Homework 1
Math 688 Graduate Combinatorics I
Fall 2020
Due Friday, September 11

To get 100% on this homework, you must correctly solve the 5 questions below for a total of 10 points.

1. **(2 points)** Let Γ be a connected graph that contains exactly two vertices of odd degrees. Prove that there exists a walk from one vertex of odd degree to the other vertex of odd degree that goes through every edge exactly once.

2. **(2 points)** If $\mathbf{d} = (d_1, \ldots, d_n)$ is graphic and $d_1 \geq d_2 \geq \cdots \geq d_n$, then $\sum_{i=1}^{n} d_i$ is even and for any $1 \leq k \leq n$,

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(k, d_i).$$

   Does the converse hold ?

3. **(2 points)** Let n be a natural number. Show that if a graph has n vertices and $\binom{n-1}{2}+1$ edges, then it must be connected.

4. **(2 points)** Let Γ be a connected graph on $n \geq 2$ vertices. Prove that Γ contains two distinct vertices that have the same degree. Prove that there is exactly one connected graph on n vertices whose degree sequence contains every integer between 1 and $n-1$.

5. **(2 points)** Let Γ be a graph with n vertices and e edges. Show that there exists at least one edge uv such that

$$d(u) + d(v) \geq 4e/n.$$

   If Γ does not contain any triangles, prove that

$$e \leq \lfloor n^2/4 \rfloor.$$

   For every $n \geq 2$, give an example of a graph on n vertices and $\lfloor n^2/4 \rfloor$ edges that contains no triangles.

   Reward

   After you finish the homework, you can relax playing this game:

   `https://apps.apple.com/us/app/one-touch-drawing/id494856551`

Homework 2
Math 688 Graduate Combinatorics I
Due Wednesday, September 23

To get 100% on this homework, you must solve correctly the 5 questions for a total of 10 points.

1. **(2 points)** Give a combinatorial and an algebraic proof of the identity below for any natural number n:
$$\sum_{k=1}^{n} k^2 \binom{n}{k} = n2^{n-1} + n(n-1)2^{n-2}.$$

2. **(2 points)** Consider walks in the cartesian plane where each step is either $R : (x, y) \to (x + 1, y)$ or $U : (x, y) \to (x, y + 1)$. Show that the number of walks (where each step is R or U) from $(0, 0)$ to (m, n) equals $\binom{m+n}{m}$. If $m = n$ and in addition, no walk can go above the line $y = x$, how many walks are there from $(0, 0)$ to (n, n) ?

3. **(2 points)** Let $n \geq 3$ be a natural number. Prove that any permutation in S_n can be written as a product of transpositions from each of the following sets:
   (a) $\{(1, 2), (1, 3), \ldots , (1, n)\}$,
   (b) $\{(1, 2), (2, 3), \ldots , (n - 1, n)\}$.

4. **(2 points)** If $f(x)$ is the generating function for the sequence $(a_n)_{n\geq 0}$, determine the generating function for the following sequences:
   (a) $(a_{n+1})_{n\geq 0}$ and $(a_{n+2})_{n\geq 0}$.
   (b) $(na_n)_{n\geq 0}$.

   Use part 1 to determine the generating function for the Fibonacci sequence $(F_n)_{n\geq 0}$ defined by $F_0 = F_1 = 1, F_n = F_{n-1} + F_{n-2}, n \geq 2$. Use part 2 to determine the generating function for the sequence $(b_n)_{n\geq 0}$ defined by $b_0 = 1$ and $b_{n+1} = 2b_n + n$ for $n \geq 0$. Determine formulas for both F_n and b_n for $n \geq 0$.

5. **(2 points)** Show that the number of solutions $(x_1, \ldots , x_k)$ of the equation
$$x_1 + \cdots + x_k = n$$
   in positive integers is $\binom{n-1}{k-1}$. A solution of this equation is called a composition of n into k parts. Show that the total number of compositions of n is 2^{n-1}. Prove that for $n \geq 4$, in the list of all compositions of n, the number 3 appears exactly $n \cdot 2^{n-5}$ times.

Homework 3
Math 688 Graduate Combinatorics I
Due Wednesday, October 7 in class

To get 100% on this homework, you must solve correctly the 5 questions for a total of 10 points.

1. **(2 points)** Let $s(n,k)$ denote the Stirling number of the first kind. Prove that for any natural number n and any real number t,
$$t(t+1)\dots(t+n-1) = \sum_{k=1}^{n} |s(n,k)| t^k.$$

2. **(2 points)** Let $S(n,k)$ denote the Stirling number of the second kind. If $n \geq k$ are natural numbers, show that
$$S(n+1,k) = \sum_{j=k-1}^{n} \binom{n}{j} S(j,k-1).$$
Prove that $S(n,k) \geq \binom{n}{k-1}$.

3. **(2 points)** Define $p_k(n)$ as the number of integer solutions $(x_1,\dots,x_k)$ of
$$n = x_1 + \dots + x_k, x_1 \geq \dots \geq x_k \geq 1.$$
For example, $7 = 5+1+1 = 4+2+1 = 3+3+1 = 3+2+2$ so $p_3(7) = 4$. If k is fixed, prove that
$$\lim_{n\to\infty} \frac{p_k(n)}{\frac{n^{k-1}}{k!(k-1)!}} = 1.$$

4. **(2 points)** Let A be a real and symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geq \dots \geq \lambda_n$ and corresponding orthonormal eigenvectors $u_1,\dots,u_n$. Prove that for any $1 \leq j \leq n$:

   (a)
$$\lambda_j = \max_{\dim W = j} \min_{u\in W: u\neq 0} \frac{u^T A u}{u^T u}. \tag{1}$$

   (b)
$$\lambda_j = \min_{\dim U = n-j+1} \max_{u\in U: u\neq 0} \frac{u^T A u}{u^T u}. \tag{2}$$

5. **(2 points)** Let A be a real and symmetric $n \times n$ matrix. For each eigenvalue θ of A, let U_θ be a matrix whose columns form an orthonormal basis for the eigenspace corresponding to θ and denote $E_\theta := U_\theta U_\theta^T$. The matrices E_θ are called the principal idempotents of A. The set of (distinct) eigenvalues of A is denoted by $ev(A)$. Prove that

(a) $E_\theta^2 = E_\theta$ and $E_\theta E_\tau = 0$ if $\theta \neq \tau$.

(b) $A = \sum_{\theta \in ev(A)} \theta E_\theta$ and $I = \sum_{\theta \in ev(A)} E_\theta$.

(c) For any polynomial p, $p(A) = \sum_{\theta \in ev(A)} p(\theta) E_\theta$.

(d) For any rational function f which is defined at every eigenvalue of A, $f(A) = \sum_{\theta \in ev(A)} f(\theta) E_\theta$.

(e) Each E_θ is a polynomial in A.

Homework 4
Math 688 Graduate Combinatorics I
Due Wednesday, October 21

To get 100% on this homework, you must solve correctly the 5 questions for a total of 10 points.

1. **(2 points)** Let $\Gamma = (V, E)$ be a graph. We choose an orientation of Γ by assigning a direction to each edge; for each edge, one endpoint is the head and the other is the tail. The oriented incidence matrix N of Γ is the $V \times E$ matrix defined as follows:

$$N(x, f) = \begin{cases} +1, \text{if } x \text{ is the tail of } f \\ -1, \text{if } x \text{ is the head of } f \\ 0, \text{ otherwise.} \end{cases}$$

   (a) Show that $NN^T = D - A$, where D is the diagonal degree matrix whose diagonal entries are the degrees of Γ and A is the adjacency matrix of Γ.

   (b) Let $L = D - A$ denote the matrix from above. It is called the Laplacian matrix of Γ. For any vector $u \in \mathbb{R}^V$, show that

$$u^T L u = \sum_{xy \in E} (u_x - u_y)^2.$$

   (c) Prove that 0 is the smallest eigenvalue of L.

   (d) Show that Γ is connected if and only if the multiplicity of 0 as an eigenvalue of Γ is 1.

2. **(2 points)** Find the Prüfer code of the tree below. Determine the trees whose Prüfer codes are $(1, 1, 2, 2, 3, 3)$ and $(7, 1, 7, 2, 7, 4, 5)$.

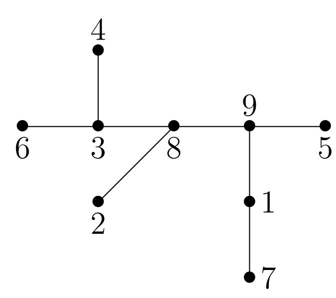


Figure 1: A tree on 9 vertices

3. **(2 points)** Let $t(n; d_1, \ldots, d_n)$ denote the number of labeled trees with vertex set $[n]$, where vertex j has degree d_j for any $1 \le j \le n$. If $d_1, d_2, \ldots, d_n \ge 1$ and $d_1 + \cdots + d_n = 2n - 2$, show that

$$t(n; d_1, \ldots, d_n) = \frac{(n-2)!}{(d_1 - 1)!(d_2 - 1)! \ldots (d_n - 1)!}.$$

4. **(2 points)** Let T be a tree on $n \geq 2$ vertices with vertex set $V(T)$. A function $f : V(T) \to [n]$ is called *good* if the numbers $|f(x) - f(y)|$ computed for all the edges $xy \in E(T)$, are all different. A tree is called *good* if it has at least one good function. Show that the path on n vertices is good and that any tree on 7 vertices or less is good.

5. **(2 points)** How many spanning trees does the complete bipartite graph $K_{a,b}$ have ? Let Γ be the graph obtained from K_n by removing one edge. How many spanning trees does Γ have ? Show your work.

Homework 5
Math 688 Graduate Combinatorics I
Due Friday, November 13

To get 100% on this homework, you must solve correctly the 5 questions for a total of 10 points.

1. **(2 points)** Let Γ be a graph with chromatic number k. Show that in any proper k-coloring of Γ, there is a vertex of each color which is adjacent to vertices of every other color. Deduce that Γ contains at least k vertices of degree at least $k-1$ and that Γ contains at least $\binom{k}{2}$ edges.

2. **(2 points)** Let $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ be two graphs with disjoint vertex sets.

   (a) The union $\Gamma + \Gamma'$ is the graph with vertex set $V \cup V'$ and edge set $E \cup E'$. Show that
   $$\chi(\Gamma + \Gamma') = \max(\chi(\Gamma), \chi(\Gamma')).$$

   (b) The join $\Gamma \vee \Gamma'$ is the graph obtained from the union $\Gamma + \Gamma'$ by adding all the edges with one endpoint in V and another in V'. Prove that
   $$\chi(\Gamma \vee \Gamma') = \chi(\Gamma) + \chi(\Gamma').$$

   (c) The cartesian product $\Gamma \Box \Gamma'$ is the graph with vertex set $V \times V'$ where (x, x') is adjacent to (y, y') if $x = y$ and x' is adjacent to y' in Γ' or x is adjacent to y in Γ and $x' = y'$. Prove that
   $$\chi(\Gamma \Box \Gamma') = \max(\chi(\Gamma), \chi(\Gamma')).$$

   (d) The weak product $\Gamma \times \Gamma'$ is the graph with vertex set $V \times V'$ where (x, x') is adjacent to (y, y') if x is adjacent to y in Γ and x' is adjacent to y' in Γ'. Prove that
   $$\chi(\Gamma \times \Gamma') \leq \min(\chi(\Gamma), \chi(\Gamma')).$$

3. **(2 points)** Let $n \geq r \geq 1$ be two natural numbers. The Turán graph $T_{n,r}$ is the complete r-partite graph with n vertices whose partite sets differ in size by at most 1. Prove that among r-partite graphs on n vertices, $T_{n,r}$ is the unique graph with the largest number of edges. Let $a = \lfloor n/r \rfloor$ and $b = n - ra$. Prove that $T_{n,r}$ has b partite sets of size $a+1$ and $r-b$ partite sets of size a. Show that
   $$e(T_{n,r}) = (1 - 1/r)n^2/2 - b(r-b)/2r$$
   and that
   $$e(T_{n,r}) \leq \lfloor (1 - 1/r)n^2/2 \rfloor.$$
   Determine when strict inequality occurs.

4. **(2 points)** Let $\Gamma = (V, E)$ be a graph on n vertices, e edges that does not contain any cycle on 4 vertices. Prove that

$$\sum_{x \in V} \binom{d(x)}{2} \leq \binom{n}{2}.$$

Using this inequality, show that $e < \frac{n^{3/2}}{2} + \frac{n}{4}$.

5. **(2 points)** Let $n \geq 6$. If Γ is a connected graph with n vertices and n edges, prove that Γ contains exactly one cycle. If Γ is a connected graph with n vertices and $n+4$ edges, show that Γ contains two edge-disjoint cycles.

Homework 6
Math 688 Graduate Combinatorics I
Due Friday, December 11

To get 100% on this homework, you must solve correctly the 5 questions for a total of 10 points.

1. **(2 points)** Prove that a tree T contains a perfect matching if and only if $odd(T \setminus v) = 1$ for every vertex v of T, where $odd(T \setminus v)$ denotes the number of components of odd order in $T \setminus v$.

2. **(2 points)** Let Γ be a bipartite graph with partite sets A and B.

   (a) If $|N(S)| > |S|$ for any $S \subseteq A$, prove that for any edge e of Γ, there exists a matching M_e of Γ such that M_e saturates A and $e \in M_e$.

   (b) If t is a natural number such that $|N(S)| \geq t|S|$ for any $S \subseteq A$, prove that for any $x \in A$, there exists a set L_x of t neighbors of x such that $L_x \cap L_y = \emptyset$ for any $x \neq y \in A$.

3. **(2 points)** Let m and n be two natural numbers. Consider two partitions of $[mn] = \{1, \ldots, mn\}$ into m sets of size n:

$$[mn] = A_1 \cup \ldots \cup A_m = B_1 \cup \ldots \cup B_m.$$

   Show that there is a permutation $\sigma \in S_m$ such that $A_j \cap B_{\sigma(j)} \neq \emptyset$ for any $1 \leq j \leq m$.

4. **(2 points)** Using the Hungarian algorithm, find a transversal of maximum weight in the matrix below. Prove that there is no larger weight transversal by exhibiting a solution to the dual (min-cover) problem.

$$\begin{bmatrix} 3 & 1 & 4 & 4 & 5 \\ 1 & 4 & 3 & 5 & 4 \\ 7 & 6 & 8 & 7 & 2 \\ 2 & 1 & 3 & 4 & 5 \\ 6 & 3 & 2 & 8 & 7 \end{bmatrix}.$$

5. **(2 points)** Determine the stable matching resulting from the Gale-Shapley algorithm run with men proposing and with women proposing given the preference lists below:

Men	$\{u, v, w, x, y, z\}$	Women	$\{a, b, c, d, e, f\}$
$u:$	$a > b > d > c > f > e$	$a:$	$z > x > y > u > v > w$
$v:$	$a > b > c > f > e > d$	$b:$	$y > z > w > x > v > u$
$w:$	$c > b > d > a > f > e$	$c:$	$v > x > w > y > u > z$
$x:$	$c > a > d > b > e > f$	$d:$	$w > y > u > x > z > v$
$y:$	$c > d > a > b > f > e$	$e:$	$u > v > x > w > y > z$
$z:$	$d > e > f > c > b > a$	$f:$	$u > w > x > v > z > y$

Exam 1
Math 688 Graduate Combinatorics I
Due Sunday, September 27, 2020.

1. Let Q be a set of size $q \geq 2$. If $d \geq 2$ is a natural number, define the graph Q_d whose vertices are all the ordered d-tuples $(x_1, \ldots, x_d)$ with $x_1, \ldots, x_d \in Q$, where $(x_1, \ldots, x_d)$ is adjacent to $(y_1, \ldots, y_d)$ if they differ in exactly one position, meaning that there exists ℓ between 1 and d such that $x_\ell \neq y_\ell$ and $x_j = y_j$ for any $j \neq \ell$.

   (a) How many vertices and edges does Q_d have ?

   (b) If $q = 2$ and $d \geq 2$, how many C_4s does Q_d have ?

   (c) When is the graph Q_d Eulerian ?

   (d) When is it bipartite ? Justify all your answers.

2. Let $\Gamma = (V, E)$ be an undirected simple graph with $n \geq 2$ vertices and e edges. Show that there is a partition of the vertex set $V = A \cup B$ such that the number of edges between A and B (with one endpoint in A and the other in B) is at least $e/2$. If $n = 2k$ is even, show that there is a partition $V = C \cup D$ such that the number of edges between C and D is at least $\frac{k}{2k-1} \cdot e$.

3. A sequence $(b_1, \ldots, b_{2n})$ is called a b-sequence if it contains n $+1$s and n -1s and all the partial sums $a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_{2n-1}$ are non-negative. Let $(a_0, a_1, \ldots, a_{2n})$ be a sequence with $n+1$ terms equal to $+1$ and n terms equal to -1. Prove that all the $2n+1$ cyclic shifts $(a_j, a_{j+1}, \ldots, a_{j-1})$ for $0 \leq j \leq 2n$ (where the indices are taken modulo $2n+1$) are distinct. Show that exactly one of these cyclic shifts has the property that $a_j = 1$ and $(a_{j+1}, a_{j+2}, \ldots, a_{j-1})$ is a b-sequence. Use this fact to determine the number of b-sequences with $2n$ terms.

4. Determine the generating function of the sequence $(n^2)_{n \geq 0}$. Let $(a_n)_{n \geq 0}$ be a sequence defined as $a_0 = a_1 = 1$ and $a_n = a_{n-1} + (n-1)a_{n-2}$ for $n \geq 2$. If $f(t) = \sum_{n \geq 0} \frac{a_n t^n}{n!}$ is the exponential generating function of this sequence, prove that $f'(t) = (1+t)f(t)$ and $f(t) = e^{t + t^2/2}$.

5. Prove that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$. If $t \geq 1$ is fixed, calculate

$$\lim_{n \to \infty} \sqrt[n]{\sum_{k=0}^{n} \binom{n}{k}^t}.$$

Exam 2
Math 688 Graduate Combinatorics I
Due Sunday, November 1, 2020, by 11.59pm

1. Let $T \neq T'$ be two different trees on the same vertex set of cardinality $n \geq 3$.

   (a) Show that for any $e \in E(T) \setminus E(T')$, there exists $e' \in E(T') \setminus E(T)$ such that $(T \setminus \{e\}) \cup \{e'\}$ is a tree on n vertices.

   (b) Show that for any $e' \in E(T') \setminus E(T)$, there exists $e \in E(T) \setminus E(T')$ such that $(T \cup \{e'\}) \setminus \{e\}$ is a tree on n vertices.

2. For $n \geq 2$, consider a rooted tree (T, r) on n vertices. Draw the tree in the plane without crossing edges. Imagine that the edges are walls perpendicular on the plane and starting at the root, walk around this system of walls, keeping the wall always to your left. Each time we move away from the root we write down a 1 and each time we move toward the root we write down a 0. This way we end up with a tree code, namely a sequence of length $2(n-1)$ consisting of 0s and 1s.

   (a) Does there exist an unlabeled tree with the code ?

      i. 1111100000;

      ii. 101010101010;

      iii. 1100011100 ?

   If yes, draw it. If no, give a short argument why not.

   (b) For $n \geq 1$, let t_n denote the number of unlabeled trees on n vertices. Determine t_n for $1 \leq n \leq 7$. Show that for any $n \geq 3$, $\frac{n^{n-2}}{n!} < t_n < \binom{2n-2}{n-1}$. Prove that for n sufficiently large, $2^n < t_n < 4^n$. Actually, $n > 30$ would work, but you don't have to prove this stronger statements, sufficiently large will do.

3. Let $\Gamma = (V, E)$ be a graph on $n \geq 1$ vertices. Denote by Γ^c the complement of Γ which is the graph with the same vertex set as Γ such that $x \neq y$ are adjacent in Γ^c if and only if x and y are not adjacent in Γ. Denote by $L(\Gamma)$ the Laplacian of Γ and by $L(\Gamma^c)$ the Laplacian of Γ^c.

   (a) Prove that $L(\Gamma) + L(\Gamma^c) = nI_n - J_n$.

   (b) If the eigenvalues of $L(\Gamma)$ are $0 = \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$, then prove that the eigenvalues of $L(\Gamma^c)$ are

$$0 \leq n - \mu_n \leq \ldots \leq n - \mu_2.$$

   (c) Using the previous part, find the eigenvalues of the Laplacians of the complete k-partite graph $K_{a_1,\ldots,a_k}$ (there are k partite sets of orders $a_1, \ldots, a_k$ and any two vertices in different partite classes are adjacent) and the graph $K_n \setminus K_a$ obtained from removing the edges of a clique K_a in a complete graph K_n (for $n \geq a$ of course).

(d) Calculate the number of spanning trees of $K_{a_1,\ldots,a_k}$ and $K_n \setminus K_a$.

4. (a) Let m be a natural number. If $Q_m = \frac{2}{m} J_m - I_m$, show that $Q_m^2 = I_m$. Here J_m is the all one $m \times m$ matrix and I_m is the identity $m \times m$ matrix.

   (b) If X is an $m \times n$ matrix with constant row sums and constant column sums, show that $Q_m X Q_n = X$.

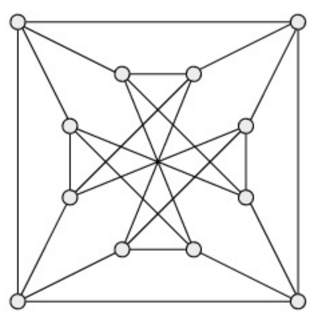
   (c) Let j_{2m} be the column vector of dimension $2m$ whose entries are all 1. If x is a vector of dimension $2m$ with half entries equal to 1 and the other half of the entries equal to 0, prove that $Q_{2m} x = j_{2m} - x$.

   (d) Let $\Gamma = (V, E)$ be a regular graph on $2n$ vertices. Let S be a subset of n vertices of Γ. Let Γ_1 be the graph obtained from Γ by adding a new vertex and making it adjacent to each vertex in S. Let Γ_2 be the graph obtained from Γ by adding a new vertex and making it adjacent to each vertex in $V \setminus S$. Show that the adjacency matrices of Γ_1 and Γ_2 are similar. Deduce that Γ_1 and Γ_2 have the same adjacency matrix eigenvalues. Construct two examples where Γ_1 and Γ_2 are non-isomorphic.

5. For both questions, make sure you justify your answers. For upper bound on χ, an explicit coloring is sufficient. For lower bound on χ, you should include a proof.

   (a) Consider the graph Γ_{13} whose vertex set is formed by the set of integers modulo 13 where x is adjacent to y if and only if $x - y \in \{1, -1, 5 - 5\}$ (mod 13). Show that Γ_{13} is a triangle-free graph and determine its chromatic number.

   (b) Determine the chromatic number of the graph below.



Exam 3
Math 688 Graduate Combinatorics I
Due Tuesday, November 24, 2020, by 11.59pm

1. Let $\Gamma = (V, E)$ be a graph on n vertices and denote by Γ^c its complement. Prove that
   (a) $\chi(\Gamma) + \chi(\Gamma^c) \leq n + 1$.
   (b) $\chi(\Gamma) \cdot \chi(\Gamma^c) \geq n$.

2. The line-graph of a graph $\Gamma = (V, E)$ is the graph whose vertices are the edges of Γ with two edges being adjacent if and only if they share an endpoint. Let T_n denote the line-graph of the complete graph on n vertices. Determine (with proof) the chromatic number of T_n and of its complement T_n^c.

3. Let $\Gamma = (V, E)$ be a connected plane graph.
   (a) Prove that a set F of edges of Γ forms a spanning tree of Γ if and only if the duals of the edges $E \setminus F$ form a spanning tree in the dual Γ^* of Γ. Using this result, prove Euler's formula.
   (b) Determine the crossing numbers of the graphs $K_6, K_{3,4}$ and $K_{2,2,3}$.

4. A point (a, b) in the Cartesian plane is called a lattice point if both a and b are integers. A lattice polygon is a polygon whose vertices are lattice points. A triangle is called primitive if it contains no lattice points in its interior or on the sides (except perhaps its vertices).
   (a) Prove that the area of any primitive lattice triangle equals 1/2.
   (b) Let S be a lattice square whose side has length 5. Prove that S has at most 36 and at least 28 lattice points. Give one example for each bound attaining equality.
   (c) Determine (with proof) the number of ways to achieve batting average 0.313 with at most 2000 at-bats ?

5. Let $\Gamma = (V, E)$ be a graph on n vertices. Let $t(\Gamma)$ denote the number of triangles in Γ. Denote by Γ^c the complement of Γ.
   (a) Prove that $t(\Gamma) + t(\Gamma^c) \geq \binom{n}{3} - \frac{\sum_{x \in V} d_\Gamma(x)(n - d_\Gamma(x) - 1)}{2}$.
   (b) Show that $t(\Gamma) + t(\Gamma^c) \geq \frac{n(n-1)(n-5)}{24}$.
   (c) Prove that in any graph with 6 vertices, there are at least 2 triangles in Γ and Γ^c.
   (d) If m is the number of edges of Γ, then prove that $t(\Gamma) \geq \frac{4m}{3n}\left(m - \frac{n^2}{4}\right)$.