Connectivity concerning the last two subconstituents of a $Q$-polynomial distance-regular graph

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Abstract

Let $\Gamma$ be a $Q$-polynomial distance-regular graph of diameter $d \geq 3$. Fix a vertex $\gamma$ of $\Gamma$ and consider the subgraph induced on the union of the last two subconstituents of $\Gamma$ with respect to $\gamma$. We prove that this subgraph is connected.

1 Introduction

All the graphs considered here will be finite and undirected, with no loops nor multiple edges. We briefly review the key definitions and basic results involving distance-regular graphs. For other notations and definitions, see [5,6,14]. Let $\Gamma$ be a connected graph with vertex set $X$. For $x, y \in X$, the distance between $x$ and $y$ is denoted by $\partial(x, y)$, and any path between $x$ and $y$ of length $\partial(x, y)$ is called geodesic. The diameter $\max_{x, y \in X} \partial(x, y)$ of $\Gamma$ is denoted by $d$. The graph $\Gamma$ is called distance-regular whenever for all integers $0 \leq h, i, j \leq d$ there exists a nonnegative integer $p_{ij}^h$ such that for all $x, y \in X$ with $\partial(x, y) = h$,

$$p_{ij}^h = |\{z \in X : \partial(z, x) = i, \partial(z, y) = j\}|.$$

For the rest of this paper we assume that $\Gamma$ is distance-regular of diameter $d \geq 2$. Let $A_0, A_1, \ldots, A_d$ denote the distance matrices of $\Gamma$ (see [5, p.127]). Then $A_0, A_1, \ldots, A_d$ form a basis for a commutative semisimple $\mathbb{R}$-algebra $M$ known as the Bose-Mesner algebra of $\Gamma$. The algebra $M$ has a second basis $E_0, E_1, \ldots, E_d$ such that

$$E_iE_j = \delta_{ij}E_i \quad (0 \leq i, j \leq d),$$
$$I = E_0 + \cdots + E_d,$$
$$E_0 = |X|^{-1}J,$$

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where $I$ is the identity matrix and $J$ is the all ones matrix (see [5, Thm 2.6.1]). We refer to $E_0, E_1, \ldots, E_d$ as the primitive idempotents of $\Gamma$. The primitive idempotent $E_0$ is called trivial. The ordering $E_0, E_1, \ldots, E_d$ is said to be $Q$-polynomial whenever for $0 \leq i \leq d$ there exists a polynomial $q_i$ of degree $i$ such that $E_i = q_i(E_1)$ (where the matrix multiplication is done entry-wise). For a primitive idempotent $E$ of $\Gamma$, we say that $\Gamma$ is $Q$-polynomial with respect to $E$ whenever there exists a $Q$-polynomial ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents such that $E = E_1$. The graph $\Gamma$ is called $Q$-polynomial whenever it is $Q$-polynomial with respect to at least one primitive idempotent.

We now recall the antipodal property. Define a binary relation $\sim$ on $X$ such that for all $x, y \in X$, $x \sim y$ whenever $x = y$ or $\partial(x, y) = d$. The graph $\Gamma$ is called antipodal whenever $\sim$ is an equivalence relation. The graph $\Gamma$ is said to be primitive whenever $\Gamma$ is not bipartite nor antipodal (see [5, Thm 4.2.]). A long-standing conjecture of Bannai and Ito [1, p. 312] states that if $\Gamma$ is primitive and $d$ is sufficiently large, then $\Gamma$ is $Q$-polynomial. For more information about the $Q$-polynomial property, see [1,5] or [14, Chapter 5].

For any vertex $\gamma$ of $\Gamma$, there exist real numbers $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ such that $\theta^*_s \neq 0$ and $\theta^*_s \leq 0$. Then for any vertex $\gamma$ of $\Gamma$ the subgraph induced on $\cup_{i=s}^d \Gamma_i(\gamma)$ is connected [10]. In [10] the authors also prove that $s \geq d/2$ and pose the following problem.

**Problem 1.1** (Cioabă-Koolen [10]). Assume that $\Gamma$ is primitive and $d \geq 3$. Is it true that for any vertex $\gamma$, the subgraph induced on $\Gamma_{d-1}(\gamma) \cup \Gamma_d(\gamma)$ is connected?

In [10], this was shown to be true if $d \in \{3,4\}$. In this note, we show that it is true for all $d \geq 3$, provided that $\Gamma$ is $Q$-polynomial. We now state our main result.

**Theorem 1.1.** Let $\Gamma$ be a $Q$-polynomial distance-regular graph of diameter $d \geq 3$. Then for any vertex $\gamma$ of $\Gamma$ the subgraph induced on $\Gamma_{d-1}(\gamma) \cup \Gamma_d(\gamma)$ is connected.

The main tool for our proof is Terwilliger’s balanced set condition (see [21,22] or Theorem 2.1 in the next section). This condition has been used by Lewis [20] to prove that the girth is at most 6 for any $Q$-polynomial distance-regular graph of valency at least 3.

## 2 Proof of the main result

For a primitive idempotent $E$ of $\Gamma$, there exist real numbers $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ (called the dual eigenvalues of $\Gamma$ with respect to $E$) such that

$$E = |X|^{-1} \sum_{h=0}^d \theta^*_h A_h.$$  \hfill (2.1)

We equip the vector space $\mathbb{R}^X$ with an inner product such that $\langle u, v \rangle = u^t v$ for all $u, v \in \mathbb{R}^X$. For $x \in X$, let $\hat{x}$ denote the vector in $\mathbb{R}^X$ with $x$-coordinate 1 and all other coordinates 0. Equation (2.1) implies that
\[ \langle E\hat{x}, E\hat{y} \rangle = |X|^{-1} \theta_i^*, \]
where $i = \partial(x, y)$. The main tool for our proof is the following theorem.

**Theorem 2.1** (Terwilliger [21, 22]). Let $\Gamma$ be a distance-regular graph with diameter $d \geq 3$, and let $E$ denote a nontrivial primitive idempotent of $\Gamma$ with dual eigenvalues $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$. Then $\Gamma$ is $Q$-polynomial with respect to $E$ if and only if $\theta_0^* \notin \{\theta_1^*, \ldots, \theta_d^*\}$ and
\[ \sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} E\hat{z} - \sum_{w \in \Gamma_i(x) \cap \Gamma_j(y)} E\hat{w} = p_{ij}^h \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_h^*} (E\hat{x} - E\hat{y}) \]
for all integers $h, i, j$ with $1 \leq h \leq d$ and $0 \leq i, j \leq d$ and all vertices $x, y$ with $\partial(x, y) = h$. Furthermore, if the conditions above hold, then $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ are mutually distinct.

The equation (2.3) is usually called the balanced set condition. We are now ready to give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Let $E$ be a primitive idempotent of $\Gamma$ with respect to which $\Gamma$ is $Q$-polynomial. We will use a proof by contradiction, and assume that there exists $\gamma \in X$ such that the subgraph induced on $\Gamma_{d-1}(\gamma) \cup \Gamma_d(\gamma)$ is disconnected. Let $C$ be the vertex set of a connected component of the subgraph induced on $\Gamma_{d-1}(\gamma) \cup \Gamma_d(\gamma)$. Let the set $\Delta$ consist of the vertices in $X$ that lie on a geodesic from $\gamma$ to $C$. The set $\Delta$ is properly contained in $X$, since $C \neq \Gamma_{d-1}(\gamma) \cup \Gamma_d(\gamma)$. We partition $\Delta = \cup_{j=0}^{d} \Delta_j$ where $\Delta_j = \Delta \cap \Gamma_j(\gamma)$ for $0 \leq j \leq d$. Note that for $0 \leq j \leq d - 1$, each vertex in $\Delta_j$ has at least one neighbor in $\Delta_{j+1}$.

A vertex in $\Delta$ will be called a *border* whenever it is adjacent to a vertex in $X \setminus \Delta$. Since $\Delta \neq X$ and $\Gamma$ is connected, $\Delta$ contains at least one border vertex. Let $t$ denote the maximal integer $j$ ($0 \leq j \leq d$) such that $\Delta_j$ contains a border vertex. By the construction $1 \leq t \leq d - 2$.

Pick a border vertex $z \in \Delta_t$. There exists $x \in \Delta_{t+2}$ such that $\partial(x, z) = 2$. Let $y \in X \setminus \Delta$ be a neighbor of $z$. Define $\xi = \partial(\gamma, y)$. By the triangle inequality $\xi \in \{t - 1, t, t + 1\}$. Note that $\xi \neq t - 1$; otherwise $y$ is on a geodesic from $\gamma$ to $C$ passing through $z$, forcing $y \in \Delta$ for a contradiction. Therefore $\xi = t$ or $\xi = t + 1$. By a routine argument using the maximality of $t$ and the definition of $\Delta$, we obtain $\partial(x, y) = 3$. Note that $\Gamma_1(x) \cap \Gamma_2(y) \subset \Gamma_{t+1}(\gamma)$ and $\Gamma_2(x) \cap \Gamma_1(y) \subset \Gamma_t(\gamma)$.

We apply the balanced set condition (2.3) to $x$ and $y$ using $h = 3, i = 1, j = 2$ and then take the inner product of each side with $E\hat{\gamma}$; this gives
\[ p_{12}^3 (\theta_{t+1}^* - \theta_t^*) = p_{12}^3 \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_3^*} (\theta_{t+2}^* - \theta_{t+1}^*). \]

There exists $y' \in \Gamma_{t-1}(\gamma) \cap \Gamma_1(z)$. We have $\partial(x, y') = 3$ and $\Gamma_1(x) \cap \Gamma_2(y') \subset \Gamma_{t+1}(\gamma)$ and $\Gamma_2(x) \cap \Gamma_1(y') \subset \Gamma_t(\gamma)$. We apply the balanced set condition (2.3) to $x$ and $y'$ using $h = 3, i = 1, j = 2$ and then take the inner product of each side with $E\hat{\gamma}$; this gives
\[ p_{12}^3 (\theta_{t+1}^* - \theta_t^*) = p_{12}^3 \frac{\theta_1^* - \theta_2^*}{\theta_0^* - \theta_3^*} (\theta_{t+2}^* - \theta_{t-1}^*). \]

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Comparing (2.4) and (2.5) we obtain $\theta^*_{\xi} = \theta^*_{t}$. We have $\xi = t - 1$ since $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ are mutually distinct. We mentioned earlier that $\xi \neq t - 1$, for a contradiction. We conclude that the subgraph induced on $\Gamma_{d-1}(\gamma) \cup \Gamma_d(\gamma)$ is connected.

To see how Theorem 1.1 is best possible, assume that $\Gamma$ is the Odd graph $O_{d+1}$ with $d \geq 3$. Recall that the vertices of $\Gamma$ are the $d$-subsets of a set $\Omega$ of size $2d + 1$. Two vertices $\alpha$ and $\beta$ are adjacent whenever $\alpha \cap \beta = \emptyset$. The diameter of $\Gamma$ is $d$ and its intersection numbers are known (see [2] or [5, Prop 9.1.7]). For $0 \leq h \leq d$, we have $p^h_{1h} = 0$ if $h < d$ and $p^h_{1h} = \lceil \frac{d+1}{2} \rceil$ if $h = d$. So with respect to any vertex of $\Gamma$, the $h$-subconstituent has no edges if $h < d$ and is regular with valency $\lceil \frac{d+1}{2} \rceil$ if $h = d$.

**Lemma 2.2.** Assume that $\Gamma$ is the Odd graph $O_{d+1}$ with $d \geq 3$. For any $\gamma \in X$, the number of connected components in the $d$-subconstituent of $\Gamma$ with respect to $\gamma$ is equal to $\left( \begin{array}{c} d \\ m \end{array} \right)$, where $m = d/2$ if $d$ is even and $m = (d + 1)/2$ if $d$ is odd. Moreover, this $d$-subconstituent is not connected.

**Proof.** Using the intersection numbers of $\Gamma$ we obtain $|\Gamma_d(\gamma)| = \left( \begin{array}{c} d \\ m \end{array} \right) \left( \begin{array}{c} d+1 \\ m \end{array} \right)$. Using the results of N. Biggs [2], each connected component of $\Gamma_d(\gamma)$ is isomorphic to the bipartite double (see [5, Section 1.11]) of $O_r+1$, where $r = d/2$ if $d$ is even and $r = (d - 1)/2$ if $d$ is odd. This bipartite double has $2^\left( \begin{array}{c} r+1 \\ r \end{array} \right)$ vertices. The result follows after some routine algebra.

Note also that for $O_{d+1}$ the subgraph induced on $\Gamma_1(\gamma) \cup \Gamma_2(\gamma)$ is disconnected. Next assume that $\Gamma$ is the folded $(2d+1)$-cube. It has diameter $d$ and for $1 \leq h \leq d - 1$, the $h$-subconstituent of $\Gamma$ with respect to any vertex has no edges (see [5, p. 264]), and consequently not connected. Gardiner, Godsil, Hensel and Royle [16] proved that the diameter of the second subconstituent of a primitive strongly-regular graph is at most three. It would be interesting to extend this result to distance-regular graphs with diameter $d \geq 3$. For example, if $\Gamma$ is a distance-regular with $d = 3$, then what is the diameter of $\Gamma_3(\gamma)$ when $\Gamma_3(\gamma)$ is connected? Another related problem from [10] is to classify the distance-regular graphs $\Gamma$ of diameter 3 such that $\Gamma_3(\gamma)$ is disconnected for some vertex $\gamma$. See [19] for related results.

The vertex-connectivity of a primitive distance-regular graph is equal to its valency, as proved by Brouwer and Mesner [8] for diameter $d = 2$, and by Brouwer and Koolen [7] for $d \geq 3$. Brouwer and Haemers [6, p. 127] observed that for certain strongly-regular graphs constructed by Haemers [17, p. 76] the vertex-connectivity of their second subconstituent is strictly less than the valency. It would be interesting to determine lower bounds for the vertex-connectivity and edge-connectivity of the subconstituents for a distance-regular graph with $d \geq 3$. See [3, 11–13, 15, 18] for related connectivity results concerning distance-regular graphs and association schemes.

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