MATH888 Graduate Combinatorics II
Spring 2020

Lectures Mondays, Wednesdays, Fridays 9.05-9.55am ISE Lab 222
Instructor Dr. Sebastian Cioabă, `cioaba@udel.edu`, Ewing 506
Office Hours Mondays and Fridays 10-11am, Ewing 506 or by appointment.
Textbook There will be no textbooks for this class. However, I will use materials from various books such as:

- N. Alon and J. Spencer, *The Probabilistic Method*, 4th edition, Wiley.
- S. Ball and S. Weiner, *An Introduction to Finite Geometry*, available online at `https://mat-web.upc.edu/people/simeon.michael.ball/IFG.pdf`
- A.E. Brouwer and W. Haemers, *Spectra of Graphs*, Springer 2012.
- C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer GTM 2001.
- J.van Lint and R.M. Wilson, *A Course in Combinatorics* 2nd edition, Cambridge Univ. Press.

Topics

We will go deeper into topics and methods used in combinatorics such as algebraic combinatorics, extremal combinatorics, finite geometry, probabilistic methods, and if time permitting, topological methods.

Course Description

The objectives of this course are to study and understand combinatorics in deeper detail, learn some of the connections between combinatorics and other areas of mathematics such as algebra, linear algebra, probability, geometry and topology and some of the applications to other areas such as computer science and coding theory. Ideally, after taking this class, a student should have a solid knowledge of the fundamentals of combinatorics, be aware of some research highlights from recent years and should be able to attend a conference in discrete mathematics and understand the basic ideas and methods in most of the talks.

The lectures are important for your understanding of the material and even though attendance is not mandatory, I expect that with the exception of medical or other extraordinary circumstances, each student should attend every class.

Grading

- **Homework** (50% of the final grade). I will assign homework every week or so. The homework questions will be non-trivial and you will likely have to work hard to solve them, but you are encouraged to discuss them with me during office hours. For the homework problems, no joint work between the students is allowed.
- **Exams** (30% of the final grade). There will be three take-home exams that will take place during the weeks of March 9, April 6 and May 4. For the take-home exams, no joint work between the students is allowed.
- **Research Paper Presentation and Report** (20% of the final grade). Each student will be assigned a research paper. Each student should write a short report (at most 4 pages) summarizing the results and the ideas of the paper. Each student will give a 25 minutes presentation to the rest of the class. After each presentation, there will be 5 minutes for questions. The presentations will take place towards the end of the semester, the weeks of May 11 and May 18.
- Your letter grade for this course will be determined from the following scheme:

  A: 90-100, A-: 85-90,
  B+:80-85, B: 75-80, B-: 70-75,
  C+: 65-70, C: 60-65, C-:55-60,
  D: 50-55, F: 0-50

Homework 1
Math 888 Combinatorics II
Due Wednesday, February 19 in class.

To get 100% on this homework, you must solve correctly all 5 questions for a total of 10 points.

1. **(2 points)** Let $\mathbb{F}$ be a finite field of order q. Show that q is a prime power. Conversely, if q is a prime power, show that there exist a finite field of order q. If $\mathbb{F}$ is a finite field, prove that the multiplicative group of $\mathbb{F}$ is cyclic.

2. **(2 points)** If Γ is a strongly regular graph with parameters v, k, λ, μ, show that the complement of Γ is also strongly regular and determine its parameters in terms of v, k, λ, μ.

3. **(2 points)** Consider 25 ordered pairs (i, j) and other 25 ordered pairs $(i, j)'$, where $i, j \in \mathbb{Z}_5 = \mathbb{F}_5$. Make a graph whose vertices are these 50 ordered pairs and where we join

   (a) (i, j) with $(i, j + 1)$,

   (b) $(i, j)'$ with $(i, j + 2)'$,

   (c) (i, k) with $(j, ij + k)'$

   for any $i, j, k \in \mathbb{Z}_5$. Prove that the resulting graph is a strongly regular graph and determine its parameters v, k, λ, μ and eigenvalues. For each i, show that $(i, *)$ induces a C_5 and $(i, *)'$ induces a C_5. For each i, j, show that the union of $(i, *)$ and $(j, *)'$ induces a Petersen graph.

4. **(2 points)** For $n \geq 2$, a Latin square of order n is an $n \times n$ array with entries $1, \ldots, n$, where each row and each column is a permutation of $\{1, \ldots, n\}$. Equivalently, this is an $n \times n$ array where each $j \in [n]$ appears exactly once in each row and in each column. Given a Latin square L of order n, construct a graph Γ as follows. The vertices of Γ are the $n \times n$ coordinates $\{(i, j) : 1 \leq i, j \leq n\}$ where (a, b) is adjacent to (c, d) if $a = c$ or $b = d$ or $L_{a,b} = L_{c,d}$. Prove that Γ is a strongly regular graph and determine its parameters v, k, λ, μ and its eigenvalues.

5. **(2 points)** Consider the following two Latin squares of order 4:

$$L_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}, L_2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

   For $j \in \{1, 2\}$, let Γ_j be the strongly regular graph associated with L_j. Determine the parameters of the complement of Γ_1 and the complement of Γ_2 and show that they are not isomorphic. What is the complement of Γ_2 ?

Homework 2
Math 888 Combinatorics II
Due Wednesday, March 4 in class.

To get 100% on this homework, you must solve correctly all 5 questions for a total of 10 points.

1. **(2 points)** A generalized quadrangle $GQ(s,t)$ is an incidence structure of points and lines such that

   (a) each line contains $s+1$ points,

   (b) each point lies on $t+1$ lines,

   (c) any two points are on at most one line,

   (d) if P is a point not on a line ℓ, then there exists a unique point on ℓ collinear with P.

   The point-graph of a $GQ(s,t)$ is the graph with the points of the quadrangle as vertices, where two points are adjacent if they are collinear. Show that the point-graph is strongly regular and determine its parameters and eigenvalues in terms of s and t.

2. **(2 points)** Consider the point-line incidence structure of a 3×3 grid (9 points and 6 lines horizontal and vertical). Show this is $GQ(2,1)$ and determine the parameters of its point graph. Consider the point-line structure where the points are the edges of K_6 and the lines are the perfect matchings of K_6, where incidence means inclusion. Show that this is a $GQ(2,2)$ and determine the parameters of its point graph. Consider the graph whose vertices are the 16 vectors of $\mathbb{F}_2^5$ of even weight. Two vectors are adjacent if they differ in an odd number (1 or 3) of positions. Show that this graph is strongly regular and determine its parameters and eigenvalues.

3. **(2 points)** Consider a $t-(v,k,\lambda)$ design with a set of points $\mathcal{P}$. Let J be a subset of $\mathcal{P}$ with $|J| = j \leq t$. Show that the number of blocks that are disjoint from J equals $\frac{\lambda\binom{v-j}{k}}{\binom{v-t}{k-t}}$.

4. **(2 points)** Let $n = 2t+1 \geq 3$ be an odd integer. Define $\mathcal{P} = \mathbb{Z}_n \times \mathbb{Z}_3$. Consider the design with point set $\mathcal{P}$ whose blocks are triples of the form $\{(x,0),(x,1),(x,2)\}$ with $x \in \mathbb{Z}_n$ and all triples $\{(x,i),(y,i),\left(\frac{x+y}{2}, i+1\right)\}$ with $x \neq y \in \mathbb{Z}_n$ and $i \in \mathbb{Z}_3$. Show that this block design is a $2-(6t+3,3,1)$ design.

5. **(2 points)** Let q be a prime power and $n \geq 1$ be an integer.Let $\mathbb{F}_q^n$ denote the n-dimensional vector space over the finite field $\mathbb{F}_q$. Denote by $\begin{bmatrix} n \\ k \end{bmatrix}_q$ the number of k-

dimensional subspaces of $\mathbb{F}_q^n$. Show that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)\dots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\dots(q-1)}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q .$$

Homework 3
Math 888 Combinatorics II
Due Wednesday, March 18 in class.

To get 100% on this homework, you must solve correctly all 5 questions for a total of 10 points.

1. **(2 points)** Let $\mathcal{O}$ be a subset of a projective plane of order k such that no three points of $\mathcal{O}$ are on a line. Show that $|\mathcal{O}| \leq k+1$ if k is odd and $|\mathcal{O}| \leq k+2$ if k is even. A set of $k+1$ points, no three on a line, is called an *oval.* A set of $k+2$ points, no three on a line, is called a *hyperoval.* For $k \in \{2, 3, 4\}$, find a subset $S_k = \{s_1, \dots, s_{k+1}\}$ of $\mathbb{Z}_{k^2+k+1}$ such that the elements of $\mathbb{Z}_{k^2+k+1}$ as points and the k^2+k+1 blocks $S + x = \{s_1 + x, \dots, s_{k+1} + x\}$ form a projective plane of order k. In each of the cases of the previous exercise, construct a hyperoval or an oval.

2. **(2 points)** Let $n \geq 2$ be an integer. A Hadamard matrix of order n is an $n \times n$ matrix H with entries $+1$ or -1 is called if $HH^T = nI_n$. A Hadamard matrix is called normalized if its first row is formed by all 1s. Show if H is a Hadamard matrix of order n then $n \in \{1, 2\}$ or $n \equiv 0 \pmod 4$. Show that if H is Hadamard matrix of order n and K is a Hadamard matrix of order m, then $H \otimes K$ is a Hadamard matrix of order nm.

3. **(2 points)** Let H be a normalized Hadamard matrix of order $4k$. Delete the first row and the first column. Identify points with the rows of this matrix. Each column defines a subset of the rows, namely those rows for which there is a $+1$ in that column. These subsets are the blocks. Prove that this is a $2-(4k-1, 2k-1, k-1)$-design. Consider the same matrix H, but now delete only the first row. Each of the other rows determines a $2k$-subset of the set of columns corresponding to the $+1$ entries. Show that this gives a $3-(4k, 2k, k-1)$ design.

4. **(2 points)** Let $n \geq 5$ be a integer. Denote by Γ the Cayley graph of the cyclic group $\mathbb{Z}_n$ with generating set $S = \{\pm 1, \pm 2\}$. Determine the eigenvalues of the adjacency matrix of this graph. What are the largest, 2nd largest and smallest eigenvalues ?

5. **(2 points)** Let $n \geq 2$ be an integer. Denote by Q_n the n-dimensional cube graph which is the Cayley graph of the abelian group $\mathbb{Z}_2^n$ whose generating set consists of the n vectors of weight 1. Determine the eigenvalues of Q_n and their multiplicities.

Homework 5
Math 888 Combinatorics II
Due Wednesday, May 6

1. **(2 points)** Let $n > s \geq 1$ be two integers. Let $\mathcal{F} \subset \mathcal{P}([n])$ be a collection of sets of $[n]$ such that $\mathcal{F}$ does not contain $s+1$ nested sets: $A_1 \subset A_2 \subset \cdots \subset A_{s+1}$. For $0 \leq k \leq n$, let $f_k = |\mathcal{F} \cap \binom{[n]}{k}|$. Show that
$$\sum_{k=0}^{n} \frac{f_k}{\binom{n}{k}} \leq s.$$
[This is a generalization of Sperner Theorem which is the case $s = 1$].

2. **(2 points)** Let $n \geq 1$ be an integer. A family $\mathcal{A} \subset \mathcal{P}([n])$ of subsets of $[n]$is called an ideal (or simplicial complex) if $A \in \mathcal{A}$ and $B \subset A$ imply that $B \in \mathcal{A}$. Use Local LYM inequality to prove that the average size of a set in $\mathcal{A}$ is at most $n/2$.

3. **(2 points)** Let $n \geq r \geq 1$ be two integers. For $1 \leq i \neq j \leq n$ and $A \subset [n]$, define the i,j-shift or compression $S_{i,j}(A)$ of A as follows:
$$S_{i,j}(A) = \begin{cases} (A \setminus \{j\}) \cup \{i\} & \text{if } i \notin A, j \in A \\ A, \text{ otherwise.} \end{cases}$$
For a collection of r-subsets $\mathcal{A} \subset \binom{[n]}{r}$, define
$$S_{i,j}(\mathcal{A}) = \{A : A \in \mathcal{A}, S_{i,j}(A) \in \mathcal{A}\} \cup \{S_{i,j}(A) : A \in \mathcal{A}, S_{i,j}(A) \notin \mathcal{A}\}.$$
Prove that
   (a) $|S_{i,j}(\mathcal{A})| = |\mathcal{A}|$.
   (b) If $\mathcal{A}$ is t-intersecting, then $\mathcal{A}$ is t-intersecting.
   (c) $|\partial\mathcal{A}| \geq |\partial S_{i,j}(\mathcal{A})|$.

4. **(2 points)** What is the value $b^{(4)}(14, 6, 5, 2)$? Describe the set $\mathcal{B}^{(4)}(14, 6, 5, 2)$. What is the 1001-th element of $\binom{\mathbb{N}}{4}$? Prove that if $\mathcal{A} \subset \binom{\mathbb{N}}{2}, |\mathcal{A} = 4$, then $|\partial A| \geq 4$. How many non-isomorphic collections $\mathcal{A}$ are there with $|\mathcal{A}| = |\partial\mathcal{A}| = 4$?

5. **(2 points)** Let $n \geq r \geq t \geq 1$ be three integers such that $n > 2r - t$. For $0 \leq s \leq r - t$, define
$$\mathcal{F}_s = \{A \in \binom{[n]}{r} : |A \cap [t + 2s]| \geq t + s\}.$$
Prove that
   (a) Each $\mathcal{F}_s$ is a t-intersecting collection of r-subsets. What is $\mathcal{F}_0$?
   (b) For each $0 \leq s \leq r - t$, $|\mathcal{F}_s| = \sum_{j=t+s}^{t+2s} \binom{t+2s}{j}\binom{n-t-2s}{r-j}$.
   (c) Determine the largest size of $\mathcal{F}_s$ when $(r, t) = (5, 3)$ and $n \in \{8, 11, 13\}$.

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Math 888 Combinatorics II
Due Wednesday, May 6

1. **(2 points)** Let $n > s \geq 1$ be two integers. Let $\mathcal{F} \subset \mathcal{P}([n])$ be a collection of sets of $[n]$ such that $\mathcal{F}$ does not contain $s+1$ nested sets: $A_1 \subset A_2 \subset \cdots \subset A_{s+1}$. For $0 \leq k \leq n$, let $f_k = |\mathcal{F} \cap \binom{[n]}{k}|$. Show that
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Prove that
   (a) $|S_{i,j}(\mathcal{A})| = |\mathcal{A}|$.
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   (a) Each $\mathcal{F}_s$ is a t-intersecting collection of r-subsets. What is $\mathcal{F}_0$?
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   (c) Determine the largest size of $\mathcal{F}_s$ when $(r, t) = (5, 3)$ and $n \in \{8, 11, 13\}$.