The eigenvalues of strongly regular graphs are used in Chapter 4 of this thesis. If $A$ denotes the adjacency matrix of a strongly regular graph, then we have the relation

$$A^2 = (\lambda - \mu)A + (k - \mu)I + \mu J.$$ 

Consider an eigenvector $x$ of $A$ orthogonal to the all ones vector $1$ (which is an eigenvector of $A$ corresponding to $k$) with corresponding eigenvalue $\gamma$. Applying $x$ to the above equation reduces to

$$\gamma^2 - (\lambda - \mu)\gamma - (k - \mu) = 0.$$ 

The two solutions $\theta, \tau$ to this quadratic are

$$\theta, \tau = \frac{\lambda - \mu \pm \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2}.$$ 

Since $x$ was arbitrary, we deduce $k, \theta, \text{and} \tau$ are precisely the three distinct eigenvalues of a strongly regular graph.
Let $m_\theta$ and $m_\tau$ denote the multiplicity of the eigenvalues $\theta$ and $\tau$, respectively, with $\theta > \tau$. Since $k$ has multiplicity 1 and the trace of $A$ is 0,

$$m_\theta + m_\tau = n - 1 \quad \text{and} \quad m_\theta \theta + m_\tau \tau = -k.$$ 

This yields

$$m_\theta = \frac{(n - 1) \tau + k}{\tau - \theta} \quad \text{and} \quad m_\tau = -\frac{(n - 1) \theta + k}{\tau - \theta}.$$ 

Substituting the values from Equation 1.5,

$$m_\theta = \frac{1}{2} \left( n - 1 - \frac{2k + (n - 1)(\lambda - \mu)}{\sqrt{(\lambda - \mu)^2 + 4(k - \mu)}} \right),$$

$$m_\tau = \frac{1}{2} \left( n - 1 + \frac{2k + (n - 1)(\lambda - \mu)}{\sqrt{(\lambda - \mu)^2 + 4(k - \mu)}} \right).$$
Chapter 2

SPECTRAL CONDITIONS FOR EDGE-DISJOINT SPANNING TREES

Most of the results of this chapter have appeared in [24].

2.1 Introduction

The Kirchhoff Matrix Tree Theorem [45] (see also [34] for a short proof) is one of the classical results of combinatorics. It states that the number of spanning trees of a graph $G$ with $n$ vertices is equal to any principal minor of the Laplacian matrix $L$ of the graph and consequently, equals $\prod_{i=2}^{n} \mu_i$. In particular, if $G$ is a $k$-regular graph, then the number of spanning trees of $G$ is $\prod_{i=2}^{n} (k-\lambda_i)^{n}$.

Motivated by these facts and by a question of Seymour [23], we find relations between the maximum number of edge-disjoint spanning trees (also called the spanning tree packing number or tree packing number; see Palmer [56] for a survey) and the eigenvalues of a regular graph. Let $\sigma(G)$ denote the maximum number of edge-disjoint spanning trees of $G$. Obviously, $G$ is connected if and only if $\sigma(G) \geq 1$.

A classical result, due to Nash-Williams [55] and independently, Tutte [60] (see [44] for a recent short constructive proof), states that a graph $G$ contains $m$ edge-disjoint spanning trees if and only if for any partition of its vertex set $V(G) = X_1 \cup \cdots \cup X_t$ into $t$ non-empty subsets, the following inequality is satisfied:

$$\sum_{1 \leq i < j \leq t} e(X_i, X_j) \geq m(t - 1).$$  (2.1)

A simple consequence of the Nash-Williams/Tutte Theorem is that if $G$ is a $2m$-edge-connected graph, then $\sigma(G) \geq m$ (see Kundu [48]). Catlin [16] improved this result and showed that a graph $G$ is $2m$-edge-connected if and only if the graph
obtained from removing any $m$ edges from $G$ contains at least $m$ edge-disjoint spanning trees.

An obvious approach to finding relations between $\sigma(G)$ and the eigenvalues of $G$ is by using the relations between eigenvalues and edge-connectivity of a regular graph as well as the previous observations relating the edge-connectivity to $\sigma(G)$. Cioabă [21] has proven that if $G$ is a $k$-regular graph and $2 \leq r \leq k$ is an integer such that $\lambda_2 < k - \frac{2(r-1)}{k+1}$, then $G$ is $r$-edge-connected. A consequence of these facts is that if $G$ is a $k$-regular graph with $\lambda_2 < k - \frac{2(2m-1)}{k+1}$ for some integer $m$, $2 \leq m \leq \lfloor \frac{k}{2} \rfloor$, then $G$ is $2m$-edge-connected and consequently, $G$ contains $m$-edge-disjoint spanning trees.

We improve the bound above as follows.

**Theorem 2.1.1.** If $k \geq 4$ is an integer and $G$ is a $k$-regular graph such that $\lambda_2(G) < k - \frac{3}{k+1}$, then $G$ contains at least 2 edge-disjoint spanning trees.

The proof of this result is contained in Section 2.2. In Section 2.2, it is shown that Theorem 2.1.1 is essentially best possible by constructing examples of $k$-regular graphs $W_k$ such that $\sigma(W_k) = 1$ and $\lambda_2(W_k) \in (k - \frac{3}{k+2}, k - \frac{3}{k+3})$. We also improve the bound for a $k$-regular graph to have 3 edge-disjoint spanning trees.

**Theorem 2.1.2.** If $k \geq 6$ is an integer and $G$ is a $k$-regular graph such that $\lambda_2(G) < k - \frac{5}{k+1}$, then $G$ contains at least 3 edge-disjoint spanning trees.

The proof of this result is contained in Section 2.3. In Section 2.3, it is shown that Theorem 2.1.2 is essentially best possible by constructing examples of $k$-regular graphs $Z_k$ such that $\sigma(Z_k) = 2$ and $\lambda_2(Z_k) \in [k - \frac{5}{k+1}, k - \frac{5}{k+3})$. We will also answer a question of Palmer [56] (Section 3.7, page 19) by proving that the minimum number of vertices of a $k$-regular graph with edge-connectivity 2 and spanning tree number 1 is $3(k+1)$.

The main tools used are the Nash-Williams/Tutte Theorem stated above and eigenvalue interlacing. If $S$ and $T$ are disjoint subsets of the vertex set of a graph $G$, then let $E(S,T)$ denote the set of edges with one endpoint in $S$ and another endpoint
in $T$. Also, let $e(S, T) = |E(S, T)|$. If $H$ is an induced subgraph of a graph $G$, then the eigenvalues of $H$ interlace the eigenvalues of $G$. This implies that if $A$ and $B$ are two disjoint subsets of a graph $G$ such that $e(A, B) = 0$, then the eigenvalues of $G[A \cup B]$ interlace the eigenvalues of $G$. As the spectrum of $G[A \cup B]$ is the union of the spectrum of $G[A]$ and the spectrum of $G[B]$ (this holds because $e(A, B) = 0$), it follows that

$$
\lambda_2(G) \geq \lambda_2(G[A \cup B]) \geq \min(\lambda_1(G[A]), \lambda_1(G[B])) \geq \min(\bar{d}(A), \bar{d}(B)),
$$

where $\bar{d}(S)$ denotes the average degree of $G[S]$ (the last inequality is true by Lemma 1.4.1).

Cioabă \cite{21} proved a relationship between edge-connectivity and the second largest eigenvalue of a $k$-regular graph, which will be used in the proofs of Theorem 2.1.1 and Theorem 2.1.2.

**Theorem 2.1.3** (\cite{21}). Let $G$ be a connected $k$-regular graph, and let $\kappa'(G)$ denote the edge-connectivity of $G$.

1. If $\kappa'(G) = 1$, then $\lambda_2(G) \geq k - \frac{2}{k+5}$.

2. If $\kappa'(G) \leq 2$, then $\lambda_2(G) \geq k - \frac{4}{k+3}$.

### 2.2 Eigenvalue condition for 2 edge-disjoint spanning trees

In this section, we give a proof of Theorem 2.1.1 showing that if $G$ is a $k$-regular graph such that $\lambda_2(G) < k - \frac{3}{k+1}$, then $G$ contains at least 2 edge-disjoint spanning trees. We show that the bound $k - \frac{3}{k+1}$ is essentially best possible by constructing examples of $k$-regular graphs $W_k$ having $\sigma(W_k) = 1$ and $k - \frac{3}{k+2} < \lambda_2(W_k) < k - \frac{3}{k+3}$.

**Proof of Theorem 2.1.1.** We prove the contrapositive. Assume that $G$ does not contain 2-edge-disjoint spanning trees. We will show that $\lambda_2(G) \geq k - \frac{3}{k+1}$.

By the Nash-Williams/Tutte Theorem, there exists a partition of the vertex set of $G$ into $t$ subsets $X_1, \ldots, X_t$ such that

$$
\sum_{1 \leq i < j \leq t} e(X_i, X_j) \leq 2(t - 1) - 1 = 2t - 3.
$$

(2.3)
It follows that
\[ \sum_{i=1}^{t} r_i \leq 4t - 6 \quad (2.4) \]
where \( r_i = e(X_i, V \setminus X_i) \).

Let \( n_i = |X_i| \) for \( 1 \leq i \leq t \). It is easy to see that \( r_i \leq k - 1 \) implies \( n_i \geq k + 1 \) for each \( 1 \leq i \leq t \).

If \( t = 2 \), then \( e(X_1, V \setminus X_1) = 1 \). By Theorem 2.1.3, it follows that \( \lambda_2(G) \geq k - \frac{2}{k+5} > k - \frac{3}{k+1} \) and this finishes the proof of this case. We may assume \( r_i \geq 2 \) for every \( 1 \leq i \leq t \) since \( r_i = 1 \) and Theorem 2.1.3 together imply \( \lambda_2(G) \geq k - \frac{2}{k+5} > k - \frac{3}{k+1} \).

If \( t = 3 \), then \( r_1 + r_2 + r_3 \leq 6 \) which implies \( r_1 = r_2 = r_3 = 2 \). The only way this can happen is if \( e(X_i, X_j) = 1 \) for every \( 1 \leq i < j \leq 3 \). Consider the partition of \( G \) into \( X_1, X_2 \) and \( X_3 \). The quotient matrix of this partition is
\[
A_3 = \begin{bmatrix}
k - \frac{2}{n_1} & \frac{1}{n_1} & \frac{1}{n_1} \\
\frac{1}{n_2} & k - \frac{2}{n_2} & \frac{1}{n_2} \\
\frac{1}{n_3} & \frac{1}{n_3} & k - \frac{2}{n_3}
\end{bmatrix}.
\]

The largest eigenvalue of \( A_3 \) is \( k \) and the second eigenvalue of \( A_3 \) equals
\[
k - \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3} + \sqrt{\frac{1}{n_1^2} + \frac{1}{n_2^2} + \frac{1}{n_3^2} - \frac{1}{n_1 n_2} - \frac{1}{n_2 n_3} - \frac{1}{n_3 n_1}},
\]
which is greater than \( k - \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3} \). Thus, eigenvalue interlacing and \( n_i \geq k + 1 \) for \( 1 \leq i \leq 3 \) imply \( \lambda_2(G) \geq \lambda_2(A_3) \geq k - \frac{3}{k+1} \). This finishes the proof of the case when \( t = 3 \).

Assume \( t \geq 4 \) from now on. Let \( a \) denote the number of \( r_i \)'s that equal 2 and \( b \) denote the number of \( r_j \)'s that equal 3. Using Equation (2.4), we get
\[
4t - 6 \geq \sum_{i=1}^{t} r_i \geq 2a + 3b + 4(t - a - b) = 4t - 2a - b,
\]
which implies \( 2a + b \geq 6 \).

Recall that \( \overline{d}(A) \) denotes the average degree of the subgraph of \( G \) induced by the subset \( A \subset V(G) \).
If $a = 0$, then $b \geq 6$. This implies that there exist two indices $1 \leq i < j \leq t$ such that $r_i = r_j = 3$ and $e(X_i, X_j) = 0$. Eigenvalue interlacing (2.2) implies $\lambda_2(G) \geq \lambda_2(G[X_i \cup X_j]) \geq \min(\lambda_1(G[X_i]), \lambda_1(G[X_j])) \geq \min(\lambda_1(G[X_i]), \lambda_1(G[X_j])) \geq \min(k - \frac{2}{m}, k - \frac{3}{n_j}) \geq k - \frac{3}{k+1}$.

If $a = 1$, then $b \geq 4$. This implies there exist two indices $1 \leq i < j \leq t$ such that $r_i = 2$, $r_j = 3$ and $e(X_i, X_j) = 0$. Eigenvalue interlacing (2.2) implies $\lambda_2(G) \geq \lambda_2(G[X_i \cup X_j]) \geq \min(\lambda_1(G[X_i]), \lambda_1(G[X_j])) \geq \min(\lambda_1(G[X_i]), \lambda_1(G[X_j])) \geq \min(k - \frac{2}{m}, k - \frac{2}{n_j}) \geq k - \frac{2}{k+1} > k - \frac{3}{k+1}$. Otherwise, there exist two indices $1 \leq p < q \leq t$ such that $r_p = 2$, $r_q = 3$ and $e(X_p, X_q) = 0$. By a similar eigenvalue interlacing argument, we get $\lambda_2(G) \geq k - \frac{3}{k+1}$ in this case as well.

If $a \geq 3$, then there exist two indices $1 \leq i < j \leq t$ such that $r_i = r_j = 2$ and $e(X_i, X_j) = 0$ because $t \geq 4$. Therefore as before, eigenvalue interlacing (2.2) implies $\lambda_2(G) \geq k - \frac{2}{k+1} > k - \frac{3}{k+1}$. This finishes the proof of Theorem 2.1.1.

We show that our bound is essentially best possible by presenting a family of $k$-regular graphs $W_k$ with $k - \frac{3}{k+2} < \lambda_2(W_k) < k - \frac{3}{k+3}$ and $\sigma(W_k) = 1$, for every $k \geq 4$.

For $k \geq 4$, consider three vertex disjoint copies $G_1, G_2, G_3$ of $K_{k+1}$ minus one edge. Let $a_i$ and $b_i$ be the two non-adjacent vertices in $G_i$ for $1 \leq i \leq 3$. Let $W_k$ be the graph with

$$V(W_k) = V(G_1) \cup V(G_2) \cup V(G_3),$$

and

$$E(W_k) = E(G_1) \cup E(G_2) \cup E(G_3) \cup \{a_1a_2, b_2b_3, a_3b_1\}.$$ 

The graph $W_k$ has $3(k+1)$ vertices and is $k$-regular. The partition of the vertex set of $W_k$ into $V(G_1), V(G_2), V(G_3)$ has the property that the number of edges between the
Figure 2.1: The 4-regular graph $W_4$ with $\sigma(W_4) = 1$ and $3.5 = 4 - \frac{3}{4+2} < \lambda_2(W_4) \approx 3.569 < 4 - \frac{3}{4+3} \approx 3.571$

parts equals $3 < 2(3 - 1)$. By the Nash-Williams/Tutte Theorem, $W_k$ does not contain 2 edge-disjoint spanning trees.

For $k \geq 4$, denote by $\theta_k$ the largest root of the cubic polynomial

$$P_3(x) = x^3 + (2 - k)x^2 + (1 - 2k)x + 2k - 3. \quad (2.5)$$

Lemma 2.2.1. For every integer $k \geq 4$, the second largest eigenvalue of $W_k$ is $\theta_k$.

Proof. Consider the following partition of the vertex set of $W_k$ into nine parts: $V(G_1) \setminus$
\{a_1, b_1\}, V(G_2) \setminus \{a_2, b_2\}, V(G_3) \setminus \{a_3, b_3\}, \{a_1\}, \{b_1\}, \{a_2\}, \{b_2\}, \{a_3\}, \{b_3\}. This partition is equitable and its quotient matrix is the following

\[
A_9 = \begin{bmatrix}
  k - 2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
  0 & k - 2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
  0 & 0 & k - 2 & 0 & 0 & 0 & 1 & 1 & 0 \\
  k - 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
  k - 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
  0 & k - 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
  0 & k - 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
  0 & 0 & k - 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & k - 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}.
\]

(2.6)

The characteristic polynomial of \(A_9\) is

\[P_9(x) = (x - k)(x + 1)^2[x^3 + (2 - k)x^2 + (1 - 2k)x + 2k - 3]^2.\]  

(2.7)

Let \(\lambda_2 \geq \lambda_3 \geq \lambda_4\) denote the solutions of the equation \(x^3 + (2 - k)x^2 + (1 - 2k)x + 2k - 3 = 0\). Because the partition is equitable, it follows that \(k, \lambda_2, \lambda_3, \lambda_4\) and \(-1\) are eigenvalues of \(W_k\). We claim the spectrum of \(W_k\) is

\[k^{(1)}, \lambda_2^{(2)}, \lambda_3^{(2)}, \lambda_4^{(2)}, (-1)^{3k-4},\]

(2.8)

where the exponents denote the multiplicities. It suffices to obtain \(3k - 4\) linearly independent eigenvectors corresponding to \(-1\). Consider two distinct vertices \(u_1\) and \(u_2\) in \(V(G_1) \setminus \{a_1, b_1\}\). Define an eigenvector where the entry corresponding to \(u_1\) is 1, the entry corresponding to \(u_2\) is \(-1\), and all other entries 0. We create \(k - 2\) eigenvectors by redefining \(u_2\) to be each of the \(k - 2\) vertices in \(V(G_1) \setminus \{a_1, b_1, u_1\}\). This can also be done to 2 vertices \(u'_1, u'_2\) and \(u''_1, u''_2\) in \(V(G_2) \setminus \{a_2, b_2\}\) and \(V(G_3) \setminus \{a_3, b_3\}\), respectively. We obtain a total of \(3k - 6\) eigenvectors. Furthermore, define an eigenvector with entries at vertices \(u_1, u'_1,\) and \(u''_1\) to be \(-1\), and \(a_1, b_2,\) and \(a_3\) to be 1. To obtain the final eigenvector, define entries at vertices \(u_1, u'_1,\) and \(u''_1\) to be \(-1\), and \(b_1, a_2,\) and \(b_3\) to be 1.
It is easy to check all $3k - 4$ vectors are linearly independent eigenvectors corresponding to eigenvalue -1. By obtaining the entire spectrum of $W_k$, we conclude that the second largest eigenvalue of $W_k$ must be $\theta_k$. 

Lemma 2.2.2. For every integer $k \geq 4$,

$$k - \frac{3}{k + 2} < \theta_k < k - \frac{3}{k + 3}.$$ 

Proof. We find that for $k \geq 4$,

$$P_3 \left( k - \frac{3}{k + 2} \right) = -\frac{3(9 + k(-2 + k + k^2))}{(2 + k)^3} < 0,$$

$$P_3 \left( k - \frac{3}{k + 3} \right) = \frac{-81 + 6k^2}{(3 + k)^3} > 0,$$

and $P_3'(x) > 0$ beyond $x = \frac{1}{3}(-1 + 2k) < k - \frac{3}{k + 3}$. Hence,

$$k - \frac{3}{k + 2} < \theta_k < k - \frac{3}{k + 3} \quad (2.9)$$

for every $k \geq 4$. 

Palmer [56] asked whether or not the graph $W_4$ has the smallest number of vertices among all 4-regular graphs with edge-connectivity 2 and spanning tree number 1. We answer this question affirmatively below.

Proposition 2.2.3. Let $k \geq 4$ be an integer. If $G$ is a $k$-regular graph such that $\kappa'(G) = 2$ and $\sigma(G) = 1$, then $G$ has at least $3(k + 1)$ vertices. The only graph with these properties and $3(k + 1)$ vertices is $W_k$.

Proof. As $\sigma(G) = 1 < 2$, by the Nash-Williams/Tutte Theorem, there exists a partition $V(G) = X_1 \cup \cdots \cup X_t$ such that $e(X_1, \ldots, X_t) \leq 2t - 3$. This implies $r_1 + \cdots + r_t \leq 4t - 6$.

As $\kappa'(G) = 2$, $r_i \geq 2$ for each $1 \leq i \leq t$, which implies $4t - 6 \geq 2t$ and thus, $t \geq 3$.

If $t = 3$, then $r_i = 2$ for each $1 \leq i \leq 3$ and thus, $e(X_i, X_j) = 1$ for each $1 \leq i \neq j \leq 3$. As $k \geq 4$ and $r_i = 2$, we deduce that $|X_i| \geq k + 1$. Equality happens if and only if $X_i$ induces a $K_{k+1}$ without one edge. Thus, we obtain that $|V(G)| = |X_1| + |X_2| + |X_3| \geq 3(k + 1)$ with equality if and only if $G = W_k$. 

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If $t \geq 4$, then let $\alpha$ denote the number of $X_i$’s such that $|X_i| \geq k + 1$. If $\alpha \geq 3$, then $|V(G)| > 3(k + 1)$ and we are done. Otherwise, $\alpha \leq 2$. Note that if $|X_i| \leq k$, then $r_i \geq k$. Thus,

$$4t - 6 \geq r_1 + \cdots + r_t \geq 2\alpha + k(t - \alpha) = kt - (k - 2)\alpha$$

which implies $(k - 2)\alpha \geq (k - 4)t + 6$. As $\alpha \leq 2$ and $t \geq 4$, we obtain $2(k - 2) \geq (k - 4)4 + 6$ which is equivalent to $2k \leq 6$, contradiction. This finishes our proof.

2.3 Eigenvalue condition for 3 edge-disjoint spanning trees

In this section, we give a proof of Theorem 2.1.2 showing that if $G$ is a $k$-regular graph such that $\lambda_2(G) < k - \frac{5}{k+1}$, then $G$ contains at least 3 edge-disjoint spanning trees. We show that the bound $k - \frac{5}{k+1}$ is essentially best possible by constructing examples of $k$-regular graphs $Z_k$ having $\sigma(Z_k) = 2$ and $k - \frac{5}{k+1} \leq \lambda_2(Z_k) < k - \frac{5}{k+3}$.

Proof of Theorem 2.1.2. We prove the contrapositive. We assume that $G$ does not contain 3-edge-disjoint spanning trees and we prove that $\lambda_2(G) \geq k - \frac{5}{k+1}$.

By the Nash-Williams/Tutte Theorem, there exists a partition of the vertex set of $G$ into $t$ subsets $X_1, \ldots, X_t$ such that

$$\sum_{1 \leq i < j \leq t} e(X_i, X_j) \leq 3(t - 1) - 1 = 3t - 4.$$  

It follows that $\sum_{i=1}^{t} r_i \leq 6t - 8$, where $r_i = e(X_i, V \setminus X_i)$.

If $r_i \leq 2$ for some $i$ between 1 and $t$, then by Theorem 2.1.3, it follows that $\lambda_2(G) \geq k - \frac{4}{k+3} > k - \frac{5}{k+1}$.

Assume $r_i \geq 3$ for each $1 \leq i \leq t$ from now on. Let $a = |\{i : r_i = 3\}|, b = |\{i : r_i = 4\}|$ and $c = |\{i : r_i = 5\}|$. We get that

$$6t - 8 \geq r_1 + \cdots + r_t \geq 3a + 4b + 5c + 6(t - a - b - c)$$

which implies

$$3a + 2b + c \geq 8. \quad (2.10)$$
If for some $1 \leq i < j \leq t$, we have $e(X_i, X_j) = 0$ and $\max(r_i, r_j) \leq 5$, then eigenvalue interlacing (2.2) implies $\lambda_2(G) \geq \lambda_2(G[X_i \cup X_j]) \geq \min(\lambda_2(G[X_i]), \lambda_2(G[X_j])) \geq \min(d(X_i), d(X_j)) \geq k - \frac{5}{k+1}$.

Thus, we may assume that

$$e(X_i, X_j) \geq 1 \quad (2.11)$$

for every $1 \leq i < j \leq t$ whenever $\max(r_i, r_j) \leq 5$.

These arguments imply for example that

$$a + b + c \leq 6, \quad a + b \leq 5, \quad a \leq 4. \quad (2.12)$$

For the rest of the proof, we have to consider the following cases:

**Case 1.** $a \geq 2$.

The inequality $\sum_{1 \leq i < j \leq t} e(X_i, X_j) \leq 3t - 4$ implies $t \geq 3$.

As $a = |\{i : r_i = 3\}|$, assume without loss of generality that $r_1 = r_2 = 3$. Because $G$ is connected, this implies $e(X_1, X_2) < 3$. Otherwise, $e(X_1 \cup X_2, V(G) \setminus (X_1 \cup X_2)) = 0$, contradiction.

If $e(X_1, X_2) = 2$, then $e(X_1 \cup X_2, V(G) \setminus (X_1 \cup X_2)) = 2$. Using Theorem 2.1.3, this implies $\lambda_2(G) \geq k - \frac{4}{k+3} > k - \frac{5}{k+1}$, as required.

Now suppose $e(X_1, X_2) = 1$. Let $Y_3 = V(G) \setminus (X_1 \cup X_2)$. As $r_1 = r_2 = 3$, we deduce that $e(X_1, Y_3) = e(X_2, Y_3) = 2$. This means $e(Y_3, V(G) \setminus Y_3) = 4$ and since $k \geq 6$, this implies $n'_3 := |Y_3| \geq k + 1$.

Consider the partition of the vertex set of $G$ into three parts: $X_1, X_2$ and $Y_3$. The quotient matrix of this partition is

$$B_3 = \begin{bmatrix}
  k - \frac{3}{n_1} & \frac{1}{n_1} & \frac{2}{n_1} \\
  \frac{1}{n_2} & k - \frac{3}{n_2} & \frac{2}{n_2} \\
  \frac{2}{n_3} & \frac{2}{n_3} & k - \frac{4}{n_3}
\end{bmatrix}. $$
The largest eigenvalue of $B_3$ is $k$. Eigenvalue interlacing and $n_1, n_2, n'_3 \geq k + 1$ imply

$$
\lambda_2(G) \geq \lambda_2(B_3) \geq \frac{tr(B_3) - k}{2} \geq k - \frac{3}{2n_1} - \frac{3}{2n_2} - \frac{2}{n'_3} \\
\geq k - \frac{3}{2(k + 1)} - \frac{3}{2(k + 1)} - \frac{2}{k + 1} = k - \frac{5}{k + 1}.
$$

This finishes the proof of this case.

**Case 2.** $a = 1$.

Inequalities (2.10) and (2.12) imply $2b + c \geq 5 \geq b + c$. Actually, because we assumed that $e(X_i, X_j) \geq 1$ for every $1 \leq i \neq j \leq t$ with $\max(r_i, r_j) \leq 5$, we deduce that $b + c \leq 3$. Otherwise, if $b + c \geq 4$, then there exists $i \neq j$ such that $r_i = 3, r_j \in \{4, 5\}$ and $e(X_i, X_j) = 0$.

The only solution of the previous inequalities is $b = 2$ and $c = 1$. Without loss of generality, we may assume $r_1 = 3, r_2 = r_3 = 4$ and $r_4 = 5$. Using the facts of the previous paragraph, we deduce that $e(X_i, X_j) = 1$ for each $2 \leq j \leq 4$ and $e(X_i, X_j) \geq 1$ for each $2 \leq i \neq j \leq 4$.

If $e(X_2, X_3) \geq 3$, then $e(X_2, X_4) = 0$ which is a contradiction with 2.11.

If $e(X_2, X_3) = 2$, then $t \geq 5$ and $e(X_1 \cup X_2 \cup X_3 \cup X_4, V(G) \setminus (X_1 \cup X_2 \cup X_3 \cup X_4)) = 2$. Using Theorem 2.1.3, it follows that $\lambda_2(G) \geq k - \frac{4}{k+3} > k - \frac{5}{k+1}$, as required.

If $e(X_2, X_3) = 1$, then there are some subcases to consider:

1. If $e(X_2, X_4) = e(X_3, X_4) = 1$, then $t \geq 5$. Set $Y_5 := V(G) \setminus (X_1 \cup X_2 \cup X_3 \cup X_4)$. Then $e(X_4, Y_5) = 2, and e(X_3, Y_5) = e(X_2, Y_5) = 1$. These facts imply $e(Y_5, V(G) \setminus Y_5) = 4$ and $e(X_1, Y_5) = 0$. As $k \geq 6$, it follows that $n'_5 := |Y_5| \geq k + 1$. Eigenvalue interlacing (2.2) implies

$$
\lambda_2(G) \geq \lambda_2(G[X_1 \cup Y_5]) \geq \min(\lambda_1(G[X_1]), \lambda_1(G[Y_5])) \geq \min(\overline{d}(X_1), \overline{d}(Y_5)) \\
\geq \min \left( \frac{k - 3}{n_1}, k - \frac{4}{n'_5} \right) \geq k - \frac{4}{k + 1} > k - \frac{5}{k + 1},
$$

as required.
2. If $e(X_2, X_4) = 2$ and $e(X_3, X_4) = 1$, then $t \geq 5$. Set $Y_5 := V(G) \setminus (X_1 \cup X_2 \cup X_3 \cup X_4)$. Then $e(X_2, Y_5) = e(X_4, Y_5) = 1$. These facts imply $e(Y_5, V(G) \setminus Y_5) = 2$. Using Theorem 2.1.3, we obtain $\lambda_2(G) \geq k - \frac{4}{k+3} > k - \frac{5}{k+1}$, as required.

3. If $e(X_2, X_4) = 1$ and $e(X_3, X_4) = 2$, then $t \geq 5$. Set $Y_5 := V(G) \setminus (X_1 \cup X_2 \cup X_3 \cup X_4)$. Using Theorem 2.1.3, we obtain $\lambda_2(G) \geq k - \frac{4}{k+3} > k - \frac{5}{k+1}$, as required.

4. If $e(X_2, X_4) = e(X_3, X_4) = 2$, then $t = 4$. Consider the partition of the vertex set of $G$ into three parts: $X_1, X_2, X_3 \cup X_4$. The quotient matrix of this partition is

$$C_3 = \begin{bmatrix}
\frac{k-3}{n_1} & \frac{1}{n_1} & \frac{2}{n_1} \\
\frac{1}{n_2} & \frac{k-4}{n_2} & \frac{3}{n_2} \\
\frac{2}{n'_3} & \frac{3}{n'_3} & \frac{k-5}{n'_3}
\end{bmatrix}$$

where $n'_3 = |X_3 \cup X_4| = |X_3| + |X_4| \geq 2(k+1)$.

The largest eigenvalue of $C_3$ is $k$. Eigenvalue interlacing and $n_1, n_2 \geq k+1$, $n'_3 \geq 2(k+1)$ imply

$$\lambda_2(G) \geq \lambda_2(C_3) \geq \frac{tr(C_3) - k}{2} \geq k - \frac{3}{2n_1} - \frac{2}{n_2} - \frac{5}{2n'_3} \geq k - \frac{3}{2(k+1)} - \frac{2}{k+1} - \frac{5}{4(k+1)} > k - \frac{5}{k+1}.$$  

Case 3. $a = 0$.

Inequalities (2.10) and (2.12) imply $2b + c \geq 8, b + c \leq 6, b \leq 5$. Clearly $b \geq 2$.

If $b = 2$, then we must have $c = 4$. Hence, there exists $i \neq j$ such that $e(X_i, X_j) = 0$ and $r_i = 4$ and $r_j \in \{4, 5\}$, contradicting (2.11).

If $b = 3$, then $2 \leq c \leq 3$. Assume that $c = 2$ first. Without loss of generality, assume $r_1 = r_2 = r_3 = 4$ and $r_4 = r_5 = 5$. Equation (2.11) implies that $e(X_i, X_j) = 1$ for each $1 \leq i < j \leq 5$ except when $i = 4$ and $j = 5$, where $e(X_4, X_5) = 2$.
Consider the partition of the vertex set of $G$ into three parts: $X_1, X_2 \cup X_3,$ and $X_4 \cup X_5$. The quotient matrix of this partition is

$$D_3 = \begin{bmatrix}
  k - \frac{4}{n_1} & \frac{2}{n_1} & \frac{2}{n_1} \\
  \frac{2}{n_2} & k - \frac{6}{n_2} & \frac{4}{n_2} \\
  \frac{2}{n_3} & \frac{4}{n_3} & k - \frac{6}{n_3}
\end{bmatrix},$$

where $n'_2 = |X_2 \cup X_3| = |X_2| + |X_3| \geq 2(k+1)$ and $n'_3 = |X_4 \cup X_5| = |X_4| + |X_5| \geq 2(k+1)$.

The largest eigenvalue of $D_3$ is $k$. Eigenvalue interlacing and $n_1 \geq k + 1$, $n'_2, n'_3 \geq 2(k+1)$ imply

$$\lambda_2(G) \geq \lambda_2(D_3) \geq \frac{tr(D_3) - k}{2} \geq k - \frac{2}{n_1} - \frac{3}{n'_2} - \frac{3}{n'_3} \geq k - \frac{2}{k+1} - \frac{3}{2(k+1)} - \frac{3}{2(k+1)} = k - \frac{5}{k+1},$$

as required.

If $c = 3$, then since $b = 3$, it follows that there exists $i \neq j$ such that $e(X_i, X_j) = 0$ and $r_i = 4$ and $r_j \in \{4, 5\}$, contradicting (2.11).

If $b = 4$, we have inequality (2.12) implies $c \leq 2$. If $c = 2$, then there exist $i \neq j$ such that $e(X_i, X_j) = 0, r_i = 4$ and $r_j \leq 5$. This contradicts (2.11) and finishes the proof of this subcase.

Suppose $c = 0$. Without loss of generality, assume that $r_i = 4$ for $1 \leq i \leq 4$. If $t = 4$, then (2.11) implies that the graph $G$ is necessarily of the form shown in Figure 2.2.

Consider the partition of the vertex set of $G$ into three parts: $X_1, X_2, X_3 \cup X_4$. The quotient matrix of this partition is

$$E_3 = \begin{bmatrix}
  k - \frac{4}{n_1} & \frac{2}{n_1} & \frac{2}{n_1} \\
  \frac{2}{n_2} & k - \frac{4}{n_2} & \frac{2}{n_2} \\
  \frac{2}{n_3} & \frac{2}{n_3} & k - \frac{4}{n_3}
\end{bmatrix},$$

where $n'_3 = |X_3 \cup X_4| = |X_3| + |X_4| \geq 2(k+1)$.  

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The largest eigenvalue of $E_3$ is $k$. Eigenvalue interlacing and $n_1, n_2 \geq k + 1$, $n_3' \geq 2(k + 1)$ imply
\[
\lambda_2(G) \geq \lambda_2(E_3) \geq \frac{\text{tr}(E_3) - k}{2} \geq k - \frac{2}{n_1} - \frac{2}{n_2} - \frac{2}{n_3'}
\geq k - \frac{2}{k + 1} - \frac{2}{k + 1} - \frac{2}{2(k + 1)} = k - \frac{5}{k + 1},
\]
as desired.

If $t \geq 5$, then there are two possibilities: either $e(X_i, X_j) = 1$ for each $1 \leq i < j \leq 4$ or without loss of generality, $e(X_i, X_j) = 1$ for each $1 \leq i < j \leq 4$ except for $i = 1$ and $j = 2$, where $e(X_1, X_2) = 2$.

In the first situation, if $Y_5 := V(G) \setminus (X_1 \cup X_2 \cup X_3 \cup X_4)$, then $e(X_i, Y_5) = 1$ for each $1 \leq i \leq 4$ and thus, $e(Y_5, V(G) \setminus Y_5) = 4$. This implies $|Y_5| \geq k + 1$. Consider the partition of $V(G)$ into three parts $X_1, X_2 \cup X_3, X_4 \cup Y_5$. The quotient matrix of this partition is
\[
F_3 = \begin{bmatrix}
  k - \frac{4}{n_1} & \frac{2}{n_1} & \frac{2}{n_1} \\
  \frac{2}{n_2'} & k - \frac{6}{n_2} & \frac{4}{n_2} \\
  \frac{2}{n_3'} & \frac{4}{n_3} & k - \frac{6}{n_3}
\end{bmatrix},
\]
where $n_2' = |X_2 \cup X_3| = |X_2| + |X_3| \geq 2(k + 1)$ and $n_3' = |X_4 \cup Y_5| = |X_4| + |Y_5| \geq 2(k + 1)$.

The largest eigenvalue of $F_3$ is $k$. Eigenvalue interlacing and $n_1 \geq k + 1, n_2', n_3' \geq$
2(k + 1) imply

\[ \lambda_2(G) \geq \lambda_2(F_3) \geq \frac{tr(F_3) - k}{2} \geq k - \frac{2}{n_1} - \frac{3}{n_2} - \frac{3}{n_3} \geq k - \frac{2}{k + 1} - \frac{3}{2(k + 1)} - \frac{3}{2(k + 1)} = k - \frac{5}{k + 1}, \]

as required.

In the second situation, if \( Y_5 := V(G) \setminus (X_1 \cup X_2 \cup X_3 \cup X_4) \) then \( e(X_1, Y_5) = e(X_2, Y_5) = 0 \) and \( e(X_3, Y_5) = e(X_4, Y_5) = 1 \). This implies \( e(Y_5, V(G) \setminus Y_5) = 2 \). By Theorem 2.1.3, we deduce that \( \lambda_2(G) \geq k - \frac{4}{k + 3} > k - \frac{5}{k + 1} \) which finishes the proof of this subcase.

Assume that \( c = 1 \). Without loss of generality, assume that \( r_i = 4 \) for \( 1 \leq i \leq 4 \), and \( r_5 = 5 \). Equation (2.11) implies that the graph is necessarily of the form shown in Figure 2.3, where \( Y \) is a component that necessarily joins to \( X_5 \). By Theorem 2.1.3, it follows that \( \lambda_2(G) \geq k - \frac{2}{k + 5} > k - \frac{5}{k + 1} \), as required.

**Figure 2.3:** The structure of \( G \) when \( a = 0, b = 4, c = 1, \) and \( t \geq 5 \).

If \( b = 5 \), then \( c = 0 \) by (2.11). Also, by (2.11), it follows that \( t = 5 \) and \( e(X_i, X_j) = 1 \) for each \( 1 \leq i < j \leq 5 \). Consider the partition of the vertex set of \( G \) into three parts: \( X_1, X_2 \cup X_3, X_4 \cup X_5 \). The quotient matrix of this partition is

\[
G_3 = \begin{bmatrix}
    k - \frac{4}{n_1} & \frac{2}{n_1} & \frac{2}{n_1} \\
    \frac{2}{n_2} & k - \frac{6}{n_2} & \frac{4}{n_2} \\
    \frac{2}{n_3} & \frac{4}{n_3} & k - \frac{6}{n_3}
\end{bmatrix},
\]

which is identical to the quotient matrix \( F_3 \) in a previous case, which yields \( \lambda_2(G) \geq k - \frac{5}{k + 1} \).
If \( b > 5 \), then (2.11) will yield a contradiction. This finishes the proof of Theorem 2.1.2.

We show that our bound is essentially best possible by presenting a family of \( k \)-regular graphs \( Z_k \) with \( k - \frac{5}{k+1} \leq \lambda_2(Z_k) < k - \frac{5}{k+3} \) and \( \sigma(Z_k) = 2 \), for every \( k \geq 6 \).

For \( k \geq 6 \), consider 5 vertex disjoint copies \( H_1, H_2, H_3, H_4, H_5 \) of \( K_{k+1} \) without two vertex-disjoint edges. In each \( H_i \), let the induced subgraph on these 4 vertices be a \( C_4 \) of the form \( a_i, b_i, c_i, d_i \). Let \( A = E(H_1) \cup E(H_2) \cup E(H_3) \cup E(H_4) \cup E(H_5) \). We define

\[
E(Z_k) = A \cup \{b_1a_2, b_2a_3, b_3a_4, b_4a_5, c_1d_3, c_3d_5, c_5d_2, c_2d_4, c_4d_1\}.
\]

**Figure 2.4:** The 10-regular graph \( Z_{10} \), with \( \sigma(Z_{10}) = 2 \) and \( 9.545 \approx 10 - \frac{5}{10+1} < \lambda_2(Z_{10}) \approx 9.609 < 10 - \frac{5}{10+3} \approx 9.615 \).

The partition of the vertex set of \( Z_k \) into the five parts: \( V(H_1), V(H_2), V(H_3), V(H_4), V(H_5) \) has the property that the number of edges between the parts equals \( 10 < 12 = 3(5-1) \).
By the Nash-Williams/Tutte Theorem, \( Z_k \) does not contain 3 edge-disjoint spanning trees.

Denote by \( \gamma_k \) the largest root of the polynomial

\[
\begin{align*}
&x^{10} + (8 - 2k)x^9 + (k^2 - 16k + 30)x^8 + (8k^2 - 50k + 58)x^7 + (20k^2 - 66k + 36)x^6 + \\
&(8k^2 + 18k - 70)x^5 + (-29k^2 + 140k - 146)x^4 + (-20k^2 + 57k - 21)x^3 + (14k^2 - 83k + 109)x^2 + \\
&(4k^2 - 13k + 5)x - k^2 + 5k - 5.
\end{align*}
\]

**Lemma 2.3.1.** For every integer \( k \geq 6 \), the second largest eigenvalue of \( Z_k \) is \( \gamma_k \).

**Proof.** Consider the following partition of the vertex set of \( Z_k \) into 25 parts: 5 parts of the form \( V(H_i) \setminus \{a_i, b_i, c_i, d_i\} \), \( i = 1, 2, 3, 4, 5 \). The remaining 20 parts consist of the 20 individual vertices \( \{a_i\}, \{b_i\}, \{c_i\}, \{d_i\}, i = 1, 2, 3, 4, 5 \). This partition is equitable and the characteristic polynomial of its quotient matrix is

\[
P_{25}(x) = (x - k)(x - 1)(x + 1)^2(x + 3)[x^{10} + (8 - 2k)x^9 + (k^2 - 16k + 30)x^8 + \\
(8k^2 - 50k + 58)x^7 + (20k^2 - 66k + 36)x^6 + (8k^2 + 18k - 70)x^5 + \\
(-29k^2 + 140k - 146)x^4 + (-20k^2 + 57k - 21)x^3 + (14k^2 - 83k + 109)x^2 + \\
+ (4k^2 - 13k + 5)x - k^2 + 5k - 5]^2.
\]

Let \( \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_{11} \) denote the solutions of the degree 10 polynomial \( P_{10}(x) \). Because the partition is equitable, it follows that these 10 solutions, \( k, 1, -1, \) and \( -3 \) are eigenvalues of \( Z_k \), including multiplicity.

We claim the spectrum of \( Z_k \) is

\[
k^{(1)}, 1^{(1)}, -3^{(1)}, -1^{(5k-18)}, \lambda_i^{(2)} \quad \text{for } i = 2, 3, \ldots, 11. \tag{2.13}
\]

It suffices to obtain \( 5k - 18 \) eigenvectors corresponding to eigenvalue \(-1\). Consider two distinct vertices \( u_1^i \) and \( u_2^i \) in \( V(H_1) \setminus \{a_1, b_1, c_1, d_1\} \). Define an eigenvector where the entry corresponding to \( u_1^i \) is 1, the entry corresponding to \( u_2^i \) is -1, and all other entries 0. We create \( k - 4 \) eigenvectors by redefining \( u_2^i \) to be each of the \( k - 4 \) vertices in \( V(H_1) \setminus \{a_1, b_1, c_1, d_1, u_1\} \). This can also be applied to 2 vertices \( u_1^i, u_2^i \) in \( V(H_1) \setminus \{a_i, b_i, c_i, d_i\} \), for \( i = 2, 3, 4, 5 \). We obtain a total of \( 5k - 20 \) eigenvectors.
Furthermore, define an eigenvector with entries at vertices $u_1$ to be -2, $a_i$ and $d_i$ to be 1, for $i = 1, 2, 3, 4, 5$, and 0 in all other entries. To obtain the final eigenvector, define entries at vertices $u_1$ to be -2, $b_i$ and $c_i$ to be 1, for $i = 1, 2, 3, 4, 5$, and 0 in all other entries. It is easy to check all $5k-18$ vectors are linearly independent eigenvectors corresponding to eigenvalue -1.

By obtaining the entire spectrum of $Z_k$, we conclude that the second largest eigenvalue of $Z_k$ must lie in the degree 10 polynomial.

\[ \text{Lemma 2.3.2. For every integer } k \geq 6, \]

\[ \frac{k - 5}{k + 1} \leq \gamma_k < \frac{k - 5}{k + 3}. \]

\[ \text{Proof. The lower bound follows directly from Theorem 2.1.2 as } \sigma(Z_k) < 3. \text{ Moreover, by technical calculations (see Appendix A.0.2) } \]

\[ P_{10}^{(n)} \left( k - \frac{5}{k + 3} \right) > 0, \text{ for } n = 0, 1, ..., 10. \]

Descartes' Rule of Signs [58] implies $\gamma_k < k - \frac{5}{k+3}$. Hence,

\[ \frac{k - 5}{k + 1} \leq \gamma_k < \frac{k - 5}{k + 3} \tag{2.14} \]

for every $k \geq 6$. \qed

### 2.4 A family of graphs without 4 edge-disjoint spanning trees

Based on Theorems 2.1.1 and 2.1.2, we conjecture if a $k$-regular graph $G$ has $\lambda_2(G) < k - \frac{2m-1}{k+1}$, for $k \geq 2m$, then $G$ contains at least $m$ edge-disjoint spanning trees (see Section 6.1). Moreover, we believe this bound is best possible. We support this claim further for $m = 4$ by providing a family of $k$-regular graphs $A_k$ with $k - \frac{7}{k+1} < \lambda_2(A_k) < k - \frac{7}{k+3}$ and $\sigma(A_k) = 3$.

For $k \geq 8$, consider 7 vertex disjoint copies $H_1, H_2, H_3, H_4, H_5, H_6, H_7$ of $K_{k+1}$ without 3 vertex-disjoint edges. Denote the three missing edges by $a_ib_i$, $c_id_i$, and $e_if_i$, $1 \leq i \leq 7$.  

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Let \( A = E(H_1) \cup E(H_2) \cup E(H_3) \cup E(H_4) \cup E(H_5) \cup E(H_6) \cup E(H_7) \). To define the other edges of \( A_k \), consider a cyclic construction joining the \( H_i \)'s as follows: create 3 cycles, each of length 14, alternating with an edge and missing edge. The first cycle is of the form
\[ a_1, a_2, b_2, a_3, b_3, a_4, b_4, a_5, b_5, a_6, b_6, a_7, b_7, b_1. \]

Repeat this construction on the \( c_i, d_i \) and \( e_i, f_i \) to create two other cycles of length 14, alternating between an edge and missing edge. The edge set of this family will be \( A \) with the 21 edges created from these 3 cycles. The graph \( A_k \) is a connected \( k \)-regular graph on \( 7(k + 1) \) vertices (see Figure 2.5).

The partition of the vertex set of \( A_k \) into the seven parts \( V(H_i) \), \( 1 \leq i \leq 7 \), has the property that the number of edges between the parts equals 21 < 24 = 4(7 - 1).

By the Nash-Williams/Tutte Theorem, \( A_k \) does not contain 4 edge-disjoint spanning trees. It is not difficult to check that \( \sigma(A_k) = 3 \).

**Theorem 2.4.1.** For every integer \( k \geq 8 \), \( \lambda_2(A_k) \) is the largest root of
\[ P_7(x) = x^7 - (k - 6)x^6 - (6k - 15)x^5 - (8k - 13)x^4 - (27 - 8k)x^3 - (36 - 13k)x^2 - (6k - 29)x - k. \]

Furthermore, \( k - \frac{7}{k+1} < \lambda_2(A_k) < k - \frac{7}{k+3} \).

**Proof.** Many of the calculations are done through Mathematica, and can be found in Appendix A.1. Consider the following partition of \( V(A_k) \) into 49 parts: 7 parts of the form \( V(H_i) \setminus (\{a_i\}, \{b_i\}, \{c_i\}, \{d_i\}, \{e_i\}, \{f_i\}) \), \( 1 \leq i \leq 7 \). The remaining 42 parts consist of the 42 individual vertices \( \{a_i\}, \{b_i\}, \{c_i\}, \{d_i\}, \{e_i\}, \{f_i\} \), \( 1 \leq i \leq 7 \). This partition is equitable and the characteristic polynomial of its quotient matrix is
\[ (k - x)(1 + x)^6 \left( x^7 - (k - 6)x^6 - (6k - 15)x^5 - (8k - 13)x^4 - (27 - 8k)x^3 - (36 - 13k)x^2 - (6k - 29)x - k \right)^6. \]

Let \( \lambda_2 \geq \lambda_3 \geq ... \geq \lambda_8 \) denote the distinct solutions of the degree 7 polynomial. Because the partition is equitable, it follows that these 7 solutions, \( k \), and \(-1\) are eigenvalues of \( A_k \), including multiplicity. We claim the spectrum of \( A_k \) is
\[ k^{(1)}, -1^{(7k-36)}, \lambda_i^{(6)} \quad \text{for } 2 \leq i \leq 8. \]
Figure 2.5: The cyclic construction of $\mathcal{A}_k$ with $k - \frac{7}{k+1} < \lambda_2(\mathcal{G}_k) < k - \frac{5}{k+3}$ and $\sigma(\mathcal{A}_k) = 3$

It suffices to obtain $7k - 36$ eigenvectors corresponding to $-1$.

Consider two distinct vertices $u^1_1$ and $u^2_1$ in $V(H_1) \setminus \{a_1, b_1, c_1, d_1, e_1, f_1\}$. Define an eigenvector where the entry corresponding to $u^1_1$ is $-1$, the entry corresponding to $u^1_2$ is $1$, and all other entries $0$. We create $k - 6$ eigenvectors by redefining $u^1_2$ to be each of the $k - 6$ vertices in $V(H_1) \setminus \{a_1, b_1, c_1, d_1, e_1, f_1, u^1_1\}$. This can also be applied to $2$ vertices $u^1_i, u^2_i$ in $V(H_i) \setminus \{a_i, b_i, c_i, d_i, e_i, f_i\}$, for $2 \leq i \leq 7$. We obtain a total of $7k - 42$ eigenvectors.

Define an eigenvector whose entries at $u^1_i$ are $-1$, for $1 \leq i \leq 7$ (as above). In
addition, define the entries in positions $a_1, b_2, b_3, b_4, b_5, b_6,$ and $b_7$ to be 1. All other entries are 0. This creates an eigenvector corresponding to eigenvalue -1. Similarly, define another eigenvector whose entries at $u_i^1$ are -1, for $1 \leq i \leq 7$ (as above). Let the entries in positions $b_1, a_2, a_3, a_4, a_5, a_6,$ and $a_7$ to be 1 (on alternating vertices in the cycle of length 14). All other entries are 0. This is another eigenvector corresponding to eigenvalue -1. We can create 4 more eigenvectors in a similar manner by defining $c_i, d_i$ and $e_i, f_i$ entries as 1, accordingly. These six eigenvectors, along with the previous $7k - 42$ eigenvectors, combine to form $7k - 36$ vectors that are linearly independent eigenvectors corresponding to eigenvalue -1.

By obtaining the entire spectrum of $A_k$, we conclude $\lambda_2(A_k)$ is a root of $P_7(x)$. We find that

$$P_7^n \left( k - \frac{7}{k+3} \right) > 0, \text{ for } n = 0, 1, ..., 7.$$ 

Descartes’ Rule of Signs [58, p. 89-93] implies $\lambda_2(A_k) < k - \frac{7}{k+3}$. Furthermore,

$$P_7(k - \frac{7}{k+1}) < 0.$$ 

Hence,

$$k - \frac{7}{k+1} < \lambda_2(A_k) < k - \frac{7}{k+3} \quad (2.15)$$

for every $k \geq 8$. \hfill \Box
Chapter 3

SPECTRAL CONDITIONS FOR TOUGHNESS

Most of the results of this chapter can be found in [25].

3.1 Introduction

The toughness $t(G)$ of a connected graph $G$ is the minimum of $\frac{|S|}{c(G \setminus S)}$, where the minimum is taken over all vertex subsets $S$ whose removal disconnects $G$, and $c(G \setminus S)$ denotes the number of components of the graph obtained by removing the vertices of $S$ from $G$. A graph $G$ is called $t$-tough if $t(G) \geq t$. Chvátal [20] introduced this parameter in 1973 to capture combinatorial properties related to the cycle structure of a graph. The toughness of a graph is related to many other important properties of a graph such as Hamiltonicity, and the existence of various factors, cycles or spanning trees. It is a difficult parameter to determine exactly (see [4] for a survey). A graph is Hamiltonian if there exists a cycle that traverses every vertex of the graph exactly once (except for the first and last vertex, which will be the same). Chvátal proved a simple relationship between Hamiltonicity and toughness.

Lemma 3.1.1 ([19]). If $G$ is Hamiltonian, then for any non-empty set $S \subset V(G)$, $c(G \setminus S) \leq |S|$. Namely, $G$ is 1-tough.

Two of Chvátal conjectures from [20] motivated a lot of subsequent work. The first conjecture is related to the converse of Lemma 3.1.1: does there exist some $t_0 > 0$ such that any graph with toughness greater than $t_0$ is Hamiltonian? This conjecture is open at the present time and Bauer, Broersma, and Veldman [5] showed that if such a $t_0$ exists, then it must be at least $9/4$. The second conjecture of Chvátal asserted the existence of $t_1 > 0$ such that any graph with toughness greater than $t_1$
is pancyclic, meaning the graph contains a cycle of length $i$, for all $3 \leq i \leq |V(G)|$.

This was disproved by several authors including Alon [1], who showed that there are graphs of arbitrarily large girth and toughness. Alon’s results relied heavily on some connections between the toughness of a regular graph and its eigenvalues. Around the same time and independently, Brouwer [11] discovered slightly better relations between the toughness of a regular graph and its eigenvalues.

Theorem 3.1.2 (Brouwer [11]). If $G$ is a connected $k$-regular graph with eigenvalues $k = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n$ and $\lambda = \max(|\lambda_2|, |\lambda_n|)$, then

$$t(G) > \frac{k}{\lambda} - 2.$$  \hfill (3.1)

Brouwer [11] conjectured that the lower bound of the previous theorem can be improved to $t(G) \geq \frac{k}{\lambda} - 1$ for any connected $k$-regular graph $G$. This bound would be best possible as there exists regular bipartite graphs with toughness arbitrarily close to 0 (see Figure 3.1).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.1}
\caption{Consider $d$ copies of $K_{d,d}$ missing an edge $e$, and 2 isolated vertices, each joining to a missing edge of the $d$ copies. Then the toughness is at most $\frac{2}{d}$, which can be arbitrarily close to $\frac{4}{d} - 1 = 0$ when $d$ is large.}
\end{figure}

In this chapter, we improve Brouwer’s inequality $t(G) \geq \frac{k}{\lambda} - 2$ in certain cases. For small $\tau$, we obtain a better eigenvalue condition that implies a regular graph is $\tau$-tough, and we determine a best possible sufficient eigenvalue condition for a regular
graph to be 1-tough. We note here that Bauer, van den Heuvel, Morgana, and Schmeichel [6, 7] proved that recognizing 1-tough graphs is an NP-hard problem for regular graphs of valency at least 3. Our results also imply some results of Liu and Chen for regular graphs [49, Theorem 2.7, Corollary 2.8, Theorem 3.2].

Our improvements of Brouwer’s bound (3.1) are the following two results.

**Theorem 3.1.3.** Let $G$ be a connected $k$-regular graph on $n$ vertices, $k \geq 3$, with adjacency eigenvalues $k = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and edge-connectivity $\kappa'$. If $\tau \leq \kappa'/k$ is a positive number such that $\lambda_2(G) < k - \frac{\tau k}{k+1}$, then $t(G) \geq \tau$.

**Theorem 3.1.4.** If $G$ is a connected $k$-regular graph and

$$\lambda_2(G) < \begin{cases} \frac{k-2+\sqrt{k^2+8}}{2} & \text{when } k \text{ is odd} \\ \frac{k-2+\sqrt{k^2+12}}{2} & \text{when } k \text{ is even} \end{cases}$$

then $t(G) \geq 1$.

The proofs of Theorem 3.1.3 and Theorem 3.1.4 are contained in Section 3.2. We show that Theorem 3.1.4 is best possible in the sense that for each $k \geq 3$, we will construct examples of $k$-regular graphs whose second largest eigenvalue equals the right hand-side of inequality (3.2), but whose toughness is less than 1. These examples are described in Section 3.3. Our examples are regular graphs of diameter 4 and their existence also answers a question of Liu and Chen [49, p. 1088] about the minimum possible diameter of a regular graph with toughness less than 1. Lastly, we provide a spectral condition for the existence of a $d$-tree, which is a spanning tree with maximum degree $d$ for $d \geq 3$.

### 3.2 Proofs of Theorem 3.1.3 and Theorem 3.1.4

We first give a short proof of Theorem 3.1.3.

**Proof of Theorem 3.1.3.** We prove the contrapositive. Assume $\tau \leq \kappa'/k \leq 1$ is a positive number and that $G$ is a connected $k$-regular graph such that $t(G) < \tau$. Then
there exists \( S \subset V(G) \) of size \( s \) such that \( c = c(G \setminus S) > s/\tau \geq s \) (as \( \tau \in (0,1] \)). We denote by \( H_1, \ldots, H_c \) the components of \( G \setminus S \). Let \( n_i \) be the number of vertices in \( H_i \) and let \( t_i \) be the number of edges between \( H_i \) and \( S \). The number of edges between \( S \) and \( H_1 \cup \cdots \cup H_c \) is \( t_1 + \cdots + t_c \leq ks < k\tau c \). We claim that there exist at least two \( t_i \)'s such that \( t_i \leq \tau k \). Assume otherwise. Then at least \( c - 1 \) of the \( t_i \)'s are greater than \( \tau k \) and thus, \( t_1 + \cdots + t_c \geq \kappa' + (c - 1)\tau k = k\tau c + \kappa' - \tau k \geq k\tau c \), a contradiction. The last inequality follows since \( \tau \leq \kappa'/k \). Without loss of generality, assume that \( \max(t_1, t_2) \leq \tau k < k \). If \( n_1 \leq k \), then \( t_1 \geq n_1(k - n_1 + 1) \geq k \) which is a contradiction. Thus, \( n_1 \geq k + 1 \) and by a similar argument \( n_2 \geq k + 1 \). For \( i \in \{1, 2\} \), the average degree of the component \( H_i \) is \( \frac{kn_i - t_i}{n_i} = k - \frac{t_i}{n_i} \geq k - \frac{\tau k}{k+1} \). As the largest eigenvalue of the adjacency matrix of a graph is at least its average degree (see Lemma 1.4.1), we obtain that \( \min(\lambda_1(H_1), \lambda_1(H_2)) \geq k - \frac{\tau k}{k+1} \). By eigenvalue interlacing (see Section 1.3.2), we obtain

\[
\lambda_2(G) \geq \lambda_2(H_1 \cup H_2) \geq \min(\lambda_1(H_1), \lambda_1(H_2)) \geq k - \frac{\tau k}{k+1}.
\] (3.3)

This finishes our proof. \( \Box \)

In order to prove Theorem 3.1.4, we need to do some preliminary work. If \( t \) is a positive even integer, denote by \( M_t \) a perfect matching on \( t \) vertices. If \( G \) and \( H \) are two vertex disjoint graphs, the join \( G \lor H \) of \( G \) and \( H \) is the graph obtained by taking the union of \( G \) and \( H \) and adding all the edges between the vertex set of \( G \) and the vertex set of \( H \). Recall that the complement of a graph \( G \) is denoted by \( \overline{G} \).

**Lemma 3.2.1.** For \( k \geq 3 \), define

\[
X_k = \begin{cases} 
M_{k-1} \lor K_2 & \text{if } k \text{ is odd} \\
M_{k-2} \lor K_3 & \text{if } k \text{ is even.}
\end{cases}
\] (3.4)

Then

\[
\lambda_1(X_k) = \begin{cases} 
\frac{k-2+\sqrt{k^2+8}}{2} & \text{if } k \text{ is odd} \\
\frac{k-2+\sqrt{k^2+12}}{2} & \text{if } k \text{ is even.}
\end{cases}
\] (3.5)
Proof. If $k \geq 3$ is odd, then the partition of the vertex set of $X_k$ into the 2 vertices of degree $k$ and the $k-1$ vertices of degree $k-1$ is an equitable partition whose quotient matrix is \[
\begin{bmatrix}
1 & k-1 \\
2 & k-3
\end{bmatrix} \] . The largest eigenvalue of $X_k$ equals the largest eigenvalue of this quotient matrix which is $\frac{k-2+\sqrt{k^2+8}}{2}$ (see \cite[Section 2.3]{12}). If $k$ is even, the proof is similar.  

Let $k \geq 3$ be an integer. Denote by $X(k)$ the family of all connected irregular graphs with maximum degree $k$, order $n \geq k+1$ and size $e$ with $2e \geq kn - k + 1$. Assume also that each graph in $X(k)$ has at least 2 vertices of degree $k$ when $k$ is odd and at least 3 vertices of degree $k$ when $k$ is even. Denote by $\gamma(k)$ the minimum spectral radius among all graphs in $X(k)$.

It is clear that $X_k \in X(k)$ for any $k \geq 3$ and consequently, $\gamma(k) \leq \lambda_1(X_k)$. We now show that actually $\gamma(k) = \lambda_1(X_k)$ and that $X_k$ is the only graph in $X(k)$ whose largest eigenvalue is $\gamma(k)$.

Lemma 3.2.2. Let $k \geq 3$ be an integer. If $X \in X(k) \setminus \{X_k\}$, then $\lambda_1(X) > \lambda_1(X_k)$ and thus,

$$
\gamma(k) = \begin{cases} 
\frac{k-2+\sqrt{k^2+12}}{2} & \text{if } k \text{ is even}, \\
\frac{k-2+\sqrt{k^2+8}}{2} & \text{if } k \text{ is odd}.
\end{cases}
$$

Proof. When $k$ is even, this was proved in \cite[Theorem 2]{22} so for the rest of the proof, we assume that $k$ is odd. Let $X \in X(k) \setminus \{X_k\}$. We will prove that $\lambda_1(X) > \lambda_1(X_k)$.

Assume first that $n \geq k+2$, where $n$ is the number of vertices of $X$. As $X \in X_k$, we know that $2e \geq kn - k + 1$, where $e$ is the number of edges of $X$. Because the largest eigenvalue of $X$ is at least the average degree of $X$ (Lemma 1.4.1), after some straightforward calculations we obtain that

$$
\lambda_1(X) \geq \frac{2e}{n} \geq k - \frac{k-1}{n} \geq k - \frac{k-1}{k+2} > \frac{k-2+\sqrt{k^2+8}}{2} = \lambda_1(X_k).
$$

(3.7)
Assume now that \( n = k + 1 \). If \( 2e > kn - k + 1 \), then, since \( k \) is odd, we must have \( 2e \geq kn - k + 3 \). We obtain that

\[
\lambda_1(X) \geq \frac{2e}{n} \geq k - \frac{k - 3}{k + 1} > \frac{k - 2 + \sqrt{k^2 + 8}}{2} = \lambda_1(X_k).
\] (3.8)

The only possible case remaining is \( n = k + 1 \) and \( 2e = kn - k + 1 \). Let \( V_1 \) be a subset of two vertices of degree \( k \) and let \( V_2 \) the complement of \( V_1 \). The quotient matrix of the partition of the vertex set into \( V_1 \) and \( V_2 \) is

\[
B = \begin{bmatrix} 1 & k - 1 \\ 2 & k - 3 \end{bmatrix}.
\] (3.9)

Eigenvalue interlacing gives

\[
\lambda_1(X) \geq \lambda_1(B) = \frac{k - 2 + \sqrt{k^2 + 8}}{2} = \lambda_1(X_k),
\] (3.10)

with equality if and only if the partition \( V_1 \cup V_2 \) is equitable. This happens if and only if \( X = X_k \). As \( X \in X_k \setminus \{X_k\} \), we get that \( \lambda_1(X) > \frac{k - 2 + \sqrt{k^2 + 8}}{2} = \lambda_1(X_k) \), which finishes our proof.

We are ready now to prove Theorem 3.1.4. The proof is similar to the one for Theorem 3.1.3.

Proof of Theorem 3.1.4. We prove the contrapositive. We show that \( t(G) < 1 \) implies that \( \lambda_2(G) \geq \gamma(k) \). If \( t(G) < 1 \), then there exists an \( S \subset V(G) \) of size \( s \) such that \( c = c(G \setminus S) > |S| = s \). As in the proof of Theorem 3.1.3, let \( H_1, H_2, \ldots, H_c \) be the components of \( G \setminus S \). For \( 1 \leq i \leq c \), denote by \( t_i \geq 1 \) the number of edges between \( S \) and \( H_i \), and denote \( n_i \) to be the number of vertices of \( H_i \). Then \( \sum_{i=1}^{c} t_i = e(S, V \setminus S) \leq ks \). We claim there exists two \( t_i \)’s such that \( t_i < k \). Indeed, if there exists at most one such \( t_i \), then at least \( c - 1 \) of the \( t_i \)’s are greater than \( k \) and thus, \( ks \geq \sum_{i=1}^{c} t_i \geq (c-1)k+1 > ks \) which is a contradiction. Thus, there exist at least two \( t_i \)’s (say \( t_1 \) and \( t_2 \)) such that \( \max(t_1, t_2) < k \). If \( n_1 \leq k \), then \( t_1 \geq n_1(k - n_1 + 1) \geq k \) which is a contradiction. Thus, \( n_1 \geq k + 1 \) and by a similar argument \( n_2 \geq k + 1 \). As \( t_i \leq k - 1 \) for \( i \in \{1, 2\} \), we get \( 2e(H_i) = kn_i - t_i \geq kn_i - k + 1 \). Also, if \( k \) is even,
then $2e_i = kn_i - t_i$ implies that $t_i$ is even. For $i \in \{1, 2\}$, this means $t_i \leq k - 2$ as $t_i < k$. Hence, $\max(t_1, t_2) \leq k - 2$ when $k$ is even and $\max(t_1, t_2) \leq k - 1$ when $k$ is odd. This implies $H_i$ contains at least two vertices of degree $k$ when $k$ is odd and at least 3 vertices of degree $k$ when $k$ is even. Thus, $H_i \in \mathcal{X}_k$ for $i \in \{1, 2\}$. By Lemma 3.2.2, we get that $\min(\lambda_1(H_1), \lambda_1(H_2)) \geq \gamma(k)$. By Cauchy eigenvalue interlacing (see Section 1.3.2), we obtain $\lambda_2(G) \geq \lambda_2(H_1 \cup H_2) \geq \min(\lambda_1(H_1), \lambda_1(H_2)) \geq \gamma(k)$ which finishes our proof.

The following is an immediate corollary of Theorem 3.1.4 that can be used to determine the toughness of many bipartite regular graphs, such as the $n$-dimensional cube $H(n, 2)$.

**Corollary 3.2.3.** If $G$ is a bipartite $k$-regular graph with

$$\lambda_2(G) < \begin{cases} \frac{1}{2} (k - 2 + \sqrt{k^2 + 8}) & k \text{ is odd} \\ \frac{1}{2} (k - 2 + \sqrt{k^2 + 12}) & k \text{ is even} \end{cases}$$

(3.11)

then $t(G) = 1$.

**Proof.** Let $S$ be the set of vertices of one part of the bipartition. Then $\frac{|S|}{\sigma(G \setminus S)} = 1$, and so $t(G) \leq 1$. Theorem 3.1.4 implies that $t(G) = 1$. \qed

We believe that Corollary 3.2.3 can be improved by proving a similar result to Lemma 3.2.2 for bipartite graphs.

### 3.3 Examples showing Theorem 3.1.4 is best possible

In this section, we construct two families of connected $k$-regular graphs $G_k$ and $H_k$ such that $\lambda_2(G_k) = \gamma(k)$ and $t(G_k) < 1$ for every odd $k \geq 3$, and $\lambda_2(H_k) = \gamma(k)$ and $t(G_k) < 1$ for every even $k \geq 3$. This shows that the bound contained in Theorem 3.1.4 is best possible.
3.3.1 Examples for odd $k$

If $k \geq 3$ is odd, consider $k$ pairwise vertex disjoint copies of $\overline{M}_{k-1} \lor K_2$. Consider also an independent set $T$ of size $k-1$. The graph $G_k$ is obtained by adding a matching of size $k-1$ between the vertex set of $T$ and the $k-1$ vertices of degree $k-1$ in each of the $k-1$ copies of $\overline{M}_{k-1} \lor K_2$. The graph $G_k$ is a connected $k$-regular graph on $n = k(k+1)+k-1 = k^2+2k-1$ vertices. By choosing $S \subset V(G_k)$ as the independent set $T$ of $k-1$ vertices, the graph $G_k$ has toughness at most $\frac{k-1}{k} < 1$. The graph $G_5$ is shown in Figure 3.2.

![Graph](image)

**Figure 3.2:** The graph $G_5$ is 5-regular, has $\lambda_2(G_5) = \gamma(5)$, but $t(G_5) < 1$.

We claim $\lambda_2(G_k) = \frac{1}{2} \left(k - 2 + \sqrt{k^2 + 8}\right)$. The vertices of $X_k = \overline{M}_{k-1} \lor K_2$ can be ordered so that its adjacency matrix is partitioned in the form

$$A(G_k) = \begin{bmatrix}
A(\overline{M}_{k-1}) & J \\
J^T & A(K_2)
\end{bmatrix},$$
where $J$ denotes an all-ones matrix of appropriate size. The $n = k^2 + 2k - 1$ vertices of $G_k$ can be ordered so that its adjacency matrix is partitioned in the form

$$A(G_k) = \begin{bmatrix}
0 & B & B & \cdots & B \\
B^T & A(X_k) & 0 & \cdots & 0 \\
B^T & 0 & A(X_k) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
B^T & 0 & 0 & \cdots & A(X_k)
\end{bmatrix},$$

where 0 denotes a zero matrix of appropriate size, and $B = [I \ 0]$, where $I$ is an identity matrix of order $k - 1$.

The eigenvectors of $X_k$ that are constant on each part of its 2-part equitable partition have eigenvalues given by the quotient matrix

$$\begin{bmatrix}
k - 3 & 2 \\
k - 1 & 1
\end{bmatrix}.$$  

These eigenvalues are $\frac{1}{2}(k - 2 \pm \sqrt{k^2 + 8})$, and the positive eigenvalue is $\lambda_1(X_k) = \gamma(k)$ from Lemma 3.2.2. Let $x$ and $y$ be eigenvectors of $G_k$ associated with these two eigenvalues.

If $u$ is a column eigenvector of $A(X_k)$, consider the $n$-dimensional $k - 1$ column vectors

$$\begin{bmatrix}
0 \\
u \\
-u \\
\vdots \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
u \\
0 \\
\vdots \\
0
\end{bmatrix}, \cdots, \begin{bmatrix}
0 \\
-u \\
\vdots \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
0 \\
\vdots \\
-u
\end{bmatrix},$$

where the zero vectors are of appropriate size, and the first zero vector is always present (corresponding to the $k - 1$ isolated vertices). It is straightforward to check that these $k - 1$ vectors are linearly independent eigenvectors of $A(G_k)$ corresponding to the same eigenvalue as $u$. Thus each eigenvalue of $A(X_k)$ of multiplicity $t$ yields an eigenvalue
of $A(G_k)$ of multiplicity at least $t(k-1)$. In particular, taking $u = x$ and $u = y$, we see that the two eigenvalues of $A(X_k)$ yield eigenvalues of $A(G_k)$ of multiplicity at least $k - 1$ each. Thus $\lambda_1(X_k) = \gamma(k)$ is an eigenvalue of $G_k$ with multiplicity at least $k - 1$.

Now consider the $(2k + 1)$-part equitable partition of $G_k$ obtained by extending the 2-part partition of the $k$ copies of $X_k$ in $G_k$ (a generalization of Figure 3.3). If $W$ is the vector space spanned by the $n$-dimensional vectors that are constant on each part of the partition, the dimension of $W$ is $2k + 1$. Note that each of the $2(k - 1)$ independent eigenvectors of $G_k$ inherited from the eigenvectors $x$ and $y$ of $X_k$ are in $W$. The natural 3-part equitable partition of $G_k$ (Figure 3.3) has quotient matrix

$$
\begin{bmatrix}
0 & k & 0 \\
1 & k-3 & 2 \\
0 & k-1 & 1
\end{bmatrix},
$$

with eigenvalues $k$ and $-1 \pm \sqrt{2}$. These three eigenvalues are different from those above. Each corresponding eigenvector of this quotient matrix $u$ lifts to an eigenvector $v$ of $G_k$ in $W$ as follows: For any entry in $u$, the value extends to all entries in $v$ in the corresponding section of the partition. Thus the three lifted eigenvectors, together with the previous $2(k - 1)$ eigenvectors of $G_k$ inherited from $X_k$, form a basis for $W$.

The remaining eigenvectors in a basis of eigenvectors for $G_k$ may be chosen orthogonal to be to the vectors in $W$. They may be chosen to be orthogonal to the characteristic vectors of the parts of the $(2k + 1)$-part partition because these characteristic vectors are also a basis for $W$. Therefore, they will be some of the eigenvectors of the matrix $A(\hat{G}_k)$ obtained from $A(G_k)$ by replacing each all-ones block in each diagonal block of $A(X_k)$ by an all-zeros matrix. But $A(\hat{G}_k)$ is the adjacency matrix of a graph $\hat{G}_k$ with $k + 1$ connected components, one of which is the graph $G'$ obtained by attaching $k$ copies of $M_{k-1}$ to a set of $k - 1$ independent vertices by perfect matchings. Each of the remaining $k$ components is a copy of $K_2$. It follows that the greatest eigenvalue of $\hat{G}_k$ is that of a component of $G'$. Because $G'$ has a 2-part
Figure 3.3: The equitable partition of $G_5$ in three parts.

equitable partition with quotient matrix

$$
\begin{bmatrix}
0 & k \\
1 & k - 3
\end{bmatrix},
$$

its greatest eigenvalue is $\frac{1}{2} (k - 3 + \sqrt{k^2 - 2k + 9})$, which is less than $\frac{1}{2} (k - 2 + \sqrt{k^2 + 8}) = \gamma(k)$.

3.3.2 Examples for even $k$

If $k \geq 4$ is even, consider $k - 1$ vertex disjoint copies of $M_{k-2} \vee K_3$. Consider also a vertex disjoint copy $T$ of $M_{k-2}$. The graph $H_k$ is obtained by adding a matching of size $k - 2$ between the vertices in $T$ and the $k - 2$ vertices of degree $k - 1$ in each of the $k - 1$ copies of $M_{k-2} \vee K_3$. The graph $H_k$ is a connected $k$-regular graph on $n = (k - 1)(k + 1) + k - 2 = k^2 + k - 3$ vertices. By choosing $S \subset V(H_k)$ as the vertices in $T$, it is easy to check that $H_k$ has toughness at most $\frac{k-2}{k-1} < 1$. The graph $H_6$ is shown in Figure 3.4.
Figure 3.4: The graph $\mathcal{H}_6$ is 6-regular, has $\lambda_2(\mathcal{H}_6) = \gamma(6)$, but $t(\mathcal{H}_6) < 1$

We now show that $\lambda_2(\mathcal{H}_k) = \frac{1}{2} \left( k - 2 + \sqrt{k^2 + 12} \right)$. The vertices of $X_k = \overline{M}_{k-2} \vee K_3$ can be ordered so that its adjacency matrix is partitioned in the form

$$A(X_k) = \begin{bmatrix} A(\overline{M}_{k-2}) & J \\ J^T & A(K_3) \end{bmatrix},$$

where $J$ denotes an all-ones matrix of appropriate size. The $n = k^2 + k - 3$ vertices of $\mathcal{H}_k$ can be ordered so that its adjacency matrix is partitioned in the form

$$A(\mathcal{H}_k) = \begin{bmatrix} 0 & B & B & \cdots & B \\ B^T & A(X_k) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B^T & 0 & A(X_k) & \cdots & 0 \\ B^T & 0 & 0 & \cdots & A(X_k) \end{bmatrix}.$$
where 0 denotes a zero matrix of appropriate size, and \( B = \begin{bmatrix} I & 0 \end{bmatrix} \), where \( I \) is an identity matrix of order \( k - 2 \).

The eigenvectors of \( X_k \) that are constant on each part of its 2-part equitable partition have eigenvalues given by the quotient matrix

\[
\begin{bmatrix}
  k - 4 & 3 \\
  k - 2 & 2
\end{bmatrix}.
\]

These are \( \frac{1}{2} \left( k - 2 \pm \sqrt{k^2 + 12} \right) \), and the positive eigenvalue is \( \lambda_1(X_k) = \gamma(k) \) from Lemma 3.2.2. Let \( x \) and \( y \) be eigenvectors of \( X_k \) associated with these two eigenvalues.

If \( u \) is a column eigenvector of \( A(X_k) \), consider the \( n \)-dimensional \( k - 2 \) column vectors

\[
\begin{bmatrix}
  0 \\
  u \\
  -u \\
  \vdots \\
  0
\end{bmatrix}, \quad \begin{bmatrix}
  0 \\
  u \\
  0 \\
  \vdots \\
  0
\end{bmatrix}, \quad \begin{bmatrix}
  0 \\
  u \\
  -u \\
  \vdots \\
  -u
\end{bmatrix}, \quad \begin{bmatrix}
  0 \\
  u \\
  0 \\
  \vdots \\
  u
\end{bmatrix}, \quad \begin{bmatrix}
  0 \\
  u \\
  0 \\
  \vdots \\
  -u
\end{bmatrix}, \quad \begin{bmatrix}
  0 \\
  u \\
  0 \\
  \vdots \\
  u
\end{bmatrix}, \quad \begin{bmatrix}
  0 \\
  u \\
  -u \\
  \vdots \\
  -u
\end{bmatrix}, \quad \begin{bmatrix}
  0 \\
  u \\
  0 \\
  \vdots \\
  u
\end{bmatrix},
\]

where the zero vectors are of appropriate size, and the first zero vector is always present (corresponding to the perfect matching on \( k - 2 \) vertices). It is straightforward to check that these \( k - 2 \) vectors are linearly independent eigenvectors of \( A(H_k) \) with the same eigenvalue as \( u \). Thus each eigenvalue of \( A(H_k) \) of multiplicity \( t \) yields an eigenvalue of \( A(H_k) \) of multiplicity at least \( t(k - 2) \). In particular, taking \( u = x \) and \( u = y \), we see that the two eigenvalues of \( A(X_k) \), \( \frac{1}{2} \left( k - 2 \pm \sqrt{k^2 + 12} \right) \), yield eigenvalues of \( A(H_k) \) of multiplicity at least \( k - 2 \) each. Thus \( \lambda_1(X_k) = \gamma(k) \) is an eigenvalue of \( H_k \) with multiplicity at least \( k - 2 \).

Now consider the \( (2k - 1) \)-part equitable partition of \( H_k \) obtained by extending the 2-part partition of the \( k - 1 \) copies of \( X_k \) in \( H_k \) (a generalization of Figure 3.5). If \( W \) is the vector space spanned by \( n \)-dimensional vectors that are constant on each part of the partition, \( W \) has dimension \( 2k - 1 \). Note that each of the \( 2(k - 2) \) independent
eigenvectors of $\mathcal{H}_k$ inherited from the eigenvectors $x$ and $y$ of $H_k$ are in $W$. The natural 3-part equitable partition of $\mathcal{H}_k$ (Figure 3.5) has quotient matrix

$$
\begin{bmatrix}
1 & k - 1 & 0 \\
1 & k - 4 & 3 \\
0 & k - 2 & 2
\end{bmatrix},
$$

with eigenvalues $k$ and $\frac{1}{2} (-1 \pm \sqrt{13})$. These three eigenvalues are different from those above. Each corresponding eigenvector $u$ lifts to an eigenvector $v$ of $\mathcal{H}_k$ in $W$ as follows: For any entry in $u$, the value extends to all entries in $v$ in the corresponding section of the partition. Thus the three lifted eigenvectors, together with the previous $2(k - 2)$ eigenvectors of $\mathcal{H}_k$ inherited from $X_k$, form a basis for $W$.

![Figure 3.5: The equitable partition of $\mathcal{H}_6$ in three parts.](image)

The remaining eigenvectors in a basis of eigenvectors for $\mathcal{H}_k$ may be chosen orthogonal to the vectors in $W$. They may be chosen to be orthogonal to the characteristic vectors of the parts of the $(2k - 1)$-part partition because the characteristic vectors are also a basis for $W$. Therefore, they will be some of the eigenvectors of the matrix $A(\hat{\mathcal{H}}_k)$ obtained from $A(\mathcal{H})$ by replacing each all-ones block in each diagonal block of $A(X_k)$ by an all-zeros matrix. But $A(\hat{\mathcal{H}}_k)$ is the adjacency matrix of a graph.
\( \hat{H}_k \) with \( k \) connected components, one of which is the graph \( \mathcal{H}' \) obtained by attaching \( k - 1 \) copies of \( M_{k-2} \) to an \( M_{k-2} \). Each of the remaining \( k - 1 \) components is a copy of \( K_3 \). It follows that the greatest eigenvalue of \( \hat{H}_k \) is that of a component of \( \mathcal{H}' \). Because \( \mathcal{H}' \) has a 2-part equitable partition with quotient matrix

\[
\begin{bmatrix}
1 & k - 1 \\
1 & k - 4 \\
\end{bmatrix},
\]

its greatest eigenvalue is \( \frac{1}{2} \left( k - 3 + \sqrt{k^2 - 6k + 21} \right) \), which is less than \( \frac{1}{2} \left( k - 2 + \sqrt{k^2 + 12} \right) = \gamma(k) \).

### 3.4 Sufficiency for a \( d \)-spanning tree

A \( d \)-tree is a tree with maximum degree at most \( d \), \( d \geq 2 \). Win [64] independently, and later Ellingham and Zha [30] provided a sufficient condition for a spanning \( d \)-tree in a graph.

**Theorem 3.4.1** ([30], [64]). Let \( c(G \setminus S) \) denote the number of components of \( G \setminus S \). Let \( s = |S| \), and assume \( d \geq 3 \). If \( G \) is a connected graph with \( c(G \setminus S) \leq \left( d - 2 \right)s + 2 \), \( \forall S \subseteq V(G) \), then \( G \) has a \( d \)-tree.

**Theorem 3.4.2.** Let \( G \) be a \( k \)-regular graph and \( d \geq 3 \) be an integer. If \( \lambda_4 < k - \frac{k}{(d-2)(k+1)} \), then \( G \) has a spanning \( d \)-tree.

**Proof.** Suppose \( G \) does not have a spanning \( d \)-tree. Then Theorem 3.4.1 implies there exists a subset of vertices \( S \) such that \( c = c(G \setminus S) \geq (d - 2)s + 3 \). Let \( t_i > 0 \) denote the number of edges between \( S \) and component \( i \), \( 1 \leq i \leq c \). Then

\[
ks \geq t_1 + t_2 + \ldots + t_c.
\]

We claim there must be at least 4 \( t_i \)'s such that \( t_i < \frac{k}{d-2} \). If there are at most 3 such \( t_i \)'s, then there are at least \( c - 3 \) \( t_i \)'s such that \( t_i \geq \frac{k}{d-2} \), say \( t_1, t_2, \ldots, t_{c-3} \). Then

\[
t_1 + \ldots + t_{c-3} \geq \frac{(c - 3)k}{d - 2} \geq \frac{(d - 2)sk}{d - 2} = ks.
\]
Since $t_i > 0$ for all $i$, $t_1 + ... + t_c > ks$, this is a contradiction.

Without loss of generality, let $H_1, H_2, H_3,$ and $H_4$ denote the 4 components of $G \setminus S$ with $t_i < \frac{k}{d-2}$, $1 \leq i \leq 4$. Let $n_i$ denote the order of $H_i$, $1 \leq i \leq 4$. $t_i < \frac{k}{d-2}$ implies $n_i \geq k + 1$. Then eigenvalue interlacing implies

$$
\lambda_4 \geq \lambda_4(H_1 \cup H_2 \cup H_3 \cup H_4) \geq \min_{1 \leq i \leq 4} (\lambda_1(H_i)) > k - \frac{k}{(d-2)n_i} \geq k - \frac{k}{(d-2)(k+1)}.
$$

□
4.1 Introduction

We recall the toughness \( t(G) \) of a connected graph \( G \) is the minimum of \( \frac{|S|}{c(G \backslash S)} \), where the minimum is taken over all vertex subsets \( S \) whose removal disconnects \( G \), and \( c(G \backslash S) \) denotes the number of components of the graph obtained by removing the vertices of \( S \) from \( G \).

Brouwer [11] stated his belief that \( t(G) = \frac{k}{\lambda} \) for many graphs \( G \), where \( \lambda = \max(|\lambda_2|, |\lambda_n|) \). Brouwer’s reasoning hinged on the fact that a connected \( k \)-regular graph \( G \) with \( n \) vertices attaining equality in the Hoffman ratio bound (meaning that the independence number \( \alpha(G) \) of \( G \) equals \( \frac{n(-\lambda_{\min})}{k-\lambda_{\min}} \); see also Section 1.3.4, or [12, Chapter 3] or [34, Chapter 9]) and having \( \lambda = -\lambda_{\min} \), is likely to have toughness equal to \( k/\lambda = k/(-\lambda_{\min}) \). Brouwer argued that for such a graph \( G \), \( k/\lambda \) is definitely an upper bound for the toughness, (as one can take \( S \) to be the complement of an independent set of maximum size \( \frac{n\lambda}{k-\lambda} \) and then \( t(G) \leq \frac{|S|}{c(G \backslash S)} = \frac{k}{\lambda} \) and Brouwer suggested that for many such graphs \( k/\lambda \) is the exact value of \( t(G) \).

In this chapter, we determine the exact value of the toughness of several families of strongly regular graphs attaining equality in the Hoffman ratio bound, including the Lattice graphs, the Triangular graphs, the complements of the Triangular graphs, and the complements of the point-graphs of generalized quadrangles. Moreover, for each graph \( G \) above, we determine the disconnecting sets of vertices \( S \) such that \( \frac{|S|}{c(G \backslash S)} \) equals the toughness of \( G \). We show that for all these graphs except the Petersen
graph, the toughness equals \( k/(-\lambda_{\text{min}}) \), where \( k \) is the degree of regularity and \( \lambda_{\text{min}} \) is the smallest eigenvalue of the adjacency matrix.

We remark here that the toughness of a claw-free strongly regular graph can be deduced by combining a theorem of Matthews and Sumner [52] with a theorem of Brouwer and Mesner [13]. The theorem of Matthews and Sumner states that the toughness of a connected non-complete graph not containing any induced \( K_{1,3} \) (claw) equals half of its vertex-connectivity. The theorem of Brouwer and Mesner states that the vertex-connectivity of a connected strongly regular graph equals its degree. However, this does not determine the disconnecting sets of vertices \( S \) for when the minimum of \( \frac{|S|}{c(G \setminus S)} \) is attained.

4.2 The toughness of the Lattice graphs

In this section, we determine the exact value of the toughness of the Lattice graph \( L_2(v) \) for \( v \geq 2 \). The Lattice graph \( L_2(v) \) is the line graph of the complete bipartite graph \( K_{v,v} \) and is a strongly regular graph with parameters \((v^2, 2v-2, v-2, 2)\). The spectrum of the adjacency matrix of \( L_2(v) \) is \((2v-2)^1, (v-2)(2v-2), (-2)(v^2-2v+1)\), where the exponents denote the multiplicities of the eigenvalues. The main result of this section is the following theorem.

**Theorem 4.2.1.** For \( v \geq 2 \), \( t(L_2(v)) = v-1 \). Moreover, if \( S \) is a subset of vertices of \( L_2(v) \) such that \( \frac{|S|}{c(L_2(v) \setminus S)} = v-1 \), then \( S \) is the complement of a maximum independent set of size \( v \), or \( S \) is the neighborhood of some vertex of \( L_2(v) \).

**Proof.** The Lattice graph \( L_2(v) \) has independence number \( \alpha(L_2(v)) = v \) and attains equality in the Hoffman ratio bound as \( \frac{n(-\lambda_{\text{min}})}{k-\lambda_{\text{min}}} = \frac{v^2}{2v-2+2} = v \). By our discussion in the introduction, this implies \( t(L_2(v)) \leq k/(-\lambda_{\text{min}}) = v-1 \).

Let \( S \subset V(L_2(v)) \) be a disconnecting set of vertices and let \( c = c(G \setminus S) \). We will show that \( \frac{|S|}{c(L_2(v) \setminus S)} \geq v-1 \) with equality if and only if \( S \) is the complement of a maximum independent set of size \( v \) or \( S \) is the neighborhood of some vertex of \( L_2(v) \).

Consider the vertices of \( L_2(v) \) arranged in a \( v \times v \) grid, with two vertices adjacent if and only if they are in the same row or the same column. We may assume that the
components of $L_2(v) \setminus S$ are pairwise row disjoint and column disjoint $h_i \times w_i$ rectangular blocks for $1 \leq i \leq r$. Indeed, if some component $C$ with maximum height $h$ and width $w$ was not a rectangular block, we can redefine this component as the rectangular block of height $h$ and width $w$ that contains $C$. This will not affect any other components in $L_2(v) \setminus S$ and will decrease $|S|$, and thus will decrease $\frac{|S|}{c(L_2(v) \setminus S)}$. Obviously, any two vertices from distinct components must be in different rows and different columns. Therefore $2 \leq c \leq v$.

We may also assume that $\sum_{i=1}^{c} h_i = \sum_{i=1}^{c} w_i = v$. Otherwise, if $S$ contains an entire row (column) of vertices, we can simply add $w_i$ (or $h_i$) vertices from that row (column) to one of the components $h_i \times w_i$ of $L_2(v) \setminus S$. This will decrease $|S|$ and has no effect on $c(L_2(v) \setminus S)$, and therefore it will decrease $\frac{|S|}{c(L_2(v) \setminus S)}$. These facts imply that

$$|S| = v^2 - \sum_{i=1}^{c} h_i w_i \geq v^2 - \sum_{i=1}^{c} h_i^2 + \frac{\sum_{i=1}^{c} w_i^2}{2} \geq v^2 - (c - 1) - (v - c + 1)^2$$

$$= (c - 1)(2v - c) = c(v - 1) + (c - 2)(v - c)$$

$$\geq c(v - 1),$$

where the last inequality is true since $2 \leq c \leq v$. This proves $\frac{|S|}{c(L_2(v) \setminus S)} \geq v - 1$. Equality happens above if and only if (after an eventual renumbering of the components) $h_i = w_i = 1$ for each $1 \leq i \leq c - 1$, $h_c = w_c = v - c + 1$ and $c = 2$ or $c = v$. This means $S$ is the neighborhood of some vertex of $L_2(v)$ (when $c = 2$) or $S$ is the complement of an independent set of size $v$ (when $c = v$).

**4.3 The toughness of some graphs defined by orthogonal arrays**

In this section we obtain sufficient conditions to determine the exact value of the toughness of some graphs corresponding to orthogonal arrays. An *orthogonal array* $OA(m, v)$ is an $m \times v^2$ array with entries from a set of size $v$ such that the $v^2$ ordered pairs defined by any two rows of the array are distinct. We define a graph corresponding to an $OA(m, v)$ as follows: let the $v^2$ columns be the vertex set, and two columns adjacent if and only if they have exact one coordinate in common. An $OA(m, v)$ is a
strongly regular graph with parameters \((v^2, (v - 1)m, v - 2 + (m - 1)(m - 2), m(m - 1))\) ([34], Theorem 10.4.2). The spectrum is \((m(v - 1))^{(1)}, (v - m)^{m(v - 1)}, (-m)^{(v - m + 1)(v - 1)}\).

In particular, the graph corresponding to \(OA(2, v)\) is the Lattice graph \(L_2(v)\). We provide a sufficient condition that determines the toughness of graphs defined by an \(OA(m, v)\) for \(m > 2\).

**Theorem 4.3.1.** Let \(G\) be the graph defined by an \(OA(m, v)\). If \(\alpha(G) = v\), then \(t(G) = v - 1\). Moreover, if \(S\) is a subset of vertices of \(G\) such that \(\frac{|S|}{c(G \setminus S)} = v - 1\), then \(S\) is the complement of a maximum independent set of size \(v\).

**Proof.** If \(\alpha(G) = v\), then \(G\) attains equality in the Hoffman ratio bound as \(\frac{n(-\lambda_{\min})}{k - \lambda_{\min}} = \frac{v^2m}{mv - m + m} = v\). By our discussion in the introduction, this implies \(t(G) \leq \frac{k}{(-\lambda_{\min})} = v - 1\). Since \(G\) and the Lattice graph \(L_2(v)\) have the same vertex set, and the edge set of \(L_2(v)\) is a subset of \(G\), we must have \(t(G) \geq v - 1\). The disconnecting sets \(S\) such that \(\frac{|S|}{c(G \setminus S)} = v - 1\) can only possibly be those of \(L_2(v)\). For \(m > 2\), we find that this only occurs when \(S\) is the complement of a maximum independent set of size \(v\). \(\square\)

### 4.3.1 Graphs corresponding to Latin squares

An \(OA(3, v)\) is equivalent to a Latin square: the \(v^2\) ordered pairs in the first two rows represent the row and columns in a \(v \times v\) array, and the third row is the symbol in that entry (of course one may choose any two rows for the coordinates and the final row to represent the symbols in the entries to form different Latin squares). By the definition of an orthogonal array, a Latin square has each of the \(v\) symbols represented once in every row and column.

The graph corresponding to an \(OA(3, v)\) is strongly regular with parameters \((v^2, 3(v - 1), v, 6)\). The spectrum is \((3v - 3)^1, (v - 3)^{1\frac{1}{2}(v^2 + v - 2)}, (-3)^{\frac{1}{2}(v^2 - v)}\). A transversal of order \(v\) in a Latin square is a set of \(v\) cells, one from each row and column, such that no two cells contain the same symbol. This is equivalent to an independent set of size \(v\) in graph corresponding to an \(OA(3, v)\).
Theorem 4.3.2. Consider a Latin square that represents a multiplication table of a group. Let $G$ be the graph corresponding to this $OA(3, v)$. If $G$ has an independent set of size $v$, then $t(G) = v - 1$. Moreover, $S$ is a disconnecting set of vertices of $G$ such that $\frac{|S|}{\alpha(G \setminus S)} = v - 1$ if and only if $S$ is the complement of a maximum independent set of size $v$. In particular, if $G$ corresponds to a multiplication table of a group of odd order, $t(G) = v - 1$.

Proof. By Theorem 4.3.1, it suffices to show any Latin square corresponding to a multiplication table of a group $H$ of odd order $v$ has a transversal of order $v$. Let $g \in H$, and assume it has order $t = 2i - 1$. If $h = g^i$, then $h^2 = g$. Since $g$ was arbitrary, we see that every element of $H$ has a unique square root. Namely, the main diagonal in the Latin square corresponds to a transversal of order $v$. Taking $S$ to be the complement of this maximum of independent set, we find $|S| = c(G \setminus S) = v - 1$. See Section 6.2 for problems and results on other Latin squares.

4.4 The toughness of the Triangular graphs

In this section, we determine the exact value of the toughness of the Triangular graph $T_v$ for $v \geq 4$. The graph $T_v$ is the line graph of $K_v$; it is a strongly regular graph with parameters $((v^2), 2v - 4, v - 2, 4)$. The spectrum of $T_v$ is $(2v - 4)^{(1)}, (v - 4)^{(v-1)}, (-2)^{(v^2-3v)/2}$. The following theorem is the main result of this section.

Theorem 4.4.1. For $v \geq 4$, $t(T_v) = v - 2$. Moreover, $S$ is a disconnecting set of vertices of $T_v$ such that $\frac{|S|}{\alpha(T_v \setminus S)} = v - 2$ if and only if $S$ is the neighborhood of a vertex of $T_v$ or $S$ is the complement of a maximum independent set of size $\frac{v^2}{2}$ (when $v$ is even).

Proof. The Triangular graph $T_v$ has independence number $\alpha(T_v) = \lfloor v/2 \rfloor$ and attains equality in the Hoffman ratio bound when $v$ is even as $\frac{n(-\lambda_{\min})}{k-\lambda_{\min}} = \frac{\binom{v}{2}}{2v-4+2} = v/2$. By our discussion in the introduction, this implies $t(T_v) \leq k/(-\lambda_{\min}) = v - 2$ when $v$ is even. If $S$ is the neighborhood of some vertex of $T_v$, then $T_v \setminus S$ consists of two components: one isolated vertex and a graph isomorphic to $T_{v-2}$. This shows that $t(T_v) \leq (2v - 4)/2 = v - 2$ regardless of the parity of $v$. 

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Let $S \subset V(T_v)$ be a disconnecting set of vertices and let $c = c(T_v \setminus S)$ and $s = |S|$. We will show that $s/c \geq v - 2$ with equality if and only if $S$ is the complement of a maximum independent set of size $v$ (when $v$ is even) or $S$ is the neighborhood of some vertex of $T_v$. The vertices of the graph $T_v$ are the 2-subsets of $[v] = \{1, \ldots, v\}$, and two such subsets are adjacent if and only if they share exactly one element of $[v]$.

Thus, $2 \leq r \leq \lfloor \frac{v}{2} \rfloor$. For any component $C$ in $T_v \setminus S$ whose vertices are 2-subsets of some subset $A \subset [v]$ (where $|A|$ is minimum with this property), we can assume that $C$ contains all the $\binom{|A|}{2}$ 2-subsets of $A$. This is because including in $C$ all the 2-subsets of $|A|$ decreases $|S|$ and does not change $c(T_v \setminus S)$, therefore decreasing $\frac{|S|}{c(T_v \setminus S)}$.

Thus, without loss of generality we may assume that there exist integers $0 = a_0 < a_1 < a_2 < \cdots < a_{c-1} < a_c = v$ such that $a_j - a_{j-1} \geq 2$, and the vertex set of the $j$-th component of $T_v \setminus S$ is formed by all the 2-subsets of $\{a_j - 1, a_j + 2, \ldots, a_j\}$ for $1 \leq j \leq r$. By letting $n_j = a_j - a_{j-1} \geq 2$, we see $\sum_{j=1}^{c} n_j = v$. Thus,

$$|S| = \binom{v}{2} - \sum_{j=1}^{c} \binom{n_j}{2} \geq \binom{v}{2} - \left[ \binom{v-2(c-1)}{2} + c - 1 \right]$$

$$= (c-1)(2v - 2c) = c(v - 2) + (c - 2)(v - 2c)$$

$$\geq c(v - 2)$$

where the last inequality is true since $2 \leq c \leq \lfloor \frac{v}{2} \rfloor$. The first equality occurs when the components of $T_v \setminus S$ are $c - 1$ isolated vertices and a component consists of all the 2-subsets of a subset of size $v - 2(c - 1)$ of $[v]$. The final equality occurs when $c = 2$ or $c = \frac{v}{2}$. If $c = 2$, equality occurs when $S$ is the neighborhood of a vertex. If $c = \frac{v}{2}$, equality occurs when $v$ is even and all the components of $T_v \setminus S$ are isolated vertices, meaning $S$ is the complement of a maximum independent set.

4.5 The toughness of the complements of the Triangular graphs

The complement $\overline{T_v}$ of the Triangular graph $T_v$ has the 2-subsets of a set $[v]$ with $v$ elements as its vertices, with two subsets being adjacent if and only if they are disjoint. It is a strongly regular graph with parameters $\left( \binom{v}{2}, \binom{v-2}{2}, \binom{v-4}{2}, \binom{v-3}{2} \right)$. 

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The spectrum of $\overline{T_v}$ is $(v-2)^{(v-2)} - 1 , (v^2 - 3v)/2 , (3 - v)^(v-1)$ (see Section 1.5 or Theorem 5.2.5). When $v = 5$, $\overline{T_5}$ is the Petersen graph. If $S$ is a maximum independent set, say $S = \{1, 2, \{1, 3\}, \{1, 4\}, \{1, 5\}\}$, then $\frac{|S|}{c(\overline{T_5} \setminus S)} = \frac{4}{3}$ which shows $t(\overline{T_5}) = \frac{4}{3}$. By a simple case analysis, it can be shown that actually $t(\overline{T_5}) = \frac{4}{3} < \frac{3}{2} = \frac{k}{-\lambda_{\text{min}}}$.

**Theorem 4.5.1.** For $v \geq 6$, $t(\overline{T_v}) = \frac{v}{2} - 1 = k/(-\lambda_{\text{min}})$. Furthermore, if $S$ is a disconnecting set of vertices in $\overline{T_v}$ such that $\frac{|S|}{c(\overline{T_v} \setminus S)} = \frac{v}{2} - 1$, then $S$ is the complement of an independent set of maximum size in $\overline{T_v}$.

**Proof.** The graph $\overline{T_v}$ has independence number $\alpha(\overline{T_v}) = v - 1$, and the independent sets of maximum size are the 2-subsets containing a fixed element of $[v]$. The graph $\overline{T_v}$ attains equality in the Hoffman ratio bound as $\frac{n(-\lambda_{\text{min}})}{k-\lambda_{\text{min}}} = \frac{(\binom{v}{2})(v-3)}{(\binom{v}{2})^2+(v-3)} = v - 1$. By our discussion in the introduction, this implies $t(\overline{T_v}) \leq \frac{k}{(-\lambda_{\text{min}})} = \frac{v}{2} - 1$.

Let $S \subset V(\overline{T_v})$ be a disconnecting set of vertices and let $c = c(\overline{T_v} \setminus S)$ and $s = |S|$. We will show that $\frac{s}{c} \geq \frac{v}{2} - 1$ with equality if and only if $S$ is the complement of a maximum independent set of size $v - 1$. As $S$ is a disconnecting set of vertices, we get that $|S| \geq \binom{v-2}{2}$ with equality if and only if $S$ is the neighborhood of some vertex of $\overline{T_v}$ (see [13]). If $|S| = \binom{v-2}{2}$, then $\overline{T_v} \setminus S$ has two components: one isolated vertex and a subgraph of $\overline{T_v}$ isomorphic to the complement of a perfect matching in a complete bipartite graph on $2v - 4$ vertices. In this case, we get that $\frac{|S|}{c(\overline{T_v} \setminus S)} = \frac{(\binom{v}{2})}{2} = \frac{(v-2)(v-3)}{4} > \frac{v}{2} - 1$ as $v \geq 6$.

If $|S| > \binom{v-2}{2}$, then $\frac{|S|}{c(\overline{T_v} \setminus S)} \geq \frac{(\binom{v}{2})+1}{c} = \frac{v^2-5v+8}{2c}$. If $c \leq v - 3$, then $\frac{v^2-5v+8}{2c} > \frac{v}{2} - 1$, and we are done in this case. Also, if $c = v - 1$, then the set formed by picking a single vertex from each component of $\overline{T_v} \setminus S$ is an independent set of maximum size $v - 1$, and therefore it consists of all 2-subsets containing some fixed element of $[v]$. This implies that $\overline{T_v} \setminus S$ is actually an independent set of size $v - 1$ and we obtain equality $\frac{|S|}{c(\overline{T_v} \setminus S)} = \frac{(\binom{v}{2})-(v-1)}{v-1} = \frac{v}{2} - 1$ in this case. If $c = v - 2 > 3$, then the set formed by picking a single vertex from each component of $\overline{T_v} \setminus S$ is an independent set of size $v - 2 > 3$. This independent set consists of $v - 2$ 2-subsets containing some fixed element of $[v]$. 

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This implies that the components of $T_v \setminus S$ must be $v - 2$ isolated vertices and therefore,
\[
\frac{|S|}{c(T_v \setminus S)} = \frac{(v - 2)}{v - 2} > \frac{v}{2} - 1.
\]

4.6 The toughness of the complements of the point-graphs of generalized quadrangles

A generalized quadrangle is a point-line incidence structure such that any two points are on at most one line and if $p$ is a point not on a line $L$, then there is a unique point on $L$ that is collinear with $p$. If every line contains $s + 1$ points and every point is contained in $t + 1$ lines, then we say the generalized quadrangle has order $(s, t)$. The point-graph of a generalized quadrangle is the graph with the points of the quadrangle as its vertices, with two points adjacent if and only if they are collinear. The point-graph of a generalized quadrangle of order $(s, t)$ is a strongly regular graph with parameters $((s + 1)(st + 1), s(t + 1), s - 1, t + 1)$ (see [34, Section 10.8]). The complement of the point-graph of a generalized quadrangle of order $(s, t)$ is a strongly regular graph with parameters $((s + 1)(st + 1), s^2t, t(s^2 + 1) - s(t + 1), st(s - 1))$. Its distinct eigenvalues are $s^2t, t$, and $-s$ with respective multiplicities $1, \frac{st(s+1)(t+1)}{s+t}, \frac{s^2(st+1)}{s+t}$. Brouwer [9] observed that the toughness of the complement of the point-graph of a generalized quadrangle of order $(q, q)$ is between $q^2 - 2$ and $q^2$.

The following theorem improves Brouwer’s result and is the main result of this section.

**Theorem 4.6.1.** If $G$ is the complement of the point-graph of a generalized quadrangle of order $(s, t)$, then $t(G) = st$. Moreover, $S$ is a disconnecting set of $G$ such that $\frac{|S|}{c(G \setminus S)} = st$ if and only if $S$ is the complement of a maximum independent set or $S$ is the neighborhood of a vertex (when $s = 2$).

**Proof.** Since a generalized quadrangle is triangle-free, the points of an independent set of size at least 3 in $G$ must be on a line in $GQ(s, t)$. Hence, $\alpha(G) = s + 1$, which attains equality in the Hoffman ratio bound: $\frac{n(-\lambda_{\min})}{k-\lambda_{\min}} = \frac{(s+1)(st+1)s}{s^2t+s} = s + 1$. Taking $S$ to be the complement of a maximum independent set, we conclude that
\[
t(G) \leq \frac{(s + 1)(st + 1) - (s + 1)}{s + 1} = st.
\]
Let $S$ be a disconnecting set of vertices of $G$. We claim that if $G \setminus S$ contains at least three components, then the components of $G \setminus S$ must be isolated vertices. Otherwise, $G \setminus S$ would contain an induced subgraph with two isolated vertices, $v_1$ and $v_2$, and an edge $v_3v_4$. This would imply that $v_1$ and $v_2$ are on two different lines in $GQ(s,t)$, a contradiction.

Therefore, if $c(G \setminus S) \geq 3$, then $G \setminus S$ consists of at most $s + 1$ isolated vertices. Thus, 
\[ \frac{|S|}{c(G \setminus S)} = \frac{(s+1)(st+1)-c(G \setminus S)}{c(G \setminus S)} \geq st, \]
with equality if and only if $S$ is the complement of an independent set of maximum size in $G$. The only remaining case is when $G \setminus S$ has exactly two components. Brouwer and Mesner [13] proved that the vertex-connectivity of any connected strongly regular graph equals its degree and the only disconnecting sets are neighborhoods of vertices. This implies that $|S| \geq s^2t$ when $c(G \setminus S) = 2$. Thus, 
\[ \frac{|S|}{c(G \setminus S)} \geq \frac{s^2t}{2} \geq st \]
with equality if and only $S$ is the neighborhood of a vertex and $s = 2$. This finishes our proof. \hfill \Box

We recall here the fact (see [34, Section 10.8]) that $s = 2$ implies $t \in \{1, 2, 4\}$. The complement of the point-graph of $GQ(2,1)$ is the Lattice graph $L_2(3)$. The complement of the point-graph of $GQ(2,2)$ is the triangular graph $T_6$. The complement of the point-graph of $GQ(2,4)$ is the Schl"afli graph [10].
Chapter 5
THE TOUGHNESS OF THE KNESER GRAPH

5.1 Introduction

The vertices of the Kneser graph $K(v, r)$ are the $r$-subsets of $[v]$, with two distinct $r$-subsets being adjacent if and only if they are disjoint. We assume $v \geq 2r + 1$ so that the graph is connected. The graph $K(v, r)$ is $\binom{v-r}{r}$-regular. Brouwer [11] believe that $t(G) = \frac{k}{\lambda}$ for many graphs $G$. In this chapter, we determine the toughness of $K(v, r)$ for any fixed $r$ and sufficiently large $v$, and $r = 3, v \geq 7$. For these cases, the toughness is $\frac{k}{\lambda}$, where $\lambda = \max(|\lambda_2|, |\lambda_n|)$. The toughness of $K(v, 2)$ was determined in Section 4.5. We note that for $r > 2$, $K(v, r)$ is not strongly regular.

A family of sets $\mathcal{F}$ is said to be intersecting if for any $A, B \in \mathcal{F}$, $A \cap B \neq \emptyset$. Let $\binom{X}{r}$ denote the family of all $r$-size subsets of a set $X$. In particular, the family of sets corresponding to an independent set in $K(v, r)$ is intersecting.

Two classical results related to intersecting families are the Erdős-Ko-Rado Theorem and the Hilton-Milner Theorem.

**Theorem 5.1.1** (Erdős-Ko-Rado, [32, 63]). Let $|X| = v \geq 2r, r \geq 1$. If $\mathcal{F} \subset \binom{X}{r}$ is intersecting, then

$$|\mathcal{F}| \leq \binom{v-1}{r-1},$$

with equality if and only if all members of $\mathcal{F}$ share one element.

**Proof.** The proof of equality and a generalization of the theorem can be found in [63]. An independent set in $K(v, r)$ is equivalent to an intersecting family of $r$-subsets of
The Hoffman Ratio bound (see Theorem 1.3.4) and \( \lambda_n(K(v, r)) = -\binom{v-r-1}{r-1} \) (see Theorem 5.2.5 below) imply
\[
\alpha(K(v, r)) \leq \frac{\binom{v}{t}}{1 + \frac{\binom{v-r}{r-1}}{\binom{v-r-1}{r-1}}} = \binom{v-1}{r-1}.
\]

\[\square\]

**Theorem 5.1.2** (Hilton-Milner, [32]). Let \( F \subset \binom{X}{r} \) be an intersecting family of sets with \( |X| = v \geq 2r + 1, r \geq 2 \). Assume \( \cap_{F \in \mathcal{F}} F = \emptyset \). Then \( |F| \leq \binom{v-1}{r-1} - \binom{v-r-1}{r-1} + 1 \).

Furthermore, equality occurs only when \( A \) is an \( r \)-subset and
\[
\mathcal{F} = \{A\} \cup \{B : x \in B, B \cap A \neq \emptyset\},
\]
for any fixed \( x \notin A \), or \( r = 3 \) and
\[
\mathcal{G} = \{B : |A \cap B| \geq 2\}.
\]

### 5.2 Eigenvalues of \( K(v, r) \)

In this section, we describe the eigenvalues of the adjacency matrix of \( K(v, r) \) and their multiplicities. Our presentation follows that of [62]. An alternative proof without determining multiplicities can be found in [34, Section 9.4].

For \( t \geq 0, v \geq r \), let \( W_{tr} \) and \( \overline{W}_{tr} \) denote the \( \binom{v}{t} \times \binom{v}{r} \) matrices, with rows and columns corresponding to the \( t \)-subsets and \( r \)-subsets of \([v]\), respectively. Let
\[
W_{tr}(T, R) = \begin{cases} 1 & \text{if } T \subseteq R, \\ 0 & \text{otherwise}; \end{cases} \quad \text{and} \quad \overline{W}_{tr}(T, R) = \begin{cases} 1 & \text{if } T \cap R = \emptyset, \\ 0 & \text{otherwise}. \end{cases}
\]

In particular, \( W_{rr} = I \) and \( \overline{W}_{rr} \) is the adjacency matrix of \( K(v, r) \).

**Lemma 5.2.1.** The following relations hold:

\[
W_{ij}W_{jt} = \binom{t-i}{j-i}W_{it}; \quad (5.1)
\]
\[
W_{ij}\overline{W}_{jt} = \binom{v-t-i}{j-i}\overline{W}_{it}; \quad (5.2)
\]
\[ W_{tr} = \sum_{i=0}^{t} (-1)^i W_{it}^T W_{ir}; \quad (5.3) \]
\[ W_{tr} = \sum_{i=0}^{t} (-1)^i W_{it}^T W_{ir}. \quad (5.4) \]

**Proof.** Equation (5.1) counts in two ways, the number of subsets \( J \) contained in a subset \( T \) that contain a subset \( I \). Equation (5.2) counts in two ways, the number of subsets \( J \) that contain a subset \( I \) such that \( J \) is disjoint from a subset \( T \). Let \( R \) and \( T \) be an \( r \)-subset and \( t \)-subset, respectively. To prove Equation (5.3), we calculate the \((T, R)\)-entry of both sides of Equation (5.3). We have
\[
\left( \sum_{i=0}^{t} (-1)^i W_{it}^T W_{ir} \right)_{(T,R)} = \sum_{i=0}^{t} (-1)^i \left( \binom{|T \setminus T \cap R|}{i} \right) = \begin{cases} 
1 & \text{if } T \setminus (T \cap R) = \emptyset \\
0 & \text{if } T \setminus (T \cap R) \neq \emptyset 
\end{cases},
\]
which is the \((T, R)\)-entry of the left-hand side of (5.3). Similarly, we deduce Equation (5.4) by calculating the \((T, R)\)-entry of both sides:
\[
\left( \sum_{i=0}^{t} (-1)^i W_{it}^T W_{ir} \right)_{(T,R)} = \sum_{i=0}^{t} (-1)^i \left( \binom{|T \cap R|}{i} \right) = \begin{cases} 
1 & \text{if } T \cap R = \emptyset \\
0 & \text{if } T \cap R \neq \emptyset 
\end{cases},
\]
which is the \((T, R)\)-entry of the left-hand side of (5.4). \( \Box \)

Let \( V_i \) and \( V'_i \) denote the row space of \( W_{ir} \) and \( W'_{ir} \), respectively, for \( i = 0, 1, \ldots, r \).

**Lemma 5.2.2.** For \( t \leq r \leq v - t \), \( V_t \) has dimension \( \binom{r}{t} \).

**Proof.** We justify \( W_{tr} \) has full rank by showing \( W_{tr} \) has an inverse. We have
\[
W_{tr} \left( \sum_{i=0}^{t} \frac{(-1)^i}{\binom{v-t-i}{r-t-i}} W_{ri} W_{it} \right) = \sum_{i=0}^{t} \frac{(-1)^i}{\binom{v-t-i}{r-t-i}} W_{tr} W_{ri} W_{it}
= \sum_{i=0}^{t} (-1)^i W_{ti} W_{it},
\]
by (5.2). Applying the transpose and (5.3),
\[
\sum_{i=0}^{t} (-1)^i W_{it}^T W_{ti} = \sum_{i=0}^{t} (-1)^i W_{it}^T W_{ir} = W_{tt} = I.
\]  
\( \Box \)
Proposition 5.2.3. For fixed $v, r,$ and $t$ with $t \leq r$ and $t + r \leq v$, $V_t$ is also the row space of $W_{tr}$, and $V_0 \subseteq V_1 \subseteq V_2 \subseteq ... \subseteq V_r$.

Proof. Since the row space of $AB$ (for two matrices of appropriate dimensions) is contained in the row space of $B$, Equation (5.1) implies that $V_i$, the row space of $W_{ir}$, is contained in the row space of $V_j$, which is the row space of $W_{jr}$. Hence $V_i \subseteq V_j$ for $0 \leq i \leq j \leq r$. Similarly, Equation (5.2) implies $V'_i \subseteq V'_j$.

In Equation (5.3), the row space of $\sum_{i=0}^{t}(-1)^iW_{it}W_{ir}$ is contained in the row space of $\sum_{i=0}^{t}W_{ir}$. We have $\sum_{i=0}^{t}W_{ir} = \sum_{i=0}^{t}V'_i = V'_t$ since $V'_i \subseteq V'_j$ for $0 \leq i \leq j \leq t$. Thus $V_i \subseteq V'_t$. In Equation (5.4), the row space of $\sum_{i=0}^{t}(-1)^iW_{it}W_{ir}$ is contained in the row space of $\sum_{i=0}^{t}W_{ir}$, which equals $\sum_{i=0}^{t}V_i = V_t$ since $V_i \subseteq V_j$ for $0 \leq i \leq j \leq t$. This yields the reverse inclusion, and so $V_t = V'_t$.

Let $U_0 = V_0$ and $U_i = V_{i-1}^\perp$ for $i \geq 1$, where $V_{i-1}^\perp$ denotes the orthogonal complement of $V_{i-1}$ in $V_i$. Since $V_i = U_i \oplus V_{i-1}$ for $i \geq 1$,

$$V_r = U_r \oplus V_{r-1} = U_r \oplus U_{r-1} \oplus V_{r-2} = ... = U_r \oplus U_{r-1} \oplus ... \oplus U_0.$$

Proposition 5.2.4. If $u \in U_j$, then $u = aW_{jr}$ for some vector $a$ such that $aW_{ij}^T = a\overline{W}_{ji} = 0$, for $0 \leq i \leq j - 1$.

Proof. Let $u \in U_j$. Because $V_j = U_j \oplus V_{j-1}$, we have $u \in V_j$. By definition, $V_j$ is the row space of $W_{jr}$, and so $u = aW_{jr}$ for some $a$. Since

$$V_0 \subseteq V_1 \subseteq ... \subseteq V_j$$

is equivalent to

$$V_j^\perp \subseteq V_{j-1}^\perp \subseteq ... \subseteq V_0^\perp,$$

and $u \in U_j = V_{j-1}^\perp$, we get that $u \in V_{i}^\perp$ for $i < j$. Then $V_i = V'_i$ and (5.2) imply

$$0 = u\overline{W}_{ri} = aW_{jr}\overline{W}_{ri} = \left(\frac{v - j - i}{r - j}\right)a\overline{W}_{ji}.$$

Hence for $0 \leq i \leq j - 1$, $aW_{ij}^T = a\overline{W}_{ji} = 0$. 

**Theorem 5.2.5.** The eigenvalues of $K(v, r)$ are $(-1)^j \binom{v-r-j}{r-j}$ with multiplicities $\binom{v}{j} - \binom{v}{j-1}$, for $j = 0, 1, \ldots, r$.

**Proof.** Recall that the adjacency matrix of the Kneser graph is $W_{rr}$. We show that every vector in $U_j$ is an eigenvector of $W_{rr}$ with eigenvalues stated above. By (5.2) and (5.4),

$$W_{jr}W_{rr} = \left( v - r - j \right) \sum_{i=0}^{j} (-1)^i W_{ij}^T W_{ir}.$$  

We now apply $a$, defined in the previous proposition, to both sides. The left hand side yields

$$a W_{jr} W_{rr} = u W_{rr},$$

while the right hand side yields (using the fact that $W_{jj}^T = I$)

$$\left( v - r - j \right) \sum_{i=0}^{j} (-1)^i a W_{ij}^T W_{ir} = \left( v - r - j \right) \sum_{i=0}^{j-1} (-1)^i (0) W_{ir} + \left( v - r - j \right) (-1)^j a W_{jr}$$

$$= \left( v - r - j \right) (-1)^j u.$$  

This shows every vector in $U_j$ is an eigenvector of $W_{rr}$ with eigenvalue $(-1)^j \binom{v-r-j}{r-j}$. The eigenvalue $(-1)^j \binom{v-r-j}{r-j}$ has multiplicity equal to $\dim U_j = \dim \left( V_{j}^\perp \right) = \binom{v}{j} - \binom{v}{j-1}$, by Lemma 5.2.2. \hfill \square

### 5.3 Toughness of $K(v, r)$

Let $G$ be a vertex transitive graph of degree $k$ with order at least $k + 4$. Hamidoune, Lladó, and López [39] proved that for $v \in V(G)$, the graph induced on $V(G) \setminus \left( N(v) \cup \{v\} \right)$, with $N(u) \neq N(v)$ for all $u \neq v$, is connected if and only if $G$ has vertex connectivity $k$, and the disconnecting sets of vertices of order $k$ must be the neighborhood of a vertex. Tindell [59] proved that an edge transitive graph has the vertex connectivity equal to its minimum degree. This implies that a connected, $k$-regular edge-transitive graph has vertex connectivity $k$. Since Kneser graph $K(v, r)$ has these properties for $v \geq 2r + 1$ and $r \geq 2$, we have the following.
Lemma 5.3.1. For any vertex \( u \in V(K(v, r)) \), the graph from deleting \( u \) and its neighborhood is connected.

The following lemma will be used for Theorem 5.3.3 below, which can be verified with basic calculations.

Lemma 5.3.2. For \( r \geq 2 \),
\[
\frac{2^{r-1}(2r - 1)}{2^{r-1} - 1} < \frac{2}{\ln(2)} r^2 + \left(2 - \frac{3}{\ln(2)}\right) r + \frac{1}{\ln(2)}.
\]
(5.5)

For fixed \( r \) and sufficiently large \( v \), we determine the toughness of \( K(v, r) \).

Theorem 5.3.3. Let \( G = K(v, r) \), \( S \) be a subset of vertices, and \( c = c(G \setminus S) \).
If \( c \leq \frac{r}{v-r} \binom{v-r}{r} \), then \( \frac{|S|}{c} > \frac{k}{\lambda} = \frac{v-r}{r} - 1 \). If \( c > \frac{r}{v-r} \binom{v-r}{r} \) and \( v \geq \frac{2}{\ln(2)} r^2 + \left(2 - \frac{3}{\ln(2)}\right) r + \frac{1}{\ln(2)} \), then \( \frac{|S|}{c} \geq \frac{k}{\lambda} = \frac{v-r}{r} - 1 \), with equality if and only if \( S \) is the complement of a maximum independent set.

Proof. Assume first that \( c \leq \frac{r}{v-r} \binom{v-r}{r} \). If \( |S| = \binom{v-r}{r} \), then Lemma 5.3.1 implies
\[
\frac{|S|}{c(G \setminus S)} = \frac{\binom{v-r}{r}}{2} > \frac{v}{r} - 1
\]
for \( v \geq 2r + 1 \) and \( r \geq 3 \). Suppose \( |S| > \binom{v-r}{r} \). Then
\[
\frac{|S|}{c(G \setminus S)} > \frac{\binom{v-r}{r}}{c} \geq \frac{v}{r} - 1,
\]
and we are done.

Now assume \( c > \frac{r}{v-r} \binom{v-r}{r} \). If
\[
\frac{r}{v-r} \binom{v-r}{r} \geq \left(\frac{v-1}{r-1}\right) - \left(\frac{v-r-1}{r-1}\right) + 1,
\]
(5.6)
then we claim \( K(v, r) \setminus S \) is an independent set. Assume for the sake of contradiction that a component \( C \) of \( K(v, r) \setminus S \) has an edge \( v_1v_2 \). We can define two family of sets \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) as follows: Let \( \mathcal{F}_1 \) contain \( v_1 \) and one vertex from the remaining components. Let \( \mathcal{F}_2 = (\mathcal{F}_1 \cup \{v_2\}) \setminus \{v_1\} \). By Theorem 5.1.2, without loss of generality, we may assume \( \cap_{F \in \mathcal{F}_1} F = \{1\} \). Similarly we must have \( \cap_{F \in \mathcal{F}_2} F \neq \emptyset \). If \( \cap_{F \in \mathcal{F}_1} F = \cap_{F \in \mathcal{F}_2} F = \)}
{1}, then this contradicts the fact that \( v_1 \) is adjacent to \( v_2 \). Thus we may assume without loss of generality \( \cap_{F \in \mathcal{F}_2} F = \{2\} \). There are \( \binom{v-2}{r-2} \) sets containing both of these elements.

By assumption, there are more than

\[
\binom{v}{r-1} - \binom{v-r-1}{r-1} + 1 = \binom{v-2}{r-2} + \binom{v-2}{r-1} - \binom{v-r-1}{r-1} + 1
\]

components, and this value exceeds \( \binom{v-2}{r-2} \) for \( r > 1 \), a contradiction.

Hence \( K(v, r) \setminus S \) contains \( t \leq \binom{v-1}{r-1} \) isolated vertices. Then

\[
\frac{|S|}{c(G \setminus S)} = \frac{\binom{v}{r} - t}{t} \geq \frac{\binom{v}{r}}{\binom{v-1}{r-1}} - 1 = \frac{v}{r} - 1.
\]

Note that equality occurs if and only if \( S \) is the complement of a maximum independent set.

By assumption, \( v \geq \frac{2}{\ln(2)}r^2 + \left(2 - \frac{3}{\ln(2)}\right)r + \frac{1}{\ln(2)} \) and Lemma 5.3.2 imply \( v \geq \frac{2^{\frac{1}{r-1}}(2r)^{-1}}{2^{\frac{1}{r-1}} - 1} \). To complete the proof, we show \( v \geq \frac{2^{\frac{1}{r-1}}(2r)^{-1}}{2^{\frac{1}{r-1}} - 1} \) implies Equation (5.6).

The inequality \( v \geq \frac{2^{\frac{1}{r-1}}(2r)^{-1}}{2^{\frac{1}{r-1}} - 1} \) is equivalent to

\[
2(v-2r)^{r-1} > (v-1)^{r-1}.
\]

For \( r \geq 2 \),

\[
2\binom{v-r-1}{r-1} > \frac{2(v-2r)^{r-1}}{(r-1)!} \quad \text{and} \quad \frac{(v-1)^{r-1}}{(r-1)!} > \binom{v-1}{r-1}.
\]

Equation (5.7) now implies

\[
2\binom{v-r-1}{r-1} > \binom{v-1}{r-1}.
\]

This is equivalent to

\[
\frac{r}{v-r} \binom{v-r}{r} > \binom{v-1}{r-1} - \binom{v-r-1}{r-1} + 1,
\]

which is (5.6).

\[\square\]

By choosing \( S \) to be the complement of a maximum independent set, which has size \( \binom{v}{r} - \binom{v-1}{r-1} \), we see that \( t(G) \leq \frac{v}{r} - 1 \). Hence, Theorem 5.3.3 implies for fixed \( r \), if \( v \geq \frac{2^{\frac{1}{r-1}}(2r)^{-1}}{2^{\frac{1}{r-1}} - 1} \), then \( t(K(v, r)) = \frac{v}{r} - 1 \).
5.4 Toughness of $K(v, 3)$

Haemers [36] independently, and Helmberg, Mohar, Poljak, and Rendl [40] obtained a spectral lower bound on the size of a vertex cut $S$.

**Theorem 5.4.1** (Haemers [36]; Helmberg, Mohar, Poljak, Rendl [40]). Let $G$ be a connected $k$-regular graph on $n$ vertices, and $S, S_1, S_2 \subset V(G)$. Let $|S_1| = a$ and $|S_2| = b$. If $V(G) \setminus S = S_1 \cup S_2$ and there are no edges between $S_1$ and $S_2$, then

$$|S| \geq \frac{4(k - \lambda_2)(k - \lambda_n)ab}{n(\lambda_2 - \lambda_n)^2}.$$  

The distinct eigenvalues of $K(v, 3)$ are $(-1)^i v^{v-3-i}$, $i = 0, 1, 2, 3$. Theorem 5.4.1 then yields

$$|S| \geq \frac{8(v - 4)(v - 6)ab}{3(v - 2)^3}. \quad (5.8)$$

**Lemma 5.4.2.** Let $c \geq 2$, and $1 \leq n_1 \leq n_2 \leq \ldots \leq n_c$ be integers. If $\sum_{i=1}^{c} n_i \geq 2c$, then there exists a $j$, $1 \leq j \leq c$, such that

$$\min \left( \sum_{i=1}^{j} n_i, \sum_{i=j+1}^{c} n_i \right) \geq c - 1.$$

**Proof.** We use induction on $c$.

**Case 1:** $n_i \geq 2$, for $1 \leq i \leq c$.

Let $j = \lceil \frac{c}{2} \rceil$. Then

$$\sum_{i=1}^{\lceil \frac{c}{2} \rceil} n_i \geq 2 \left\lfloor \frac{c}{2} \right\rfloor > c - 1,$$

and

$$\sum_{i=\lceil \frac{c}{2} \rceil + 1}^{c} n_i \geq 2 \left\lfloor \frac{c}{2} \right\rfloor \geq c - 1.$$

**Case 2:** $n_1 = 1$.

If $c = 2$, then since $1 = n_1 \leq n_2$, the result is clear. Assume the statement holds for some $c \geq 2$. By assumption $\sum_{i=1}^{c+1} n_i \geq 2c + 2$, and so we can apply the induction hypothesis to the $c$ integers $n_2, n_3, \ldots, n_{c+1}$ (we may need to return to Case 1). Therefore there exists a $j$, $2 \leq j \leq c + 1$ such that

$$\min \left( \sum_{i=2}^{j} n_i, \sum_{i=j+1}^{c+1} n_i \right) \geq c - 1.$$
Since $\sum_{i=1}^{c+1} n_i \geq 2c + 2$, both sums cannot equal $c - 1$. If both sums are at least $c$, we are done. Otherwise, we can add $n_1$ to the sum that equals to $c - 1$, and we are done.

In particular, this lemma implies that if $G \setminus S$ is disconnected into $c$ components such that $G \setminus S$ has at least $2c$ vertices, then there exist subsets of vertices $S_1$ and $S_2$ in $G \setminus S$ with no edges between them such that $\min(|S_1|, |S_2|) \geq c - 1$.

**Lemma 5.4.3.** Let $G$ be a non-bipartite $k$-regular graph. Assume the edge-connectivity of $G$ is $k$, and the only disconnecting sets $S$ of $k$ edges are the $k$ edges incident to a vertex. Then $|S| > c(G \setminus S)$. In particular, $|S| > c(G \setminus S)$ holds when $G = K(v, r)$ for any $v \geq 2r + 1$.

**Proof.** Since the edge connectivity is equal to the degree $k$, for any disconnecting set of edges $S$, we must have $c(G \setminus S)k \leq e(S, G \setminus S)$. Equality occurs when the components are singletons. Furthermore, it is obvious that $e(S, G \setminus S) \leq k|S|$, which occurs if and only if $S$ is an independent set. Combining yields $\frac{|S|}{c(G \setminus S)} \geq 1$, with equality if and only if $G$ is bipartite. The last statement holds since the vertex connectivity of $K(v, r)$ is $k$ [29], and $\chi(K(v, r)) = v - 2r + 2 \geq 3$ [3, 50], where $\chi(K(v, r))$ denotes the chromatic number of $K(v, r)$. \qed

**Theorem 5.4.4.** For $v \geq 8$, $t(K(v, 3)) = \frac{k}{\chi} = \frac{v}{3} - 1$. Moreover, if $S$ is a subset of vertices of $K(v, 3)$ such that $\frac{|S|}{c(K(v, 3) \setminus S)} = \frac{v}{3} - 1$, then $S$ is the complement of a maximum independent set of $K(v, 3)$.

**Proof.** We show that $\frac{|S|}{c(K(v, 3) \setminus S)} > \frac{v}{3} - 1$ except when $S$ is the complement of a maximum independent set. We may assume $|S| \geq k = \binom{v-3}{3}$. If $|S| = \binom{v-3}{3}$, then Lemma 5.3.1 implies

$$\frac{|S|}{c(G \setminus S)} = \frac{\binom{v-3}{3}}{2} = \frac{(v-3)(v-4)(v-5)}{6} > \frac{v}{3} - 1.$$  

We now assume $|S| > \binom{v-3}{3}$. Let $c = c(G \setminus S)$. If $c \leq \frac{(v-4)(v-5)}{2}$, then

$$\frac{|S|}{c(G \setminus S)} > \frac{\binom{v-3}{3}}{c} = \frac{(v-3)(v-4)(v-5)}{6c} \geq \frac{v}{3} - 1.$$  

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If \( c > \frac{(v-4)(v-5)}{2} \), then for \( v \geq 12 \),
\[
c > \frac{(v-4)(v-5)}{2} \geq \binom{v-1}{2} - \binom{v-4}{2} + 1.
\]

Theorem 5.1.2 implies \( K(v,3) \setminus S \) must be an independent set of size \( t \leq \binom{v-1}{2} \).

Then
\[
\frac{|S|}{c(G \setminus S)} = \frac{(\frac{v}{3}) - t}{t} = \frac{v(v-1)(v-2)}{6t} - 1 \geq \frac{v(v-1)(v-2)}{6(v-1)^2} - 1 = \frac{v}{3} - 1, \tag{5.9}
\]
with equality if and only if \( t = \binom{v-1}{2} \). This happens if and only if \( S \) is the complement of a maximum independent set.

We now consider \( K(v,3) \) for \( 8 \leq v \leq 11 \). In all cases, we assume \( K(v,3) \) is not \( \binom{v}{3} - 1 \)-tough and obtain a contradiction.

If \( K(11,3) \) is not \( \frac{11}{3} - 1 = \frac{8}{3} \)-tough, then there exists a subset of vertices \( S \), \( |S| > \binom{11-3}{3} \), such that
\[
\frac{|S|}{c(K(11,3) \setminus S)} < \frac{8}{3},
\]
which implies \( c(K(11,3) \setminus S) > 21 \). In addition, if
\[
c(K(11,3) \setminus S) > \binom{11-1}{2} - \binom{11-4}{2} + 1 = 25,
\]
we may apply Theorem 5.1.2 and use a similar argument above that \( V(K(11,3) \setminus S) \) is an independent set. We are then done by (5.9). Assume \( 22 \leq c(K(11,3) \setminus S) \leq 25 \). Because \( S \) is not the neighborhood of a vertex, \( \frac{|S|}{c(K(11,3) \setminus S)} < \frac{8}{3} \) implies \( 57 \leq |S| \leq 66 \). Furthermore, we have \( |V(K(11,3) \setminus S)| = 165 - |S| \geq 165 - 66 = 99 > 2(25) \geq 2c(K(11,3) \setminus S) \), so we can apply Lemma 5.4.2. There exist subsets of vertices \( S_1 \) and \( S_2 \) of \( K(11,3) \setminus S \) with sizes \( a \) and \( b \), with no edges between them, such that \( \min(a,b) \geq 21 \). Equation 5.8 implies
\[
|S| \geq \frac{280}{2187} ab \geq \frac{280}{2187} (21)(99 - 21) > 82,
\]
which is a contradiction with \( |S| \leq 66 \).
The graph $K(10, 3)$ is 35-regular and has 120 vertices. If $K(10, 3)$ is not $\frac{7}{3}$-tough, then there exists a subset of vertices $S$ such that $16 \leq c(K(10, 3) \setminus S) \leq 22$. Otherwise we are done by Theorem 5.1.2. As $S$ is not the neighborhood of a vertex, $\frac{|S|}{c(K(10, 3) \setminus S)} < \frac{7}{3}$ implies $36 \leq |S| \leq 51$. Furthermore, we have $|V(K(10, 3) \setminus S)| = 120 - |S| \geq 120 - 51 = 69 > 2(22) \geq 2c(K(10, 3) \setminus S)$, so we can apply Lemma 5.4.2. There exist subsets of vertices $S_1$ and $S_2$ of $K(10, 3) \setminus S$ with sizes $a$ and $b$, with no edges between them, such that $\min(a, b) \geq 15$. Equation 5.8 implies

$$|S| \geq \frac{ab}{8} \geq \frac{1}{8}(15)(69 - 15) > 101,$$

a contradiction with $|S| \leq 51$.

The graph $K(9, 3)$ is 20-regular and has 84 vertices. If $K(9, 3)$ is not $\frac{6}{3} = 2$-tough, then there exists a subset of vertices $S$ such that $11 \leq c(K(9, 3) \setminus S) \leq 19$. Otherwise we are done by Theorem 5.1.2. As $S$ is not the neighborhood of a vertex, $\frac{|S|}{c(K(9, 3) \setminus S)} < 2$ implies $21 \leq |S| \leq 37$. Furthermore, we have $|V(K(9, 3) \setminus S)| = 84 - |S| \geq 84 - 37 = 47 > 2(19) \geq 2c(K(9, 3) \setminus S)$, so we can apply Lemma 5.4.2. There exist subsets of vertices $S_1$ and $S_2$ of $K(9, 3) \setminus S$ with sizes $a$ and $b$, with no edges between them, such that $\min(a, b) \geq 10$. Equation 5.8 implies

$$|S| \geq \frac{40}{343}ab \geq \frac{40}{343}(10)(47 - 10) > 43,$$

a contradiction with $|S| \leq 37$.

The graph $K(8, 3)$ is 10-regular and has 56 vertices. If $K(8, 3)$ is not $\frac{5}{3}$-tough, then there exists a subset of vertices $S$ such that $7 \leq c(K(9, 3) \setminus S) \leq 16$. Otherwise we are done by Theorem 5.1.2. As $S$ is not the neighborhood of a vertex, $\frac{|S|}{c(K(8, 3) \setminus S)} < \frac{5}{3}$ implies $11 \leq |S| \leq 26$.

If $c(K(8, 3) \setminus S) = 7$ and $K(8, 3)$ is not $\frac{5}{3}$-tough, then $|S| = 11$. Furthermore, we have $|V(K(8, 3) \setminus S)| = 56 - 11 = 45 > 2(7) = 2c(K(8, 3) \setminus S)$, so we can apply Lemma 5.4.2. There exist subsets of vertices $S_1$ and $S_2$ of $K(8, 3) \setminus S$ with sizes $a$ and $b$ respectively, with no edges between them so that $\min(a, b) \geq 6$. Equation 5.8 implies

$$|S| \geq \frac{8ab}{81} \geq \frac{8}{81}(6)(39) > 23,$$
Now consider $8 \leq c(K(8, 3) \setminus S) \leq 12$. If $G$ is not $\frac{5}{3}$-tough, then $11 \leq |S| \leq 19$. Furthermore, we have $|V(K(8, 3) \setminus S)| = 56 - |S| \geq 56 - 19 = 37 > 2(12) \geq 2c(K(8, 3) \setminus S)$, so we can apply Lemma 5.4.2. There exist subsets of vertices $S_1$ and $S_2$ of $K(8, 3) \setminus S$ with sizes $a$ and $b$ respectively, with no edges between them so that $\min(a, b) \geq 7$. Equation 5.8 implies

$$|S| \geq \frac{8ab}{81} \geq \frac{8}{81}(7)(37 - 7) > 20,$$

a contradiction with $|S| \leq 19$.

Now consider $13 \leq c(K(8, 3) \setminus S) \leq 14$. If $G$ is not $\frac{5}{3}$-tough, then $13 \leq |S| \leq 23$. Furthermore, we have $|V(K(8, 3) \setminus S)| = 56 - |S| \geq 56 - 23 = 33 > 2(14) \geq 2c(K(8, 3) \setminus S)$, so we can apply Lemma 5.4.2. There exist subsets of vertices $S_1$ and $S_2$ of $G \setminus S$ with sizes $a$ and $b$ respectively, with no edges between them so that $\min(a, b) \geq 12$. Equation 5.8 implies

$$|S| \geq \frac{8ab}{81} \geq \frac{8}{81}(12)(33 - 12) > 23,$$

a contradiction with $|S| \leq 23$.

If $c(K(8, 3) \setminus S) = 15$ and $K(8, 3)$ is not $\frac{5}{3}$-tough, then $16 \leq |S| \leq 24$. Furthermore, we have $|V(K(8, 3) \setminus S)| = 56 - |S| \geq 56 - 24 = 32 > 2(15) \geq 2c(K(8, 3) \setminus S)$, so we can apply Lemma 5.4.2. There exist subsets of vertices $S_1$ and $S_2$ of $G \setminus S$ with sizes $a$ and $b$ respectively, with no edges between them so that $\min(a, b) \geq 14$. Equation 5.8 implies

$$|S| \geq \frac{8ab}{81} \geq \frac{8}{81}(14)(32 - 14) > 24,$$

a contradiction with $|S| \leq 24$.

If $c(G \setminus S) = 16$ and $K(8, 3)$ is not $\frac{5}{3}$-tough, then $17 \leq |S| \leq 26$. If there are at least 2 components that are not isolated vertices, we can apply Lemma 5.4.5. So assume $G \setminus S$ contains 15 $K_1$’s and a component of size at least 15. Since $K(8, 3)$ has girth 4, this component contains a path of length 4 or a cycle of length 4 as an induced subgraph, say $v_1v_2v_3v_4$. We can obtain an independent set $F_1$ of size
17 > \binom{8-1}{3-1} - \binom{8-3-1}{3-1} + 1 = 16, by taking \( v_1, v_3, \) and one vertex from the other 15 components. By Theorem 5.1.2, without loss of generality, \( \cap_{F \in \mathcal{F}_1} F = \{1\} \). We can define a another family of sets \( \mathcal{F}_2 = (\mathcal{F}_1 \cup v_2 \cup v_4) \setminus (v_1 \cup v_3) \). Again by Theorem 5.1.2, \( \cap_{F \in \mathcal{F}_2} F \neq \emptyset \). Arguments similar to the proof of Theorem 5.3.3 imply a contradiction. 

**Lemma 5.4.5.** Suppose \( S \) is a disconnecting set of vertices such that \( K(v, 3) \setminus S \) contains two non-singleton components \( C_1 \) and \( C_2 \). Then there is no independent set \( I \) of size \( \binom{v-1}{2} - \binom{v-4}{2} + 1 \) such that \( |I \cap C_1| = |I \cap C_2| = 1 \).

**Proof.** Assume to the contrary this was possible and we treat an independent set as a family of intersecting sets. First we consider when the family of sets corresponding to the independent set all share an element, say '1'. We can construct a new family by switching the two vertices in the components of size at least two to an adjacent vertex. Now we have an intersecting family of \( \binom{v-1}{2} - \binom{v-4}{2} + 1 \) sets such that \( \binom{v-1}{2} - \binom{v-4}{2} - 1 \) share '1', and the other 2 sets do not share this. Since the intersection of all these sets is empty, the family should satisfy one of the two cases in Theorem 5.1.2. But clearly this will not happen, a contradiction.

Assume from now on that the family of sets corresponding to an independent set do not all share an element. By Theorem 5.1.2, these \( \binom{v-1}{2} - \binom{v-4}{2} + 1 \) sets can only be of two forms. Without loss of generality, first suppose that the family contains \( \{1, 2, 3\} \), and all other 3-subsets that contain two of these elements. We can construct a new family by switching one vertex in a component of size at least two to an adjacent vertex. Since there are two such components (necessarily to assure \( \{1, 2, 3\} \) is not in the chosen component), we assume without loss of generality that \( \{1, 2, 4\} \) is replaced with \( \{3, 5, 6\} \). But then this vertex is adjacent to \( \{1, 2, 7\} \), a contradiction i.e. two vertices were chosen from this component.

In the second case, we can assume the family contains \( \{1, 2, 3\} \), and all other sets in the family contain '4' with elements of \( \{1, 2, 3\} \). Similar to the previous argument, we can construct a new family with \( \{1, 4, 5\} \) replaced with \( \{2, 3, 6\} \). Now we have an
intersecting family of \( \binom{v-1}{2} - \binom{v-4}{2} + 1 \) sets such that \( \binom{v-1}{2} - \binom{v-4}{2} - 1 \) of the sets share one element, and the other two sets do not share this. This does not satisfy either case of equality in Theorem 5.1.2, and so we have a contradiction.

Lemma 5.4.6. In \( K(7, 3) \), the neighborhood of two isolated vertices contains at least seven vertices and the neighborhood of three isolated vertices contains at least nine vertices. The neighborhood of four or five isolated vertices contains at least 10 vertices.

Proof. For two isolated vertices, \( A_1 \) and \( A_2 \), this can easily be checked with two cases: when the sets share one or two elements. The argument for three isolated vertices is similar. For the final statement, it suffices to show it is true with four vertices. In this case, the smallest size of their neighborhood is when the pairwise intersections of these four sets (vertices) are largest. This occurs when they all share 2 elements, say

\[ \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 2, 6\}. \]

It is easy to check by the Inclusion-Exclusion Principle that the neighborhood of these vertices contains 10 vertices. This implies the statement is also true for five vertices. \(\square\)

Lemma 5.4.7. In \( K(7, 3) \), the neighborhood of three \( K_2 \)'s, or two \( K_2 \)'s and a \( K_1 \), or one \( K_2 \) and three \( K_1 \)'s contains at least 12 vertices.

Proof. See Figure 5.1 for the structure of a \( K_2 \) and \( K_1 \) or \( K_2 \). We first look at the case of three \( K_2 \)'s. Since \( K(7, 3) \) has girth 6, the neighborhood of a \( K_2 \), say \( v_1v_2 \), has six vertices. Let \( v_3v_4 \) denote another \( K_2 \). If \( |N(v_3) \cap N(v_1)| \geq 2 \), then this creates a cycle of length 4, a contradiction. If \( |N(v_3) \cap N(v_1)| = |N(v_3) \cap N(v_2)| \geq 1 \), then this creates a cycle of length 5, a contradiction. Thus the neighborhood of two \( K_2 \)'s has at least 10 vertices. By these arguments, if there is a third \( K_2 \), say \( v_5v_6 \), we must have \( |N(v_5) \cap (N(v_1) \cup N(v_2))| \leq 1 \) and \( |N(v_5) \cap (N(v_3) \cup N(v_4))| \leq 1 \). Thus the neighborhood of three \( K_2 \)'s has at least 12 vertices. The proof is similar for two \( K_2 \)'s and a \( K_1 \).

In the final case, we know the neighborhood of a \( K_2 \), say \( v_1v_2 \), has 6 vertices. If a \( K_1 \), say \( v_3 \), has \( |N(v_3) \cap N(v_1)| = |N(v_3) \cap N(v_2)| = 1 \), this creates a cycle of length
5, a contradiction. Thus the neighborhood of a $K_2$ and $K_1$ has at least 9 vertices. For another $K_1$, say $v_4$, the previous argument with the fact that $|N(v_3) \cap N(v_4)| \leq 1$ (otherwise we create a cycle of length 4), the neighborhood of a $K_2$ and two $K_1$’s is at least 11. Adding a third $K_1$ creates a neighborhood of size at least 12.

\[ \square \]

\textbf{Figure 5.1:} The neighborhood of a $K_2$ and a $K_1$ or $K_2$, corresponding to Lemma 5.4.6. We see all the configurations except for the bottom right, which has 10 neighbors, contradict the fact that $K(7, 3)$ has girth 6.

The following provides a lower bound on the number of edges between two parts.
Lemma 5.4.8 ([54]). Let $0 = \mu_1 < \mu_2 \leq ... \leq \mu_n$ denote the eigenvalues of the Laplacian of $G$. If $G$ be a connected graph of order $n$ and $T$ be a subset of vertices of $G$, then

$$e(T, G \setminus T) \geq \frac{\mu_2 |T| (n - |T|)}{n}.$$  

In particular, if $G = K(v, 3)$,

$$e(T, G \setminus T) \geq \frac{2|T| (35 - |T|)}{35}. \quad (5.10)$$

Since $K(7, 3)$ has girth 6, a component with 2, 3, or 4 vertices in $K(7, 3) \setminus S$ must be a tree, and thus has 6, 8, or 10 edges incident to the vertices, respectively. We see that the right hand side of (5.10) is increasing with respect to $|T|$ for $1 \leq |T| \leq 17$. Moreover, for $|T| = 5$, $e(T, K(7, 3) \setminus T) \geq 9$. Thus any component $C$ of $K(7, 3) \setminus S$, where $S$ is a disconnecting set with $|S| \leq 17$, that has $e(C, K(7, 3) \setminus C)$ equal to 6 or 8 must be an edge or a path on three vertices, respectively. Moreover, $e(C, K(7, 3) \setminus C) \neq 7$ for any component.

Lemma 5.4.9. Let $S$ be a disconnecting set of vertices of $K(7, 3)$, and denote $c = c(K(7, 3) \setminus S)$. Let $a$ denote the number of isolated vertices in $K(7, 3) \setminus S$. Then $a \geq 3c - 2|S|$.

Proof. Let $X_1, X_2, ..., X_c$ denote the components of $K(7, 3) \setminus S$, and $e(X_i, S)$ denote the number of edges where 1 endpoint is in $X_i$ and the other endpoint is in $S$. Since $K(7, 3)$ is 4-regular,

$$\sum_{i=1}^{c} e(X_i, S) \leq 4|S|.$$ 

On the other hand, since a $K_2$ has 6 neighbors,

$$\sum_{i=1}^{c} e(X_i, S) \geq 4a + (c - a)6.$$ 

Combining these two inequalities yields the result.

\[\square\]

Theorem 5.4.10. The toughness of $K(7, 3)$ is $\frac{4}{3}$. 

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**Figure 5.2:** The distance partition of $K(7, 3)$ emphasizing the neighborhood of a $K_2$.

*Proof.* The graph $K(7, 3)$ is 4-regular and has 35 vertices. If $K(7, 3)$ is not $\frac{4}{3}$-tough, then there exists a subset of vertices $S$ such that $\frac{|S|}{c(K(7, 3) \setminus S)} < \frac{4}{3}$. With the assumption that $|S| > 4$ and by Theorem 5.1.2 and Lemma 5.4.3, we only need to consider $5 \leq |S| \leq 17$ and $5 \leq c(K(7, 3) \setminus S) \leq 13$.

For a set disconnecting set of vertices $S$, let $C_1, C_2, ..., C_c$ denote the components of $K(7, 3) \setminus S$, and let $t_i$ denote the number of edges between component $i$ and $S$ for $1 \leq i \leq c$. Furthermore, assume $t_i \leq t_{i+1}$ for $1 \leq i \leq c(K(7, 3) \setminus S) - 1$. For the rest of the proof, we assume $K(7, 3)$ is not $\frac{4}{3}$-tough and obtain a contradiction. We do a case analysis on $c(K(7, 3) \setminus S)$.

**Case 1.** Suppose $c(K(7, 3) \setminus S) = 5$. 

We must have $|S| = 6$ by Lemma 5.4.3. Lemma 5.4.9 implies $a \geq 3$, where $a$ denotes the number of isolated vertices. By Lemma 5.4.6, $|S| \geq 9$, a contradiction.

**Case 2.** Suppose $c(K(7,3) \setminus S) = 6$.

We must have $|S| = 7$ by Lemma 5.4.3. Lemma 5.4.9 implies $a \geq 4$. By Lemma 5.4.6, $|S| \geq 10$, a contradiction.

**Case 3.** Suppose $c(K(7,3) \setminus S) = 7$.

We must have $8 \leq |S| \leq 9$ by Lemma 5.4.3.

**Subcase 1.** Suppose $|S| = 8$.

Lemma 5.4.9 implies $a \geq 5$. This contradicts Lemma 5.4.6 since the size of the neighborhood of 3 isolated vertices is at least 9.

**Subcase 2.** Suppose $|S| = 9$.

Lemma 5.4.9 implies $a \geq 4$. If $a = 3$, then $\sum_{i=4}^{7} t_i \leq 36 - 12 = 24$. Necessarily the remaining components are 4 $K_2$'s. But $K(7,3) \setminus S$ should have 26 vertices, a contradiction. If $a \geq 4$, we obtain a contradiction by Lemma 5.4.6 since the size of the neighborhood of 4 isolated vertices is at least 10.

**Case 4.** Suppose $c(K(7,3) \setminus S) = 8$.

We must have $9 \leq |S| \leq 10$ by Lemma 5.4.3.

**Subcase 1.** Suppose $|S| = 9$.

Lemma 5.4.9 implies $a \geq 6$. This contradicts Lemma 5.4.6 since the size of the neighborhood of 4 isolated vertices is at least 10.

**Subcase 2.** Suppose $|S| = 10$.

Lemma 5.4.9 implies $a \geq 4$. If $a = 4$, $\sum_{i=5}^{8} t_i \leq 40 - 16 = 24$. Necessarily the remaining components are 4 $K_2$'s. But $K(7,3) \setminus S$ should have 25 vertices, a contradiction. If $a = 5$, $\sum_{i=6}^{8} t_i \leq 40 - 20 = 20$. Since $t_6, t_7, t_8 \geq 6$, none of the values can exceed 8. With 20 vertices remaining in the 3 components, one component has at least 7 vertices. Lemma 5.4.8 implies $t_8 \geq 12$, a contradiction. If $a = 6$, $t_7 + t_8 \leq 16$. Since $t_7, t_8 \geq 6$, none of the values can exceed 10. With 19 vertices remaining in the
2 components, one component has at least 10 vertices. Lemma 5.4.8 implies $t_8 \geq 15$, a contradiction. If $a = 7$, $t_8 \leq 12$, and this component contains 18 vertices. Lemma 5.4.8 implies $t_8 \geq 18$, a contradiction. It is obvious that $a = 8$ cannot occur.

**Case 5.** Suppose $c(K(7,3) \setminus S) = 9$.

We must have $10 \leq |S| \leq 11$ by Lemma 5.4.3.

**Subcase 1.** Suppose $|S| = 10$.

Lemma 5.4.9 implies $a \geq 7$. Then $t_8 + t_9 \leq 40 - 28 = 12$. Necessarily the remaining components are 2 $K_2$’s. But $K(7,3) \setminus S$ should have 25 vertices, a contradiction. If $a = 8$, then $t_9 \leq 40 - 32 = 8$, and this component has 17 vertices. Lemma 5.4.8 implies $t_9 \geq 18$, a contradiction. It is obvious that $a = 9$ cannot occur.

**Subcase 2.** Suppose $|S| = 11$.

Lemma 5.4.9 implies $a \geq 5$. If $a = 5$, $t_6 + t_7 + t_8 + t_9 \leq 44 - 20 = 24$. Since $t_6, t_7, t_8, t_9 \geq 6$, the remaining components must be 4 $K_2$’s. But $K(7,3) \setminus S$ should have 24 vertices, a contradiction. If $a = 6$, $t_7 + t_8 + t_9 \leq 44 - 24 = 20$. Since $t_6, t_7, t_8 \geq 6$, none of the values can exceed 8. With 18 vertices remaining in the 3 components, there must be a component of order at least 6. Lemma 5.4.8 implies $t_8 \geq 10$, a contradiction. If $a = 7$, $t_8 + t_9 \leq 44 - 28 = 16$. Since $t_8, t_9 \geq 6$, neither value can exceed 10. With 17 vertices remaining in the 2 components, there must be a component of order at least 9. Lemma 5.4.8 implies $t_9 \geq 14$, a contradiction. If $a = 8$, $t_9 \leq 12$, and this component contains 16 vertices. Lemma 5.4.8 implies $t_9 \geq 18$, a contradiction. It is obvious that $a = 9$ cannot occur.

**Case 6.** Suppose $c(K(7,3) \setminus S) = 10$.

We must have $11 \leq |S| \leq 13$ by Lemma 5.4.3.

**Subcase 1.** Suppose $|S| = 11$.

Lemma 5.4.9 implies $a \geq 8$. If $a = 8$, then $t_9 + t_{10} \leq 44 - 32 = 12$. Since $t_9, t_{10} \geq 6$, the remaining components must be 2 $K_2$’s. But $K(7,3) \setminus S$ should have 24 vertices, a contradiction. If $a = 9$, then $t_{10} \leq 44 - 36 = 8$, and this component contains
15 vertices. Lemma 5.4.8 implies $t_{10} \geq 18$, a contradiction. It is obvious that $a = 10$ cannot occur.

**Subcase 2.** *Suppose $|S| = 12$.*

Lemma 5.4.9 implies $a \geq 6$. If $a = 6$, $\sum_{i=7}^{10} t_i \leq 48 - 24 = 24$. Necessarily the remaining 4 components are $K_2$’s. But $K(7, 3) \setminus S$ should have 23 vertices, a contradiction. If $a = 7$, $t_8 + t_9 + t_{10} \leq 20$. The remaining components are 3 $K_2$’s or 2 $K_2$’s and a $P_3$. But $K(7, 3) \setminus S$ should have 23 vertices, a contradiction. If $a = 8$, $t_9 + t_{10} \leq 14$. The remaining components are either 2 $K_2$’s or a $K_2$ and a $P_3$, which denotes a path on three vertices. But $K(7, 3) \setminus S$ should have 23 vertices, a contradiction. If $a = 9$, then $t_{10} \leq 12$, and this component contains 14 vertices. Lemma 5.4.8 implies $t_{10} \geq 17$, a contradiction. It is obvious that $a = 10$ cannot occur.

**Subcase 3.** *Suppose $|S| = 13$.*

If $|S| = 13$, Lemma 5.4.9 implies $a \geq 4$. If $a = 4$, then $\sum_{i=5}^{10} t_i \leq 52 - 16 = 36$. Since $t_i \geq 6$ for $5 \leq i \leq 10$, $t_i = 6$ for $5 \leq i \leq 10$. Necessarily these remaining 6 components are $K_2$’s. But $K(7, 3) \setminus S$ should have 22 vertices, a contradiction. If $a = 5$, then $\sum_{i=6}^{10} t_i \leq 32$. Since $t_i \geq 6$ for $6 \leq i \leq 10$, $t_6 = 6$. Continuing this argument, we see $t_7 = t_8 = t_9 = 6$, and $t_{10} = 6$ or 8. This implies $K(7, 3) \setminus S$ contains 5 $K_1$’s, 4 $K_2$’s, and a component containing 2 or 3 vertices. But $K(7, 3) \setminus S$ should have 22 vertices, a contradiction. If $a = 6$, then $\sum_{i=7}^{10} t_i \leq 52 - 24 = 28$. Since $t_i \geq 6$ for $7 \leq i \leq 10$, we must have $t_7 = t_8 = 6$, and the remaining 2 components each contain at most 3 vertices. But $K(7, 3) \setminus S$ should have 22 vertices, a contradiction. If $a = 7$, then $\sum_{i=8}^{10} t_i \leq 52 - 28 = 24$. If $t_8 = t_9 = t_{10} = 8$, the $K(7, 3) \setminus S$ contains 7 $K_1$’s and 3 $K_2$’s, which contradicts $K(7, 3) \setminus S$ having 22 vertices. Since $t_i \neq 7$ for any $i$, we must have $t_8 = 6$, so that $t_9 + t_{10} \leq 18$. If $t_9 = 6$, this component is a $K_2$ and $t_{10} \leq 12$. This last component must contain $22 - 7 - 2 - 2 = 11$ vertices. Lemma 5.4.8 implies $t_{10} \geq 16$, a contradiction. If $t_9 = 8$, this component is a $P_3$ and $t_{10} \leq 10$. The last component must contain $22 - 7 - 2 - 3 = 10$ vertices. Lemma 5.4.8 implies $t_{10} \geq 15$, a contradiction.

The remaining case is when $t_9 = t_{10} = 9$. Since these 2 components contain a total of 13 vertices, one of these components has at least 7 vertices. By Lemma 5.4.8, $t_{10} \geq 12$,
a contradiction. If $a = 8$, then $t_9 + t_{10} \leq 52 - 32 = 20$. If $t_9 = 6$, then $t_{10} \leq 14$. The last component contains 12 vertices. Lemma 5.4.8 implies $t_{10} \geq 16$, a contradiction. If $t_9 = 8$, then $t_{10} \leq 12$. The last component contains 10 vertices. Lemma 5.4.8 implies $t_{10} \geq 15$, a contradiction. Therefore we only need to consider when $t_9 = 9$ or 10 and $9 \leq t_{10} \leq 11$. Since these 2 components contain a total of 14 vertices, one of these components has at least 7 vertices. By Lemma 5.4.8, $t_{10} \geq 16$, a contradiction. It is obvious that $a = 10$ cannot occur.

**Case 7.** Suppose $c(K(7, 3) \setminus S) = 11$.

We must have $12 \leq |S| \leq 14$ by Lemma 5.4.3.

**Subcase 1.** Suppose $|S| = 12$.

Lemma 5.4.9 implies $a \geq 9$. If $a = 9$, $t_9 + t_{10} \leq 48 - 36 = 12$. Since $t_9, t_{11} \geq 6$, we must have $t_9 = t_{11} = 6$, and these components are $K_2$’s. But $K(7, 3) \setminus S$ should have 23 vertices, a contradiction. If $a = 10$, $t_{11} \leq 8$. The last component must be a $K_2$ or $P_3$, but $K(7, 3) \setminus S$ should have 23 vertices, a contradiction. It is obvious that $a = 11$ cannot occur.

**Subcase 2.** Suppose $|S| = 13$.

Lemma 5.4.9 implies $a \geq 7$. If $a = 7$, $\sum_{i=8}^{11} t_i \leq 52 - 28 = 24$. Necessarily the remaining 4 components are $K_2$’s. But $K(7, 3) \setminus S$ should have 22 vertices, a contradiction. If $a = 8$, $t_9 + t_{10} + t_{11} \leq 52 - 32 = 20$. Since $t_9, t_{10}, t_{11} \geq 6$, we must have $t_9 = t_{10} = t_{11} = 6$, or $8$. This means $K(7, 3) \setminus S$ contains 8 $K_1$’s , 2 $K_2$’s, and a $P_3$ or $K_2$. But $K(7, 3) \setminus S$ should have 22 vertices, a contradiction. If $a = 9$, then $t_{10} + t_{11} \leq 52 - 36 = 16$. Necessarily, these two components are $K_2$’s or $P_3$’s. But $K(7, 3) \setminus S$ should have 22 vertices, a contradiction. If $a = 10$, then $t_{11} \leq 52 - 40 = 12$, and the last component must contain 12 vertices. By Lemma 5.4.8, $t_{11} \geq 16$, a contradiction.

**Subcase 3.** Suppose $|S| = 14$.
Lemma 5.4.9 implies $a \geq 5$. If $a = 5$, $\sum_{i=6}^{11} t_i \leq 56 - 20 = 36$. Necessarily the remaining 5 components are $K_2$'s. But $K(7,3) \setminus S$ should have 21 vertices, a contradiction. If $a = 6$, $\sum_{i=7}^{11} t_i \leq 56 - 24 = 32$. Since $t_i \geq 6$ for $7 \leq i \leq 11$, we must have $t_7 = t_8 = t_9 = t_{10} = 6$, and $t_{11} = 6$ or 8. This means $K(7,3) \setminus S$ contains 6 $K_1$'s, 4 $K_2$'s, and a $P_3$ or $K_2$. But $K(7,3) \setminus S$ should have 21 vertices, a contradiction. If $a = 7$, $\sum_{i=8}^{11} t_i \leq 56 - 28 = 28$. Since $t_i \neq 7$ for any $i$, $t_9 + t_{10} + t_{11} \leq 22$. If $t_9, t_{10}, t_{11} \in \{6, 8\}$, making the components $K_2$'s or $P_3$'s, $K(7,3) \setminus S$ will not contain 21 vertices. So assume $t_9 = t_{10} = 6$ and $t_{11} = 10$. Then $K(7,3) \setminus S$ contains 7 $K_1$'s, 3 $K_2$'s, and a component with 8 vertices. By Lemma 5.4.8, $t_{11} \geq 14$, a contradiction. If $a = 8$, $t_9 + t_{10} + t_{11} \leq 56 - 32 = 24$. Since $t_9, t_{10}, t_{11} \geq 6$, these components are $K_2$'s, and the remaining component contains 9 vertices. By Lemma 5.4.8, $t_{11} \geq 14$, a contradiction. If $a = 9$, $t_{10} + t_{11} \leq 56 - 36 = 20$. If $t_{10} = 6$, then $t_{11} \leq 14$. The last component must have 10 vertices. By Lemma 5.4.8, $t_{11} \geq 15$, a contradiction. If $t_{10} = 8$, then $t_{11} \leq 12$. The last component must have 9 vertices. By Lemma 5.4.8, $t_{11} \geq 14$, a contradiction. If $t_{10} = t_{11} = 10$, these 2 components each contain 4 vertices, but $K(7,3) \setminus S$ should contain 21 vertices, a contradiction. If $a = 10$, $t_{11} \leq 56 - 40 = 16$. By Lemma 5.4.8, $t_{11} \geq 16$. This implies $S$ is an independent set of size 14, and these vertices (as sets) all must share an element by Theorem 5.1.2. But the deletion of an independent set of this size should create 6 $K_2$'s and a component of order 9, a contradiction. It is obvious that $a = 11$ cannot occur.

Case 8. Suppose $c(K(7,3) \setminus S) = 12$.

We must have $13 \leq |S| \leq 15$ by Lemma 5.4.3.


Lemma 5.4.9 implies $a \geq 10$. If $a = 10$, $t_{11} + t_{12} \leq 52 - 40 = 12$. Since $t_{11}, t_{12} \geq 6$, we must have $t_{11} = t_{12} = 6$, and these components are $K_2$'s. But $K(7,3) \setminus S$ should
have 22 vertices, a contradiction. If \( a = 11, t_{12} \leq 8 \). The last component must be a \( K_2 \) or \( P_3 \), but \( K(7, 3) \setminus S \) should have 22 vertices, a contradiction. It is obvious that \( a = 12 \) cannot occur.

**Subcase 2.** Suppose \(|S| = 14\).

Lemma 5.4.9 implies \( a \geq 8 \). If \( a = 8, \sum_{i=9}^{12} t_i \leq 56 - 32 = 24 \). Necessarily the remaining 4 components are \( K_2 \)'s. But \( K(7, 3) \setminus S \) should have 21 vertices, a contradiction. If \( a = 9, t_{11} + t_{12} + t_{13} \leq 56 - 36 = 20 \). Since \( t_{10}, t_{11}, t_{12} \geq 6 \), we must have \( t_{10} = t_{11} = 6 \), and \( t_{12} = 6 \) or 8. This means \( K(7, 3) \setminus S \) contains 9 \( K_1 \)'s, 2 \( K_2 \)'s, and a \( P_3 \) or \( K_2 \). But \( K(7, 3) \setminus S \) should have 21 vertices, a contradiction. If \( a = 10 \), then \( t_{11} + t_{12} \leq 56 - 40 = 16 \). Necessarily, these two components are \( K_2 \)'s or \( P_3 \)'s. But \( K(7, 3) \setminus S \) should have 21 vertices, a contradiction. If \( a = 11 \), then \( t_{11} \leq 56 - 44 = 12 \), and the last component must contain 10 vertices. By Lemma 5.4.8, \( t_{11} \geq 15 \), a contradiction. It is obvious that \( a = 12 \) cannot occur.

**Subcase 3.** Suppose \(|S| = 15\).

Lemma 5.4.9 implies \( a \geq 6 \). If \( S \) is a maximum independent set, \( S \) contains all subsets that share one element, say 7, by Theorem 5.1.1. Then \( K(7, 3) \setminus S \) consists of all 3-size subsets of [6], which will be 10 \( K_2 \)'s. We may assume the induced subgraph on \( S \) contains at least 2 edges, so that \( e(S, G \setminus S) \leq 56 \). Otherwise, if \( S \) contained 13 \( K_1 \)'s and a \( K_2 \), then we can create 2 different independent sets of size \( 14 > (7-1) \{5-1\} - (7-3-1) + 1 \), each of which contains exactly 1 endpoint of the \( K_2 \). Theorem 5.1.2 implies both families of sets would have the same two elements. But clearly there cannot be a family of 14 sets that all contain two elements. If \( a = 6, \sum_{i=7}^{12} t_i \leq 56 - 24 = 32 \). Since \( t_i \geq 6 \) for \( 7 \leq i \leq 12 \), this cannot happen. If \( a = 7, \sum_{i=8}^{12} t_i \leq 56 - 28 = 28 \), and we have the same contradiction. If \( a = 8, \sum_{i=9}^{12} t_i \leq 56 - 32 = 24 \). Necessarily the remaining 4 components are \( K_2 \)'s. But \( K(7, 3) \setminus S \) should have 20 vertices, a contradiction. If \( a = 9, t_{10} + t_{11} + t_{12} \leq 56 - 36 = 20 \). This implies \( t_{10}, t_{11}, t_{12} \in \{6, 8\} \), making the components \( K_2 \)'s or \( P_3 \)'s, and so \( K(7, 3) \setminus S \) will not contain 20 vertices. If \( a = 10, t_{11} + t_{12} \leq 56 - 40 = 16 \). If \( t_{11}, t_{12} \in \{6, 8\} \), then the components are \( K_2 \)'s or \( P_3 \)'s. But \( K(7, 3) \setminus S \) will not contain 20 vertices. Thus assume \( t_{11} = 6 \), so the component is a
$K_2$, and $t_{12} = 10$, which is a component of size 4. But $G \setminus S$ should have 20 vertices, a contradiction. If $a = 11$, $t_{12} \leq 56 - 44 = 12$. The last component contains 9 vertices, which by Lemma 5.4.8 implies $t_{12} \geq 14$, a contradiction. It is obvious that $a = 12$ cannot occur.

**Case 9.** Suppose $c(K(7,3) \setminus S) = 13$.

If $c(K(7,3) \setminus S) = 13$, then $14 \leq |S| \leq 17$. If there are at least 2 components that are not isolated vertices, we can obtain an independent set of size 13 by picking one vertex from each component, and this is a contradiction by Lemma 5.4.5. Thus assume $K(7,3) \setminus S$ contains 12 $K_1$'s a component of size at least 6. Necessarily this component contains a $P_4$ as a subgraph, say $v_1v_2v_3v_4$. We can obtain an independent set of size 14 by taking $v_1$, $v_3$, and 1 vertex from the other 12 components. By Theorem 5.1.2, the sets corresponding to the vertices all must share an element, say '1'. We can define a new family of sets by replacing $v_1$ and $v_3$ with $v_2$ and $v_4$. The sets corresponding to the vertices should also all share an element. Similar to the proof of Theorem 5.3.3, this is a contradiction.

□
In this chapter, we discuss problems and conjectures motivated from the work in the previous chapters.

6.1 Spanning trees

In Chapter 2, we studied the relations between the eigenvalues of a regular graph and its spanning tree packing number. Based on the results, we make the following conjecture.

**Conjecture 6.1.1.** Let \( k \geq 8 \) and \( 4 \leq m \leq \left\lfloor \frac{k}{2} \right\rfloor \) be two integers. If \( G \) is a \( k \)-regular graph such that \( \lambda_2(G) < k - \frac{2m-1}{k+1} \), then \( G \) contains at least \( m \) edge-disjoint spanning trees.

We supported this conjecture further in Section 2.4 by providing a family of \( k \)-regular graphs \( A_k \) with \( k - \frac{7}{k+1} < \lambda_2(A_k) < k - \frac{7}{k+3} \) and \( \sigma(A_k) = 3 \). We attempted to generalize the construction shown in Figure 2.5 to determine the spectrum, as the characteristic polynomial of the quotient matrix corresponding to the partition in 49 parts factored very nicely. However, the factored characteristic polynomial of the quotient matrix in the next extremal case (9 copies of \( K_{k+1} \) missing 4 edges) contained a polynomial of degree 21. The computation was also done on the construction involving 11 copies of \( K_{k+1} \) missing 5 edges. The characteristic polynomial of the quotient matrix did factor nicely (the highest degree factor was a polynomial of degree 11), but the calculations became difficult with such large matrices, and we did not pursue this further.
Let $c(G)$ denote the number of components of the graph $G$. We recall the vertex-toughness of $G$ is defined as $\min \frac{|S|}{c(G \setminus S)}$, where the minimum is taken over all subsets of vertices $S$ whose removal disconnects $G$ (see Chapter 3 for more details on this parameter). For $z \geq 1$, the higher order edge-toughness $\tau_z(G)$ is defined as

$$\tau_z(G) := \min \frac{|X|}{c(G \setminus X) - z},$$

where the minimum is taken over all subsets $X$ of edges of $G$ with the property $c(G \setminus X) > z$ (see Chen, Koh and Peng [18] or Catlin, Lai and Shao [17] for more details). The Nash-Williams/Tutte Theorem states that $\sigma(G) = \lfloor \tau_1(G) \rfloor$. Cunningham [27] generalized this result and showed that if $\tau_1(G) \geq \frac{p}{q}$ for some natural numbers $p$ and $q$, then $G$ contains $p$ spanning trees (repetitions allowed) such that each edge of $G$ lies in at most $q$ of the $p$ trees. Chen, Koh and Peng [18] proved that $\tau_z(G) \geq m$ if and only if $G$ contains at least $m$ edge-disjoint forests with exactly $z$ components. It would be interesting to find connections between the eigenvalues of the adjacency matrix (or of the Laplacian) of a graph $G$ and $\tau_z(G)$.

Another question of interest is to determine sufficient eigenvalue condition for the existence of nice spanning trees in a graph. A lot of work has been done on this problem in the case of random graphs (see Krivelevich [46] for example).

In Section 3.4, we found a sufficient spectral condition for a spanning $d$-tree, which is a tree with maximum degree $d$, $d \geq 2$. It would be of interest to determine whether this bound is best possible i.e. is there a family of $k$-regular graphs $G_k$ such that $\lambda_4(G_k) = k - \frac{k}{(d-2)(k+1)}$, but $G_k$ does not contain a spanning $d$-tree?

### 6.2 Graphs corresponding to Latin squares

In Section 4.2, we determined the toughness of the Lattice graphs $L_2(v)$, which is the same as the graphs corresponding to an orthogonal array $OA(2, v)$. This motivated us to determine the toughness of the the graphs corresponding to an $OA(3, v)$, which is strongly regular with parameters $(v^2, 3(v - 1), v, 6)$. There are no known Latin squares of odd order that do not have a transversal of order $v$. Ryser [57] conjectured that
Figure 6.1: A Latin square of order 4 without a transversal.

```
1 2 3 4
4 1 2 3
3 4 1 2
2 3 4 1
```

every Latin square of odd order contains a transversal. This has been shown to be true for all odd values up to nine [53]. By arguments from Section 4.3, Ryser’s conjecture is equivalent to showing the following.

**Conjecture 6.2.1.** If $G$ is the corresponding graph of a Latin square of odd order $v$, then $t(G) = v - 1$. Moreover, $S$ is a disconnecting set of vertices of $G$ such that $\frac{|S|}{\alpha(G \setminus S)} = v - 1$ if and only if $S$ is the complement of a maximum independent set of size $v$.

When $v$ is even, there are Latin squares that do not contain transversals of order $v$ (see Figure 6.1). However, there are various sufficient or necessary conditions on the existence of transversals of order $v$ (for example see [28, p. 33-38]). Hall [38] proved that the multiplication table of an abelian group of even order has a transversal if and only if there is no unique element of order 2. By arguments similar to those in Section 4.3, the Latin square graphs corresponding to these multiplication tables also have toughness $v - 1$.

Since the maximum independent sets for the graphs of $OA(3, v)$ are not fully classified, it is likely to be even more difficult to determine the toughness for the graphs of $OA(m, v)$ for $m > 3$, and thus we have not looked into this.

### 6.3 Other problems on graph toughness

In Chapters 4 and 5, we determined the toughness value for many families of graphs. The results in the chapters were highly dependent on being able to classify the structure of maximum independents. In all these graphs except for the Petersen
graph, the toughness equaled $k/(-\lambda_{\min})$ or $k/(-\lambda_2)$. This leads to the question as to whether there are other graphs that attain equality in the Hoffman ratio bound and have toughness not equal to either of these values.

An $(n, k, \lambda)$-graph is a $k$-regular graph on $n$ vertices whose second largest eigenvalue is $\lambda$. In [47], it was conjectured that there exists a positive constant $c$ such that for sufficiently large $n$, any $(n, k, \lambda)$ graph that satisfies $\frac{k}{\lambda} > c$ contains a Hamiltonian cycle. Recall that Chvátal conjectured that if $t(G) > t_0$ for some constant $t_0$, then $G$ is Hamiltonian. Chvátal’s conjecture together with Brouwer’s result that $t(G) > \frac{k}{\lambda} - 2$ for any $k$-regular graph implies the conjecture stated in [47].

A more general problem is determining the toughness of graphs that are not regular. It is worth noting that an induced subgraph with high toughness does not provide much information about the entire graph. Given a graph $H$ with high toughness, one may create a graph $G$ by using a star $S_{n-1}$, and attaching $n - 1$ copies of $H$ in place of the $n - 1$ vertices. We see that deleting the internal vertex shows that the toughness of $G$ can be at most $\frac{1}{n-1}$.

Chapter 5 determined the toughness for the Kneser graph $K(v, r)$ for $r \in \{2, 3\}$, and sufficiently large $v$. For the latter result, the lower bound on $v$ was in terms of a quadratic of $r$, which leaves a relatively large gap of unknown values. For instance, for $r = 10$, Theorem 5.3.3 only implies $t(K(v, 10)) = \frac{v}{10} - 1$ for $v \geq 267$. As we saw in Chapter 5, the toughness was more difficult to justify as $v$ approached seven, and therefore it is of interest to find a more concise proof for determining $t(K(7, 3))$.

The tools used to show $t(K(v, 3)) = \frac{v}{3} - 1$ for $v \geq 8$ were applied for $K(v, 4)$. However, the number of cases increased with many more calculations. This was done through Mathematica, and can be found in Appendix A.2. Using computer, we determined that $t(K(v, 4)) = \frac{v}{4} - 1$ for $v \geq 15$. Thus, there is still a gap open for $9 \leq v \leq 14$. Due to the tedious calculations, we have not attempted this approach for $r > 4$.


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Appendix A

MATHEMATICA CALCULATIONS

A.0.1 Calculations for Lemma 2.3.1

The following is the characteristic polynomial of the equitable partition in 25 parts:

\[
(d-4) \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & d-4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d-4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d-4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d-4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d-4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d-4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d-4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d-4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d-4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d-4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d-4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d-4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d-4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d-4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d-4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d-4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d-4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d-4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d-4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d-4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d-4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d-4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d-4 
\end{pmatrix}
\]

\[
\frac{(d-x)(-1+x)(1+x)^2(3+x)(-5+5d-d^2+5x-13dx+4d^2x+109x^2-83dx^2+14d^2x^2-21x^3+57dx^3-20d^2x^3-146x^4+140dx^4-29d^2x^4-70x^5+18dx^5+8d^2x^5+36x^6-66x^6+20d^2x^6+58x^7-50dx^7+8d^2x^7+30x^8-16dx^8+d^2x^8+8x^9-2dx^9+x^{10})^2}{(d-x)(-1+x)(1+x)^2(3+x)(-5+5d-d^2+5x-13dx+4d^2x+109x^2-83dx^2+14d^2x^2-21x^3+57dx^3-20d^2x^3-146x^4+140dx^4-29d^2x^4-70x^5+18dx^5+8d^2x^5+36x^6-66x^6+20d^2x^6+58x^7-50dx^7+8d^2x^7+30x^8-16dx^8+d^2x^8+8x^9-2dx^9+x^{10})^2}
\]

A.0.2 Calculations for Lemma 2.3.2

The following calculations show \( P^{(n)}_{10}(d - \frac{1}{d+3}) > 0 \) for \( n = 0, 1, \ldots, 10 \).

A.0.2.1 \( n=0 \)

\[
\begin{pmatrix}
-5 + 5d - d^2 + 5x - 13dx + 4d^2x + 109x^2 - 83dx^2 + 14d^2x^2 - 21x^3 + \\
57dx^3 - 20d^2x^3 - 146x^4 + 140dx^4 - 29d^2x^4 - 70x^5 + 18dx^5 + \\
8d^2x^5 + 36x^6 + 66dx^6 + 20d^2x^6 + 58x^7 - 50dx^7 + 8d^2x^7 + \\
30x^8 - 16dx^8 + d^2x^8 + 8x^9 - 2dx^9 + x^{10} \\
\end{pmatrix}
\frac{5(209081 + 2795840 + 44259960 + 79884000 + 31409640 + 1156227d^6 + 317856d^7 + 189275d^8 + 9630d^9 + 7239d^{10} + 1412d^{11} + 79d^{12})}{(d+3)^{10}}
\]

Looking at the numerator,
\[209801 + 27898484d + 4225996d^2 - 798400d^3 - 25868960d^4 + 3149694d^5 + 1156227d^6 - 317856d^7 - 185275d^8 - 9630d^9 + 7239d^9 + 1412d^9 + 79d^{12}\]
\[\geq 209801 + 27898484d + 4225996d^2 - 798400d^3 - 25868960d^4 + 3149694d^5 + 1156227d^6 - 317856d^7 - 185275d^8 - 9630d^9 + 7239d^9 + 1412d^9 + 79d^{12}\]
\[\geq 209801 + 27898484d + 4225996d^2 + 105400584d^3 + 39037282d^4 + 1245768d^5 + 119717d^8 + 7434d^9 > 0.\]

**A.0.2.2 n=1**

\[
\begin{align*}
\text{Apart} & \quad \text{FullSimplify} \quad D \\
&= \left\{\begin{array}{l}
\left(-5 + 5d - d^2 + 5x - 13dx + 4d^2 x + 109x^2 - 83dx^2 + 14d^2x^2 - 21x^3 + 57dx^3 - 20d^2x^3 - 146x^4 + 140dx^4 - 29d^2x^4 - 70x^5 + 18dx^5 + 8d^2x^5 + 36d^3x^5 - 66dx^6 + 20dx^6 + 58x^7 - 50dx^7 + 8d^2x^7 + 30x^8 - 16dx^8 + d^2x^8. \\
\end{array}\right.
\end{align*}
\]

Looking at the fraction terms,

\[
\begin{align*}
\text{Together} & \quad \left\{\begin{array}{l}
-154125 - 6265d + 9235d^2 - 1605d^3 - 80d^4 + 40d^5 - 1953125d^6 + 525000d^7 + 4312500d^8 + 1400000d^9 + 26231250d^{10} + 2500000d^{11}
\end{array}\right.
\end{align*}
\]

The expression is positive. The only concern now are the terms \(-154125 - 6265d + 9235d^2 - 1605d^3 - 80d^4 + 40d^5\). Direct calculations for \(d = 6\) and \(7\) yield the values \(1425\) and \(184220\), respectively. For \(d \geq 8\),

\[(-154125 - 6265d + 9235d^2) - 1605d^3 - 80d^4 + 40d^5\]

\[= 9235d + (d - 1)(9235)d - 6265d - 154125 + 80d^4 + (d - 2)(40)d^3 + 40d^4 - 80d^4 - 1605d^5\]

\[\geq 9235d + (7)(9235)(8) - 6265d - 154125 + 80d^4 + (6)(40)(8)d^3 - 80d^4 - 1605d^5\]

\[= (1920 - 1605)d^3 + (9235 - 6265)d + (517160 - 154125) > 0.\]

**A.0.2.3 n=2**

\[
\begin{align*}
\text{Apart} & \quad \text{FullSimplify} \quad D \\
&= \left\{\begin{array}{l}
\left(-5 + 5d - d^2 + 5x - 13dx + 4d^2 x + 109x^2 - 83dx^2 + 14d^2x^2 - 21x^3 + 57dx^3 - 20d^2x^3 - 146x^4 + 140dx^4 - 29d^2x^4 - 70x^5 + 18dx^5 + 8d^2x^5 + 36d^3x^5 - 66dx^6 + 20dx^6 + 58x^7 - 50dx^7 + 8d^2x^7 + 30x^8 - 16dx^8 + d^2x^8. \\
\end{array}\right.
\end{align*}
\]

Looking at the fraction terms and \(2d^8\),

\[
\begin{align*}
\text{Together} & \quad \left\{\begin{array}{l}
35195225 + 5000000 + 44750000 + 50800000 + 44112500 + 41000000 + 86884000 + 574400000 + 163588075 + 3659898000d + 3340851900d^2 + 1497898000d^3 + 32873750d^4 + 25888400d^5
\end{array}\right.
\end{align*}
\]

By comparing terms, the expression is positive. The only concern now are the terms \(-501172 + 218908d - 37582d^2 - 2480d^3 + 2472d^4 - 344d^5 - 60d^6 + 16d^7\). Direct calculations for \(d = 6\) and \(7\) yield the values \(1132028\) and \(4610438\), respectively. Clearly we have for the first two terms that \(-501172 + 218908d > 0\). Now assume \(d \geq 8\). Looking at the next three terms,

\[-37582d^2 - 2480d^3 + 2472d^4 = 4944d^3 + (d - 2)(2472)d^3 - 2480d^3 - 37583d^2\]

\[\geq 4944d^3 + (6)(2472)(8)d^2 - 2480d^3 - 37583d^2 > 0\]

For the final three terms,

\[-344d^5 - 60d^6 + 16d^7 = 64d^6 + (16d - 4)d^6 - 60d^6 - 344d^5 \geq 64d^6 + (16)(8)d^5 - 60d^6 - 344d^5 > 0.\]
\[ n = 3 \]

Apart

\[
\text{PullSimplify } D \left[ \begin{array}{c}
-5 + 5d - d^2 + 5x - 13dx + 4d^2x + 109x^2 - 83dx^2 + 14d^2x^2 - 21x^3 + 57dx^3 - 20d^2x^3 - 146x^4 + 140dx^4 - 29d^2x^4 - 70x^5 + 18dx^5 + 8d^2x^5 + 36x^6 - 66dx^6 + 20d^2x^6 + 58x^7 - 50dx^7 + 8d^2x^7 + 30x^8 - 16dx^8 + d^2x^8 + 8x^9 - 24x^9 + x^{10} \\
(3+d)^3 \\
35947500 \\
2377554 - 2933224d - 71280d^2 + 40944d^3 - 5340d^4 - 1380d^5 + 336d^6 + 48d^7 + 12600000 - 65700000 - 50400000 + 5340d^4 + 11340d^5 + 736d^6 + 84d^7 + 4d^8
\end{array} \right]
\]

Looking at the fraction terms and \( 48d^7 \),

Together

\[
\frac{1}{(3+d)^3} \left[ \begin{array}{c}
-433473465 - 955900680d - 877113900d^2 - 411225900d^3 - 103585225d^4 - 13375780d^5 - 696710d^6 + 8748d^7 + 20412d^8 + 20412d^9 + 11340d^{10} + 3780d^{11} + 756d^{12} + 84d^{13} + 4d^{14}
\end{array} \right]
\]

Looking at the numerator,

\[-433473465 - 955900680d - 877113900d^2 - 411225900d^3 - 103585225d^4 - 13375780d^5 - 696710d^6 + 8748d^7 + 20412d^8 + 20412d^9 + 11340d^{10} + 3780d^{11} + 756d^{12} + 84d^{13} + 4d^{14} \geq -433473465 - 955900680d - 877113900d^2 - 411225900d^3 - 103585225d^4 - 13375780d^5 - 696710d^6 + 8748d^7 + 20412d^8 + 20412d^9 + 11340d^{10} + 3780d^{11} + 756d^{12} + 84d^{13} + 4d^{14} \]

The only concern now are the terms \( 2377554 - 2933224d - 71280d^2 + 40944d^3 - 5340d^4 - 1380d^5 + 336d^6 \). Direct calculations for \( d = 6 \) and \( 7 \) yield the values 4920342 and 14390436, respectively. We ignore the first positive constant, and assume \( d \geq 8 \). Looking at the next three terms,

\[-2933224d - 71280d^2 + 40944d^3 - 81888d^2 + (d-2)(40944)d^2 - 71280d^2 - 2933224d \geq 81888d^2 + (6)(40944)(6)d^2 - 71280d^2 - 2933224d > 0 \]

For the final three terms,

\[-5340d^4 - 1380d^5 + 336d^6 - 1680d^6 + (d-5)336d^6 - 1380d^5 - 5340d^6 \geq 1680d^6 + (3)(336)(6)d^4 - 1380d^5 - 5340d^6 > 0 \]

\[ n = 4 \]

Apart

\[
\text{PullSimplify } D \left[ \begin{array}{c}
-5 + 5d - d^2 + 5x - 13dx + 4d^2x + 109x^2 - 83dx^2 + 14d^2x^2 - 21x^3 + 57dx^3 - 20d^2x^3 - 146x^4 + 140dx^4 - 29d^2x^4 - 70x^5 + 18dx^5 + 8d^2x^5 + 36x^6 - 66dx^6 + 20d^2x^6 + 58x^7 - 50dx^7 + 8d^2x^7 + 30x^8 - 16dx^8 + d^2x^8 + 8x^9 - 24x^9 + x^{10} \\
(3+d)^4 \\
28926000 - 13488000d + 13488000d^2 - 3162000d^3 + 3162000d^4 + 87850000 - 15120000d + 15120000d^2 + 44100000 - 63840000 - 50400000 + 5340d^4 + 11340d^5 + 736d^6 + 84d^7 + 4d^8
\end{array} \right]
\]

Looking at the fraction terms and \( 672d^6 \),

Together

\[
\frac{1}{(3+d)^4} \left[ \begin{array}{c}
47(1438500 + 23499000d + 32000000d^2 + 19695000d^3 + 3631125d^4 + 2800000d^5 + 10200d^6 + 20412d^7 + 17010d^8 + 7560d^9 + 1890d^{10} + 252d^{11} + 14d^{12})
\end{array} \right]
\]

This expression is positive. The only concern now are the terms \(-285504 - 1017840d + 396024d^2 - 41280d^3 - 18000d^4 + 4032d^5 + 672d^6 + 78750000 - 15120000d + 15120000d^2 + 44100000 - 63840000 - 50400000 + 5340d^4 + 11340d^5 + 736d^6 + 84d^7 + 4d^8\). Direct calculations for \( d = 6 \) and \( 7 \) yield the values 976272 and 22383576, respectively. Now assume \( d \geq 8 \). Looking at the first 3 terms,

\[-285504 - 1017840d + 396024d^2 = 1188072d + (d-3)396024d - 1017840d - 285504 \geq 1188072d + (5)(396024d) - 1017840d - 285504 > 0 \]

For the final three terms,

\[-41280d^3 - 18000d^4 + 4032d^5 = 20160d^4 + (d-5)4032d^4 - 18000d^4 - 41280d^3 \geq 20160d^4 + (3)(4032)(d^3) - 18000d^4 - 41280d^3 > 0 \]

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A.0.2.6  n=5

\[
\text{Apart} \left[ \text{FullSimplify} \left[ \frac{-5 + 5d - d^2 + 5x - 13dx + 4d^2x + 109x^2 - 83dx^2 + 14d^2x^2 - 21x^3 + 57d^3x^3 - 20d^2x^3 + 146d^4x^4 - 29d^2x^4 - 70x^5 + 18dx^5 + 8d^2x^5 + 36d^6x^6 + 20d^2x^6 + 5d^2x^7 + 8d^2x^7 + 30d^8 + 16dx^8 + d^2x^8 + }{8x^9 - 2d^3x^9 + z^{10}} \right], \{x, 5\} \right] \rightarrow d = 5/(d + 3)
\]

Looking at the fraction terms,

\[
\text{Together} \left[ \frac{94500000}{1200} - \frac{151200000}{1200} - \frac{201600000}{1200} + \frac{61824000}{1200} + \frac{14024400}{1200}, \frac{140244000}{1200} \right]
\]

This expression is positive. The only concern now are the terms \(-8576400 + 2476080d - 152400d^2 - 162000d^3 + 33600d^4 + 6720d^5\).

Clearly for the first two terms we have \(-8576400 + 2476080d > 0\) for \(d \geq 6\). Looking at the four remaining terms,

\[-152400d^2 - 162000d^3 + 33600d^4 + 6720d^5 \geq -152400d^2 - 162000d^3 + 33600(36)d^2 + 6720(36)d^3 = -152400d^2 - 162000d^3 + 1209600d^2 + 241920d^3 > 0.\]

A.0.2.7  n=6

\[
\text{Apart} \left[ \text{FullSimplify} \left[ \frac{-5 + 5d - d^2 + 5x - 13dx + 4d^2x + 109x^2 - 83dx^2 + 14d^2x^2 - 21x^3 + 57d^3x^3 - 20d^2x^3 + 146d^4x^4 - 29d^2x^4 - 70x^5 + 18dx^5 + 8d^2x^5 + 36d^6x^6 + 20d^2x^6 + 5d^2x^7 + 8d^2x^7 + 30d^8 + 16dx^8 + d^2x^8 + }{8x^9 - 2d^3x^9 + z^{10}} \right], \{x, 6\} \right] \rightarrow d = 5/(d + 3)
\]

Looking at the fraction terms and 50400d^4 + 9349920 + 244800d,

\[
\text{Together} \left[ \frac{84500000}{1440} - \frac{120960000}{1440} - \frac{3024000}{1440} + \frac{54096000}{1440} + \frac{9349920}{1440} + \frac{2448000}{1440}, \frac{93499200}{1440} \right]
\]

This expression is clearly positive for \(d \geq 6\). The only terms left are \(-1044000d^2 + 201600d^3\), and we get

\[-1044000d^2 + 201600d^3 \geq -1044000d^2 + 201600(6)d^2 \geq -1044000d^2 + 1209600d^2 > 0.\]

A.0.2.8  n=7

\[
\text{Apart} \left[ \text{FullSimplify} \left[ \frac{-5 + 5d - d^2 + 5x - 13dx + 4d^2x + 109x^2 - 83dx^2 + 14d^2x^2 - 21x^3 + 57d^3x^3 - 20d^2x^3 + 146d^4x^4 - 29d^2x^4 - 70x^5 + 18dx^5 + 8d^2x^5 + 36d^6x^6 + 20d^2x^6 + 5d^2x^7 + 8d^2x^7 + 30d^8 + 16dx^8 + d^2x^8 + }{8x^9 - 2d^3x^9 + z^{10}} \right], \{x, 7\} \right] \rightarrow d = 5/(d + 3)
\]

Looking at the fraction terms and 8282420d^3,

\[
\text{Together} \left[ \frac{828242000}{640} - \frac{72376000}{640} + \frac{13305600}{640}, \frac{133056000}{640} \right]
\]

This expression is clearly positive for \(d \geq 6\). The only remaining terms are 5937120 – 468720d^2 + 846720d^2, we have

\[5937120 – 468720d^2 + 846720d^2 > 5937120 – 468720d^2 + 846720(6)d^2 = 5937120 – 468720d^2 + 5080320d^2 > 0.\]

A.0.2.9  n=8

\[
\text{Apart} \left[ \text{FullSimplify} \left[ \frac{-5 + 5d - d^2 + 5x - 13dx + 4d^2x + 109x^2 - 83dx^2 + 14d^2x^2 - 21x^3 + 57d^3x^3 - 20d^2x^3 + 146d^4x^4 - 29d^2x^4 - 70x^5 + 18dx^5 + 8d^2x^5 + 36d^6x^6 + 20d^2x^6 + 5d^2x^7 + 8d^2x^7 + 30d^8 + 16dx^8 + d^2x^8 + }{8x^9 - 2d^3x^9 + z^{10}} \right], \{x, 8\} \right] \rightarrow d = 5/(d + 3)
\]

At \(d = 6\), the value is 44670080. Clearly the expression is increasing for \(d \geq 6\), and hence always positive for \(d \geq 6\).
The following shows the degree 7 polynomial at $d - \frac{\pi}{2}$ is negative, and thus with the previous information, the largest root must be after this point.

The value will be 10!
the number of components for \(15 \leq v\) using the Hilton-Milner Theorem (otherwise we are done by Theorem 5.3.3).

\[ K, \text{subsets of vertices in} \ (104x208) \text{exceeds the value in the Hilton Milner Theorem (Theorem 5.1.2), we were done by} \ A.2 \text{ Calculations for determining the toughness of} \ K(v, 4) \text{ for} \ v \geq 15 \]

If \(c(K(v, 4) \setminus S) \leq \frac{1}{v-4} (\frac{v-4}{4})\) we were done by Theorem 5.3.3. When \(c(K(v, 4) \setminus S)\) exceeds the value in the Hilton Milner Theorem (Theorem 5.1.2), we were done by Theorem 5.3.3 also.

\[
\text{Reduce}[\text{Binomial}[v-5, 3] > \text{Binomial}[v-1, 4-1] - \text{Binomial}[v-5, 4-1] + 1, v, \text{Integers}]\]

\[ v \in \text{Integers} \& k v \geq 22 \]

**Theorem 5.4.1 at} r = 4.\]

\[
\text{FullSimplify}[(4+\text{Binomial}[v-r, r]-\text{Binomial}[v-r-2, r-2]+\text{Binomial}[v-r-1, r-1])]/(\text{Binomial}[v, r]\text{[Binomial}[v-r, v-2, r-2]+\text{Binomial}[v-r-1, r-1])] \]

\[
9(6(-8+v)(-5+v))\]

\[
(-3+v)(-2+v)^3 \]

Define as \(x\).

To be not \((\frac{v}{4} - 1)\)-tough, there exists a subset of vertices \(S\) such that \(|S| < (v/4 - 1)c(K(v, 4) \setminus S)\). We obtain a contradiction by Theorem 5.4.1 if there are subsets of vertices in \(K(v, 4)\) of size \(a\) and \(b\), no edges between them, such that

\[
\frac{(6(-8+v)(-5+v))ab}{(-3+v)(-2+v)^3} > \left(\frac{v}{4} - 1\right) c(K(v, 4) \setminus S). \tag{A.1} \]

Since we are done when \(c(K(v, 4) \setminus S) \leq \frac{4}{v-4} (\frac{v-4}{4})\), we have a lower bound on the number of components for \(15 \leq v \leq 21\).

\[
\text{Table}[\text{Binomial}[v-5, 3], \{v, 15, 21\}]\]

\[
(120, 165, 220, 286, 364, 455, 560)\]

The following is an upper bound on the number of components for \(15 \leq v \leq 21\) using the Hilton-Milner Theorem (otherwise we are done by Theorem 5.3.3).

\[
\text{Table}[\text{Binomial}[v-1, 4-1] - \text{Binomial}[v-5, 4-1] + 1, v, 15, 21]\]

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For $15 \leq v \leq 21$, the bounds on the number of components for each case, with the assumption that $K(v, 4)$ is not $(\frac{v}{4} - 1)$-tough, satisfies the condition of Lemma 5.4.2 that $\binom{v}{4} - |S| > \binom{v}{4} - (\frac{v}{4} - 1) c > 2c$, where $c = c(K(v, 4) \setminus S)$.

Therefore, we can apply Theorem 5.4.1 with $a = (c-1)$ and $b = \binom{v}{4} - c + 1 - |S|$ by Lemma 5.4.2. As $x = \frac{6(8+v)(5+v)}{(3+v)(2+v)^2}$, Theorem 5.4.1 states

$$|S| \geq x(c-1) \left( \frac{v}{4} - (c + 1 - |S|) \right) \iff |S| \geq \frac{x(c-1) \left( \binom{v}{4} - c + 1 \right)}{1 + x(c-1)}.$$

Therefore, if we show the right side of the last inequality exceeds the right side of inequality (A.1) for suitable $c$, we obtain a contradiction and the corresponding value of $v$ has $t(K(v, 4)) = \frac{v}{4} - 1$.

For $v = 15$,

Reduce \[:: 1564c \leq \text{Binomial}[v - 1, 4 - 4] - \text{Binomial}[v - 5, 4 - 4] + 146c > \text{Binomial}[v - 5, 3] + 6(8+v)(5+v) \left( \frac{\text{Binomial}[v, 4] - v + 1)(c-1)}{1 + x(c-1)} \right) \cdot x > (v-4)(4) + c + v, \text{Integers}\]

$(c \equiv 121)c \equiv 122)c \equiv 124)c \equiv 125)c \equiv 127)c \equiv 129)c \equiv 130)c \equiv 131)c \equiv 132)c \equiv 133)c \equiv 134)c \equiv 136)c \equiv 137)c \equiv 138)c \equiv 140)c \equiv 141)c \equiv 142)c \equiv 143)c \equiv 144)c \equiv 145)c \equiv 146)c \equiv 148)c \equiv 150)c \equiv 151)c \equiv 152)c \equiv 154)c \equiv 155)c \equiv 156)c \equiv 157)c \equiv 158)c \equiv 159)c \equiv 161)c \equiv 163)c \equiv 164)c \equiv 165)c \equiv 166)c \equiv 167)c \equiv 168)c \equiv 169)c \equiv 170)c \equiv 171)c \equiv 172)c \equiv 174)c \equiv 176)c \equiv 177)c \equiv 179)c \equiv 180)c \equiv 181)c \equiv 182)c \equiv 183)c \equiv 184)c \equiv 185)c \equiv 186)c \equiv 187)c \equiv 189)c \equiv 190)c \equiv 191)c \equiv 192)c \equiv 193)c \equiv 194)c \equiv 195)c \equiv 196)c \equiv 197)c \equiv 199)c \equiv 200)c \equiv 202)c \equiv 204)c \equiv 205)c \equiv 207)c \equiv 208)c \equiv 209)c \equiv 210)c \equiv 211)c \equiv 212)c \equiv 214)c \equiv 215)c \equiv 217)c \equiv 218)c \equiv 219)c \equiv 220)c \equiv 221)c \equiv 222)c \equiv 224)c \equiv 225)c \equiv 226)c \equiv 227)c \equiv 228)c \equiv 229)c \equiv 230)c \equiv 231)c \equiv 232)c \equiv 233)c \equiv 234)c \equiv 235)c \equiv 236)c \equiv 237)c \equiv 238)c \equiv 239)c \equiv 240)c \equiv 241)c \equiv 242)c \equiv 243)c \equiv 244)c \equiv 245)c \equiv 154)&v \equiv 15$.

For $v = 16$,

Reduce \[:: 1664c \leq \text{Binomial}[v - 1, 4 - 1] - \text{Binomial}[v - 5, 4 - 1] + 146c > \text{Binomial}[v - 5, 3] + 6(8+v)(5+v) \left( \frac{\text{Binomial}[v, 4] - v + 1)(c-1)}{1 + x(c-1)} \right) \cdot x > (v-4)(4) + c + v, \text{Integers}\]

$(c \equiv 166)c \equiv 167)c \equiv 168)c \equiv 169)c \equiv 170)c \equiv 171)c \equiv 172)c \equiv 173)c \equiv 174)c \equiv 175)c \equiv 176)c \equiv 177)c \equiv 178)c \equiv 179)c \equiv 180)c \equiv 181)c \equiv 182)c \equiv 184)c \equiv 185)c \equiv 186)c \equiv 187)c \equiv 188)c \equiv 189)c \equiv 191)c \equiv 192)c \equiv 193)c \equiv 194)c \equiv 195)c \equiv 196)c \equiv 197)c \equiv 198)c \equiv 199)c \equiv 200)c \equiv 201)c \equiv 202)c \equiv 203)c \equiv 204)c \equiv 205)c \equiv 206)c \equiv 207)c \equiv 208)c \equiv 209)c \equiv 210)c \equiv 211)c \equiv 212)c \equiv 213)c \equiv 214)c \equiv 215)c \equiv 216)c \equiv 217)c \equiv 218)c \equiv 219)c \equiv 220)c \equiv 221)c \equiv 222)c \equiv 223)c \equiv 224)c \equiv 225)c \equiv 226)c \equiv 227)c \equiv 228)c \equiv 229)c \equiv 230)c \equiv 231)c \equiv 232)c \equiv 233)c \equiv 234)c \equiv 235)c \equiv 236)c \equiv 237)c \equiv 238)c \equiv 239)c \equiv 240)c \equiv 241)c \equiv 242)c \equiv 243)c \equiv 244)c \equiv 245)c$.
\[216]c = 217|c = 218|c = 219|c = 220|c = 221|c = 222|c = 223|c = 224|c = 225|c = 226|c = 227|c = 228|c = 229|c = 230|c = 231|c = 232|c = 233|c = 234|c = 235|c = 236|c = 237|c = 238|c = 239|c = 240|c = 241|c = 242|c = 243|c = 244|c = 245|c = 246|c = 247|c = 248|c = 249|c = 250|c = 251|c = 252|c = 253|c = 254|c = 255|c = 256|c = 257|c = 258|c = 259|c = 260|c = 261|c = 262|c = 263|c = 264|c = 265|c = 266|c = 267|c = 268|c = 269|c = 270|c = 271|c = 272|c = 273|c = 274|c = 275|c = 276|c = 277|c = 278|c = 279|c = 280|c = 281|c = 282|c = 283|c = 284|c = 285|c = 286|c = 287|c = 288|c = 289|c = 290|c = 291|c = 292|c = 293|c = 294|c = 295|c = 296|c = 297|c = 298|c = 299|c = 300|c = 301|c = 302|c = 303|c = 304|c = 305|c = 306|c = 307|c = 308|c = 309|c = 310|c = 311|c = 312|c = 313|c = 314|c = 315|c = 316|c = 317|c = 318|c = 319|c = 320|c = 321|c = 322|c = 323|c = 324|c = 325|c = 326|c = 327|c = 328|c = 329|c = 330|c = 331|c = 332|c = 333|c = 334|c = 335|c = 336|c = 337|c = 338|c = 339|c = 340|c = 341|&v = 16

For \( v = 17 \),

\[
\text{Reduce} \left\{ 1 \leq 17 & k \leq 17 \binom{n - 1, 4 - 1} - \binom{n - 5, 4 - 1} + 1 \leq k \leq 17 \binom{n - 5, 3} & k \left( \frac{\binom{n - 4 - 1, 5 - 1} + 1}{1 + 1} \right) * v > (v - 4) / 4 * v, \text{Integers} \right\}
\]

\[
(c = 221|c = 222|c = 223|c = 224|c = 225|c = 226|c = 227|c = 228|c = 229|c = 230|c = 231|c = 232|c = 233|c = 234|c = 235|c = 236|c = 237|c = 238|c = 239|c = 240|c = 241|c = 242|c = 243|c = 244|c = 245|c = 246|c = 247|c = 248|c = 249|c = 250|c = 251|c = 252|c = 253|c = 254|c = 255|c = 256|c = 257|c = 258|c = 259|c = 260|c = 261|c = 262|c = 263|c = 264|c = 265|c = 266|c = 267|c = 268|c = 269|c = 270|c = 271|c = 272|c = 273|c = 274|c = 275|c = 276|c = 277|c = 278|c = 279|c = 280|c = 281|c = 282|c = 283|c = 284|c = 285|c = 286|c = 287|c = 288|c = 289|c = 290|c = 291|c = 292|c = 293|c = 294|c = 295|c = 296|c = 297|c = 298|c = 299|c = 300|c = 301|c = 302|c = 303|c = 304|c = 305|c = 306|c = 307|c = 308|c = 309|c = 310|c = 311|c = 312|c = 313|c = 314|c = 315|c = 316|c = 317|c = 318|c = 319|c = 320|c = 321|c = 322|c = 323|c = 324|c = 325|c = 326|c = 327|c = 328|c = 329|c = 330|c = 331|c = 332|c = 333|c = 334|c = 335|c = 336|c = 337|c = 338|c = 339|c = 340|c = 341|c = 342|c = 343|c = 344|c = 345|c = 346|c = 347|c = 348|c = 349|c = 350|c = 351|c = 352|c = 353|c = 354|c = 355|c = 356|c = 357|c = 358|c = 359|c = 360|c = 361|c = 362|c = 363|c = 364|c = 365|c = 366|c = 367|c = 368|c = 369|c = 370|c = 371|c = 372|c = 373|c = 374|c = 375|c = 376|c = 377|c = 378|c = 379|c = 380|c = 381|c = 382|c = 383|c = 384|c = 385|c = 386|c = 387|c = 388|c = 389|c = 390|c = 391|c = 392|c = 393|c = 394|c = 395|&v = 18
\]
For \( v = 19 \),

\[
\binom{v}{c} = \frac{(v-1)!(v-c)!}{c!(v-c-1)!} \geq \frac{\binom{v-4}{c}}{c!(v-c-1)!} \geq \frac{\binom{v-5}{c}}{c!(v-c-1)!} > 0 \quad \text{for } c \geq 19
\]

For \( v = 20 \),

\[
\binom{v}{c} = \frac{(v-1)!(v-c)!}{c!(v-c-1)!} \geq \frac{\binom{v-4}{c}}{c!(v-c-1)!} \geq \frac{\binom{v-5}{c}}{c!(v-c-1)!} > 0 \quad \text{for } c \geq 20
\]

For \( v = 21 \),

\[
\binom{v}{c} = \frac{(v-1)!(v-c)!}{c!(v-c-1)!} \geq \frac{\binom{v-4}{c}}{c!(v-c-1)!} \geq \frac{\binom{v-5}{c}}{c!(v-c-1)!} > 0 \quad \text{for } c \geq 21
\]