Homework 1
Math 688 Graduate Combinatorics I
Due Monday, September 9 in class

To get 100% on this homework, you must solve correctly the 5 questions for a total of 10 points.

1. **(2 points)** A sequence $\mathbf{d} = (d_1, \ldots, d_n)$ is called *graphic* if there is a simple graph with degree sequence $\mathbf{d}$. Show that:
   (a) The sequences $(6, 6, 4, 4, 3, 3, 2)$ and $(5, 4, 4, 3, 3, 2, 2, 1, 1, 1)$ are graphic.
   (b) The sequences $(7, 6, 5, 4, 3, 3, 2)$ and $(6, 6, 5, 4, 3, 3, 1)$ are not graphic.
   (c) If $\mathbf{d} = (d_1, \ldots, d_n)$ is graphic and $d_1 \geq d_2 \geq \cdots \geq d_n$, then $\sum_{i=1}^{n} d_i$ is even and
   $$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(k, d_i), \forall k, 1 \leq k \leq n.$$

2. **(2 points)** The n-cube Q_n $(n \geq 1)$ is the graph whose vertex set is the set of all n-tuples of 0s and 1s, where two n-tuples are adjacent if they differ in precisely one coordinate.
   (a) Draw Q_1, Q_2, Q_3 and Q_4. Use SageMath `sagemath.org` to draw the graphs
   (b) Determine the order and the size of Q_n.
   (c) Prove that Q_n is bipartite for all $n \geq 1$.

3. **(2 points)** Prove that is a graph has 10 vertices and at least 37 edges, then it is connected.

4. **(2 points)** Let G be a connected graph on $n \geq 2$ vertices. Prove that G contains two distinct vertices that have the same degree. How many connected graphs G on n vertices are such that the degree sequence of G contains every integer between 1 and $n - 1$?

5. **(2 points)** Let X be a graph with n vertices and e edges. Show that there exists at least one edge uv such that $d(u) + d(v) \geq \dfrac{4e}{n}$. If X does not contain K_3's, prove $e \leq \lfloor \frac{n^2}{4} \rfloor$. Give an example of a graph on n vertices containing no K_3's with $\lfloor \frac{n^2}{4} \rfloor$ edges.

Homework 2
Math 688 Graduate Combinatorics I
Due Wednesday, September 18 in class

To get 100% on this homework, you must solve correctly the 5 questions for a total of 10 points.

1. **(2 points)** Give a combinatorial and an algebraic proof of the identity below for any natural number n:
$$\sum_{k=1}^{n} k^2 \binom{n}{k} = n2^{n-1} + n(n-1)2^{n-2}.$$

2. **(2 points)** Consider walks in the cartesian plane where each step is either $R : (x, y) \to (x + 1, y)$ or $U : (x, y) \to (x, y + 1)$. Show that the number of walks (where each step is R or U) from $(0, 0)$ to (m, n) equals $\binom{m+n}{m}$. If $m = n$ and in addition, no walk can go above the line $y = x$, how many walks are there from $(0, 0)$ to (n, n) ?

3. **(2 points)** Give a combinatorial and an algebraic proof of the identity below for any natural number n and non-negative integers a and b:
$$\binom{n+1}{a+b+1} = \sum_{k=0}^{n} \binom{k}{a}\binom{n-k}{b}.$$

4. **(2 points)** If $f(x)$ is the generating function for the sequence $(a_n)_{n\geq 0}$, determine the generating function for the following sequences:

   (a) $(a_{n+1})_{n\geq 0}$ and $(a_{n+2})_{n\geq 0}$.

   (b) $(na_n)_{n\geq 0}$.

   Use part 1 to determine the generating function for the Fibonacci sequence $(F_n)_{n\geq 0}$ defined by $F_0 = F_1 = 1, F_n = F_{n-1} + F_{n-2}, n \geq 2$. Use part 2 to determine the generating function for the sequence $(b_n)_{n\geq 0}$ defined by $b_0 = 1$ and $b_{n+1} = 2b_n + n$ for $n \geq 0$. Determine formulas for both F_n and b_n for $n \geq 0$.

5. **(2 points)** Show that the number of solutions $(x_1, \ldots, x_k)$ of the equation
$$x_1 + \cdots + x_k = n$$
in positive integers is $\binom{n-1}{k-1}$. A solution of this equation is called a composition of n into k parts. Show that the total number of compositions of n is 2^{n-1}. Prove that for $n \geq 4$, in the list of all compositions of n, the number 3 appears exactly $n \cdot 2^{n-5}$ times.

Homework 3
Math 688 Graduate Combinatorics I
Due Wednesday, October 9 in class

To get 100% on this homework, you must solve correctly the 5 questions for a total of 10 points.

1. **(2 points)** Let A be a real and symmetric $n \times n$ matrix. For each eigenvalue θ of A, let U_θ be a matrix whose columns form an orthonormal basis for the eigenspace corresponding to θ and denote $E_\theta := U_\theta U_\theta^T$. The matrices E_θ are called the principal idempotents of A. The set of (distinct) eigenvalues of A is denoted by $ev(A)$. Prove that

   (a) $E_\theta^2 = E_\theta$ and $E_\theta E_\tau = 0$ if $\theta \neq \tau$.

   (b) $A = \sum_{\theta \in ev(A)} \theta E_\theta$ and $I = \sum_{\theta \in ev(A)} E_\theta$.

   (c) For any polynomial p, $p(A) = \sum_{\theta \in ev(A)} p(\theta) E_\theta$.

   (d) For any rational function f which is defined at every eigenvalue of A, $f(A) = \sum_{\theta \in ev(A)} f(\theta) E_\theta$.

   (e) Each E_θ is a polynomial in A.

2. **(2 points)** Let $G = ([n], E)$ be an undirected simple graph on n vertices with adjacency matrix A whose eigenvalues are $\lambda_1 \geq \cdots \geq \lambda_n$. Let H be the graph obtained from G by removing vertex 1.

   (a) If $p(G, x)$ denotes the characteristic polynomial of the adjacency matrix of G and E_θ are the principal idempotents of A, prove that

   $$\frac{p(H,x)}{p(G,x)} = ((xI - A)^{-1})_{1,1} = \sum_{\theta \in ev(A)} \frac{(E_\theta)_{1,1}}{x - \theta},$$

   for any $x \notin ev(A)$.

   (b) If $\mu_1 \geq \cdots \geq \mu_{n-1}$ are the eigenvalues of the adjacency matrix of H, prove that

   $$\lambda_j \geq \mu_j \geq \lambda_{j+1},$$

   for any $1 \leq j \leq n-1$.

3. **(2 points)** Let A be a real and symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and corresponding orthonormal eigenvectors $u_1, \ldots, u_n$. Prove that for any $1 \leq j \leq n$:

   (a)

   $$\lambda_j = \max_{\dim W = j} \min_{u \in W : u \neq 0} \frac{u^T A u}{u^T u}. \tag{1}$$

   (b)

   $$\lambda_j = \min_{\dim U = n-j+1} \max_{u \in U : u \neq 0} \frac{u^T A u}{u^T u}. \tag{2}$$

4. **(2 points)** Show that the largest eigenvalue of a graph is bounded below by its average valency, with equality holding if and only if the graph is regular. Deduce from this that we can determine if a graph is regular from the information provided by its characteristic polynomial.

5. **(2 points)** We say that a graph G is regular if all its vertices have the same degree. Let $G = (V, E)$ be a connected graph with adjacency matrix A. Prove that G is regular if and only if there exists a polynomial p such that $p(A) = J$, where J is the $|V| \times |V|$ all one matrix.

Homework 4
Math 688 Graduate Combinatorics I
Due Wednesday, October 23 in class

To get 100% on this homework, you must solve correctly the 5 questions for a total of 10 points.

1. **(2 points)** Find the Prüfer code of the tree below. Determine the trees whose Prüfer codes are $(1, 2, 3, 4, 5, 6)$ and $(7, 5, 7, 5, 7, 5, 5)$.

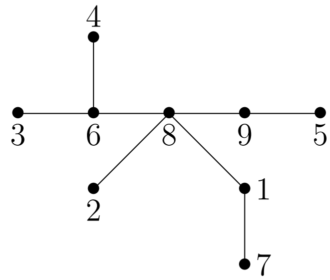

Figure 1: A tree on 9 vertices

2. **(2 points)** Let $t(n; d_1, \ldots, d_n)$ denote the number of labeled trees with vertex set $[n]$ whose degrees are $d_1, \ldots, d_n$. For example, $t(2; 1, 1) = 1, t(3; 2, 1, 1) = t(3; 1, 2, 1) = t(3; 1, 1, 2) = 1; t(4; 1, 3, 1, 1) = 3, t(4; 2, 1, 2, 1) = 2$. If $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$ and $d_1 + \cdots + d_n = 2n - 2$, show that $t(n; d_1, \ldots, d_n) = \frac{(n-2)!}{(d_1-1)!(d_2-1)!\ldots(d_n-1)!}$.

3. **(2 points)** Let T be a tree on $n \geq 2$ vertices with vertex set $V(T)$. A function $f : V(T) \to [n]$ is called *good* if the numbers $|f(x) - f(y)|$ computed for all the edges $xy \in E(T)$, are all different. A tree is called *good* if it has at least one good function. Show that the path on n vertices is good and that any tree on 6 vertices or less is good.

4. **(2 points)** Let Γ be a connected graph. The eccentricity of a vertex x is defined as the maximum of $d(x, y)$, where the maximum is taken over all vertices y in Γ. The center of Γ is the set formed by the vertices of minimum eccentricity. Show that if Γ is a tree, then its center consists of a singled vertex or two adjacent vertices.

5. **(2 points)** How many spanning trees does the complete bipartite graph $K_{a,b}$ have ? Let Γ be the graph obtained from K_n by removing one edge. How many spanning trees does Γ have ? Show your work.

Homework 5
Math 688 Graduate Combinatorics I
Due Wednesday, November 13 in class

To get 100% on this homework, you must solve correctly the 5 questions for a total of 10 points.

1. **(2 points)** Prove that a tree T contains a perfect matching if and only if $odd(T \setminus v) = 1$ for every vertex v of T, where $odd(T \setminus v)$ denotes the number of components of odd order in $T \setminus v$.

2. **(2 points)** Let $t \geq 1$ be an integer and Γ be a bipartite graph with color classes A and B such that
$$|\Gamma(S)| \geq t|S|$$
for any $S \subset A$. Show that for every vertex $x \in A$, there exists a set $L(x)$ of t neighbors of x such that $L(x) \cap L(y) = \emptyset$ for any $x \neq y \in A$.

3. **(2 points)** Let Γ be a graph with $n \geq k+1 \geq 2$ vertices and minimum degree $\delta \geq \frac{n+k-2}{2}$. Prove that G is k-connected.

4. **(2 points)** Find (with proof) the 3-regular graph of smallest order having connectivity 1. Find (with proof) the 3-regular graph of smallest order having no perfect matching.

5. **(2 points)** Consider the vertices $x = (0, \ldots, 0)$ and $y = (1, \ldots, 1)$ in the n-dimensional cube Q_n, where $n \geq 2$. Describe a maximum collection of internally-disjoint paths between x and y and a minimum disconnecting set of vertices separating x and y.

Exam 1
Math 688 Graduate Combinatorics I
Due Friday, September 27 in class

1. Let $G = (V, E)$ be an undirected simple graph with $n \geq 2$ vertices and e edges. Show that there is a partition of the vertex set $V = A \cup B$ such that the number of edges between A and B is at least $e/2$. If $n = 2k$ is even, show that there is a partition $V = C \cup D$ such that the number of edges between C and D is at least $\frac{k}{2k-1} \cdot e$.

2. For $n \geq 2$, let X_n be the graph whose vertices are the permutations in S_n with $\sigma \sim \tau$ if $\sigma\tau^{-1}$ is a transposition. Prove that the X_n is undirected ($\sigma \sim \tau$ implies $\tau \sim \sigma$), connected and determine the degree of each vertex. Show that if $\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_\ell$ are transpositions such that $\sigma_1 \ldots \sigma_k = \tau_1 \ldots \tau_\ell$, then k and ℓ have the same parity. Use this fact to prove that X_n is bipartite.

3. A sequence $(b_1, \ldots, b_{2n})$ is called a b-sequence if it contains n each of $+1$ and -1 and all the partial sums $a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_{2n-1}$ are non-negative. Let $(a_0, a_1, \ldots, a_{2n})$ be a sequence with $n+1$ terms equal to $+1$ and n terms equal to -1. Prove that all the $2n+1$ cyclic shifts $(a_j, a_{j+1}, \ldots, a_{j-1})$ for $0 \leq j \leq 2n$ (where the indices are taken modulo $2n+1$) are distinct. Show that exactly one of these cyclic shifts has the property that $a_j = 1$ and $(a_{j+1}, a_{j+2}, \ldots, a_{j-1})$ is a b-sequence. Use this fact to determine the number of b-sequences with $2n$ terms.

4. Determine the generating function of the sequence $(n^2)_{n\geq 0}$. Let $(a_n)_{n\geq 0}$ be a sequence defined as $a_0 = a_1 = 1$ and $a_n = a_{n-1} + (n-1)a_{n-2}$ for $n \geq 2$. If $f(t) = \sum_{n\geq 0} \frac{a_n t^n}{n!}$ is the exponential generating function of this sequence, prove that $f'(t) = (1+t)f(t)$ and $f(t) = e^{t+t^2/2}$.

5. Prove that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$. If $t \geq 1$ is fixed, calculate

$$\lim_{n\to\infty} \sqrt[n]{\sum_{k=0}^{n} \binom{n}{k}^t}.$$

Exam 2
Math 688 Graduate Combinatorics I
Due Monday, October 27 in class

1. Let $\Gamma = (V, E)$ be a graph on n vertices with adjacency matrix A. Let $V_1 \cup \cdots \cup V_r = V$ be a partition of its vertex set. The characteristic matrix P of this partition is the $n \times r$ matrix whose columns are the characteristic vectors of $V_1, \ldots, V_r$, respectively. This partition is called equitable if there exist number $b_{j,\ell}$ for $1 \leq j, \ell \leq r$ such that for any $j, \ell \in [r]$ and any vertex $x \in V_j$, the number of neighbors of x that are contained in V_ℓ, equals $b_{j,\ell}$ (does not depend on x, only on j and ℓ). The quotient matrix of this partition is $B = (b_{j,\ell})_{1 \leq j,\ell \leq r}$. If the partition is equitable, show that if v is an eigenvector of B with eigenvalue θ, then Pv is an eigenvector of A with eigenvalue θ. Show that the spectrum of A consists of the spectrum of the quotient matrix B (with eigenvectors in the column space of P, i.e., constant on the parts of the partition) together with the eigenvalues belonging to eigenvectors orthogonal to the columns of P (i.e., summing to zero on each part of the partition). Use this fact to determine the eigenvalues (including multiplicities) of the Petersen graph and of the friendship graph F_k (the graph on $2k+1$ vertices where k edge disjoint triangles share one vertex) for any $k \geq 1$.

2. Let $T \neq T'$ be two different trees on vertex set $[n]$ for $n \geq 3$.

   (a) Show that for any $e \in E(T) \setminus E(T')$, there exists $e' \in E(T') \setminus E(T)$ such that $(T \setminus \{e\}) \cup \{e'\}$ is a tree on n vertices.

   (b) Show that for any $e' \in E(T') \setminus E(T)$, there exists $e \in E(T) \setminus E(T')$ such that $(T \cup \{e'\}) \setminus \{e\}$ is a tree on n vertices.

3. Let $\Gamma = ([n], E)$ be a connected graph on $n \geq 2$ vertices. Let N be an arbitrary oriented incidence matrix of Γ.

   (a) Give a self-contained proof that $rank(N) = rank(NN^T) = n - 1$.

   (b) Give a self-contained proof showing that any square submatrix of N has determinant equal to $0, 1$ or -1.

   (c) Let $F \subseteq E$ be a subset of edges such that $|F| = n - 1$. Show that any square $(n-1) \times (n-1)$ submatrix of N whose columns correspond to the edges in F is invertible if and only if $([n], F)$ is a spanning tree of Γ.

4. For $n \geq 2$, consider a tree T on n vertices and specify one of its vertices as a *root*. Draw the tree in the plane without crossing edges. Imagine that the edges are walls perpendicular on the plane and starting at the root, walk around this system of walls, keeping the wall always to your right. Each time we move away from the root we write down a 1 and each time we move toward the root we write down a 0. This way we end up with a tree code, namely a sequence of length $2(n-1)$ consisting of 0s and 1s.

   (a) Does there exist an unlabeled tree with the code:

      i. 1111100000;

ii. 101010101010;

iii. 1100011100 ?

(b) For $n \geq 1$, let t_n denote the number of unlabeled trees on n vertices. Determine t_n for $1 \leq n \leq 7$. Show that for any $n \geq 3$, $\frac{n^{n-2}}{n!} < t_n < \binom{2n-2}{n-1}$. Prove that if $n > 30$, then $2^n < \frac{n^{n-2}}{n!}$. Prove that if $n \geq 2$, $\binom{2n-2}{n-1} < 4^{n-1}$. Thus, for $n > 30$, $2^n < t_n < 4^n$.

5. Let $\Gamma = (V, E)$ be a regular graph on $2n$ vertices. Let S be a subset of n vertices of Γ. Let Γ_1 be the graph obtained from Γ by adding a new vertex and making it adjacent to each vertex in S. Let Γ_2 be the graph obtained from Γ by adding a new vertex and making it adjacent to each vertex in $V \setminus S$. Show that the adjacency matrices of Γ_1 and Γ_2 are similar. Deduce that Γ_1 and Γ_2 have the same adjacency matrix eigenvalues. Construct two examples where Γ_1 and Γ_2 are non-isomorphic.

Exam 3
Math 688 Graduate Combinatorics I
Due Wednesday, November 20 in class

1. **(2 points)** Find a transversal of maximum weight in the matrix below. Prove that there is no larger weight transversal by exhibiting a solution to the dual (min-cover) problem.

$$\begin{bmatrix} 4 & 4 & 4 & 3 & 6 \\ 1 & 1 & 4 & 3 & 4 \\ 1 & 4 & 5 & 3 & 5 \\ 5 & 6 & 4 & 7 & 9 \\ 5 & 3 & 6 & 8 & 3 \end{bmatrix}.$$

2. **(2 points)** Determine the stable matching resulting from the Gale-Shapley algorithm run with men proposing and with women proposing given the preference lists below:

Men	$\{u, v, w, x, y, z\}$	Women	$\{a, b, c, d, e, f\}$
$u:$	$a > b > d > c > f > e$	$a:$	$z > x > y > u > v > w$
$v:$	$a > b > c > f > e > d$	$b:$	$y > z > w > x > v > u$
$w:$	$c > b > d > a > f > e$	$c:$	$v > x > w > y > u > z$
$x:$	$c > a > d > b > e > f$	$d:$	$w > y > u > x > z > v$
$y:$	$c > d > a > b > f > e$	$e:$	$u > v > x > w > y > z$
$z:$	$d > e > f > c > b > a$	$f:$	$u > w > x > v > z > y$

3. **(2 points)** Prove $K_{3,3}$ minus one edge is planar. Prove that K_5 minus one edge is planar. Show that the Petersen graph is not planar. If G is 3-regular graph plane graph whose faces are pentagons and hexagons, prove that the number of pentagons must be 12.

4. **(2 points)** Consider the graph G whose vertex set is formed by the set of integers modulo 13 where x is adjacent to y if and only if $x - y \in \{1, -1, 5 - 5\}$ (mod 13). Show that G is a triangle-free graph and determine its chromatic number.

5. **(2 points)** Construct a connected graph with vertex-connectivity 3, edge-connectivity 4 and minimum degree 5.