Game Theory and an Exploration of 3 x n Chomp! Boards

Senior Mathematics Project

Emily Bergman

December, 2014

Introduction:

Game theory focuses on determining if there is a best way to play a game not necessarily what the best way is. Game theory has been utilized in biology, psychology, and economics. Biologists and social scientists are using game theory to determine why characteristics like altruism, empathy, and fairness have evolutionary benefits. The benefits from every characteristic result from the ability to survive in a world of competition. In evolutionary game theory, there is a limited amount of benefit that can be achieved by improving a certain characteristic instead of another. Biologists and social scientists try to use this to explain certain social norms. In this paper we will explore a mathematical game called Chomp! and try to generalize winning strategies for a group of boards.

Background:

A game has the following features described by Anatol Rapoport in *TwoPerson Game Theory* (1820):

1. There must be at least two players.
2. Players must make moves. These moves are choices made from a group of possible alternatives. For example, in checkers a player has the option of moving any of their pieces one tile forward or to jump their opponent's checker.
3. These moves have some type of consequence that determine either who makes the next move or affects the alternatives the next player can choose from. For example, in checkers, to avoid losing a checker, a player must pay attention to where their opponent moves their pieces and adjust their strategy accordingly.
4. Choices made can be known or unknown to the other players.
   We call a game where all the choices are known to all of the other players, a game of perfect information. The game of checkers is a game of perfect information because the opponent knows where the first player chooses to move their checker.
5. Eventually the game must end. So, there must be a termination rule or a situation where the game has ended. For example, the game of checkers ends when one player has no more checkers.
6. Each player must make moves and receive payoffs. If the player makes a strategical move then they should be closer to winning or receiving profit. For example, in checkers if a player makes a strategic move they should be closer to removing an opponent's checker, getting a King, or prevent the loss of a checker.

A winning strategy is a strategy that allows a player to win regardless of their opponent moves (Davis, 324). Zermelo's Theorem states in any finite two person game of perfect information, where both players know the choices available to the other, that can end for player one in either a win or a loss has a winning strategy.

Proof of Zermelo's Theorem (Polak, "Backwards Induction: Chess, Strategies, and Credible Threats"):
We go by way of induction. Let a game be played between two people with finitely many moves and of perfect information that can end for player one in win or a loss. Show that there exists a winning strategy.

Base Case: Let the game have only one move. This means player one is the only player that makes a move; however, player one might have many options. All the options must end in a win or a loss for player one. Player one knows how each of the options will end because it is a game of perfect information; therefore, player one will always choose the move that has the best results for player one. So, if the game has at least one move that ends in a win for player one there exists a winning strategy for player one. However, if there is not a move that ends in a win for player one that means all the strategies result in a win for player two. That means there exists a winning strategy for player two. Hence, in the one move two-player game of perfect information that can only end in a win or a loss, so there exists a winning strategy.
Inductive Step: assume that for any two player game of ≤ N moves, where N is an element of the natural numbers, with perfect information that can only end in a win or a loss for player one there exists a winning strategy. Show that there exists a winning strategy for a two player game of perfect information with N+1 moves.
For a game to have N +1 moves there must be a least one sequence of choices that has N subsequent moves after the first move and all of the other possible sequences of choices has ≤ N subsequence moves after the first move. We know from the inductive hypothesis that all of the games of ≤ N moves have a winning strategy. Hence, we are left with a two player one move game of perfect information where player one either wins or losses. We know from the base case that a one move game has a winning strategy. Therefore, a game with N + 1 moves has a winning strategy.
Hence, a two player finite game of perfect information that can only end in a win or a loss for player one must have a winning strategy ■ (Polak, "Backwards Induction: Chess, Strategies, and Credible Threats").

A "position" is a unique group of options that a player has to choose from. A winning position is a position where the winning strategy can be used by the player to win. A player wants to make a move into a winning position. A losing position is the group of choices where the winning strategy cannot be used. A player wants to move out of a losing position. We will call the group of winning positions "P" and the winning strategy for player one is in P and the group of losing positions "N." To force a win we want to show that all the moves from P go to N and there always exists a move from N into P. Let us use the following example: let there be *n* subtraction signs. Players take turns changing the subtraction signs into addition signs. They can only change one subtraction sign at a time or two subtraction signs that are next to each other. The winning strategy is known:

> Case one: *n* is even. Player one's first move will be to change the two middle subtraction signs into addition signs.
>
> Case Two: *n* is odd. Player one's first move will be to change the middle subtraction sign into an addition sign.

Let L be the number of the subtraction signs on the left side of the addition sign(s). Let R be the number of subtraction signs on the right side of the subtraction sign(s). For example, let *n* = 5

$$- - - - -$$

We know from case two we know that we will change the middle subtraction sign into an addition sign.

$$- - + - -$$

Now, L=R=2. Eventually as players continue the game all of the subtraction signs will become addition signs. Hence L=R=0 is the termination rule. Once a player creates L=R=0 then that player wins the game.

P, the group of winning positions, are all of the positions where L = R and N, the group of losing positions is where L ≠ R. To show that this forces a win we want to show that all moves from P go to N and there exists a move from N into P.
Show all moves form P go to N.
Since players can only change one subtraction sign or two subtraction signs next to each other. Therefore players can only change L or R. So, L ≠ R. Therefore a move from P goes into N.

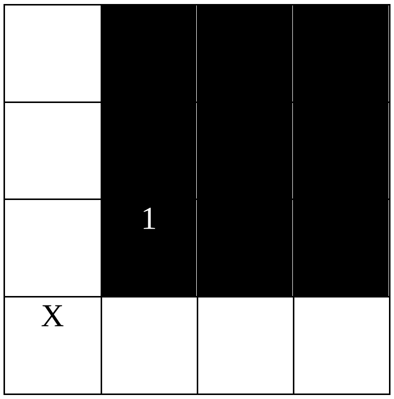
Show there exists a move from N to P.
Since players can only change one or two subtraction signs next to each other on one side. Then opponents can change the same number of subtraction signs on the opposite side. This will return L = R.
Hence, this forces a win for player one.
In any game theory problem if we can use this strategy. If we can show that a strategy forces all the moves from P move to N and there exists a move from N into P in a game then we have shown that we can force a win.

History

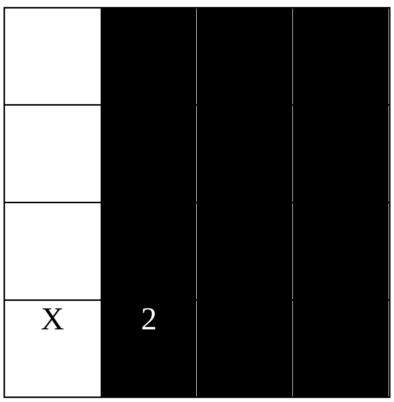
David Gale (1974) invented the game Chomp! which starts with a candy bar of size *m* x *n* where the lower left hand piece is poisoned. Each player must take a chomp out of the candy bar by choosing a piece and removing all the pieces above and to the right of the chosen one. Whoever is forced to choose the poisoned piece loses. From now on the bottom left hand piece will be denoted with an “X” for the terminating rule and loss of the game. Below is an example of a 4 x 4 Chomp! board:

<table>
<tr><td> </td><td> </td><td> </td><td> </td></tr>
<tr><td> </td><td> </td><td> </td><td> </td></tr>
<tr><td> </td><td> </td><td> </td><td> </td></tr>
<tr><td>X</td><td> </td><td> </td><td> </td></tr>
</table>

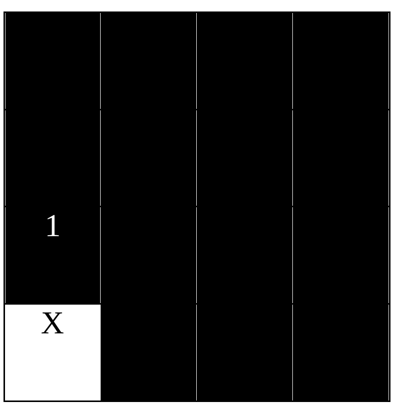
Here is an example of how a game is played. Let the first player choose the piece labeled 1 all of the pieces in black will be removed.



Then player two might choose the piece labeled 2.



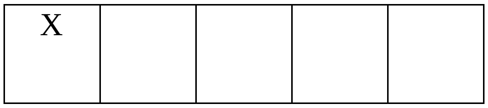
Then player one will choose the piece labeled 1 to win the game.



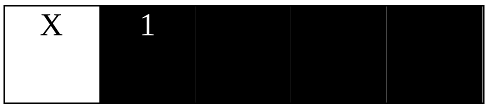
Player 2 is then forced to choose the “X” piece or the termination piece.

Original Solved Problems:
Paul R. Halmos in *Problems for Mathematicians Young and Old* posed the following types of Chomp! problems:

a. If the Chomp! board is 1 x n, can either player force a win?
Clearly, the first player can win any 1 x n board, where $n \geq 2$. The first player can win by choosing the piece directly to the right of the poisoned piece. Take for example the following 1 x 5 board:

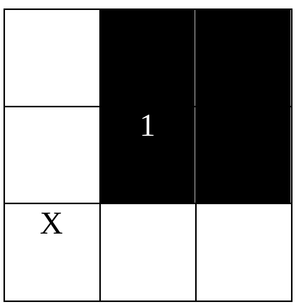


Player one to choose the piece 1

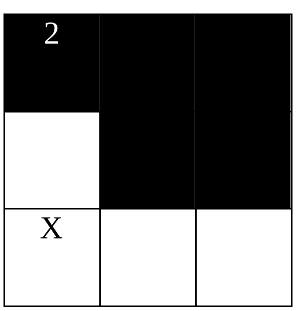


As you see the second player will have to choose the poisoned piece (Halmos,).

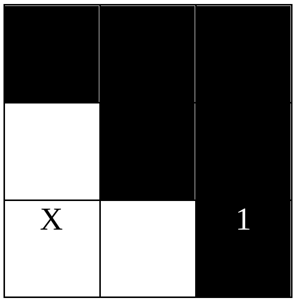
b. If the Chomp! board is square, can either player force a win?
Whenever the Chomp! board is square the first player will choose the piece in the second column and second row. Here is an example on a 3 x 3 Chomp! board:



Let C be the number of pieces in the first column and let R be the number of remaining tiles in the first row. When $C = R$, player one can mirror the moves that player two makes and win the game. So like the subtraction game, P, the group of winning positions, are positions where $C = R$ and N, the group of losing positions, are when $C \neq R$.
If a player begins in N, then the board looks like $C \neq R$ so we can remove $C - R$ or $R - C$ (depending on if the column or row has more pieces) pieces from the larger side to create a board where $C = R$ or move to P. For example,

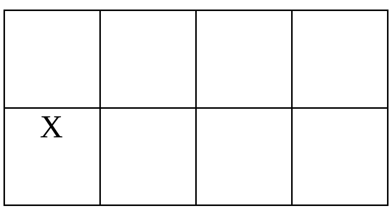


So C ≠ R because 1 ≠ 2. We can remove R – C = 2 – 1 = 1 piece from the row. This will make C = R or 1 = 1.
So, player one will create the following board by removing the piece labeled 1.

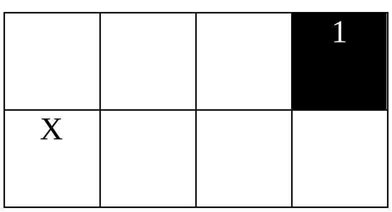


This will continue until player two is forced to choose the poison piece (Halmos,).

c. If the Chomp! board 2 x n, where n ≥ 2, can either player force a win?
If a Chomp! board is 2 x n where n is a natural number. Let F be the number of pieces in the first row and let S be the number of pieces in the second row. Here is an example of a 2 x 4 board,



The winning strategy for a 2 x n board for player one to remove the tile labeled 1.



The board with just the "X" tile is in P. The player begins in P and then they have to choose the "X" tile which moves them into N; hence losing the game. P, the group of winning positions, is the group of all positions where S = F – 1. And N, the group of losing positions, is the group of all other positions where S = F or S = F – n, where n ≥ 2.
We can show that this is a winning strategy because all the moves from P go to N and there exists a move from N to P.
Show all the moves from P go to N.
If a player begins in P the S = F – 1.
Case one: the player removes a tile on the first row and creates a new 2 x n board or S = F which would be in N.

Case two: the player removes a tile in the second row and makes S = F – n, where n ≥ 2 which is also in N.
Therefore, a game in a position from P changes to a position in N after any move.
Show there exists a move from N to P.
Suppose the player begins in N.
Case one: the player begins with a board where S = F. Then the player can remove the end piece from the second row and return to S = F – 1 or P.
Case two: the player begins with a board where S = F – n, where n ≥ 2. To return to a board where S = F – 1 we will remove n – 1 tiles from F. Then,

$$\begin{aligned} F - (n - 1) &= F - n + 1 \\ &= S + n - n + 1 \\ &= S + 1 \end{aligned}$$

So, the number of pieces in the first row is one more than the number of pieces in the second row. Hence, we can return from N to P.
Therefore, a game in a position from P changes to a position in N after any move.
Hence, removing one piece from the second row is the winning strategy for the 2 x n Chomp! board (Halmos, 212).

d. If the Chomp!board 2 x ∞, can either player force a win?
If a Chomp! board is 2 x ∞, no matter the piece player one chooses, player two has the opportunity to create a board that has the S = F – 1 configuration from above. If player one chooses a piece in the second row then player two chooses the piece below and one piece to the right of the piece chosen by player one. If player one chooses a piece in the bottom row then player two chooses the tile above and one to the left of the one chosen by player one. The player two will used the winning strategy proven in e. Hence, player two can always force a win in the 2 x ∞ Chomp! board (Halmos,212).

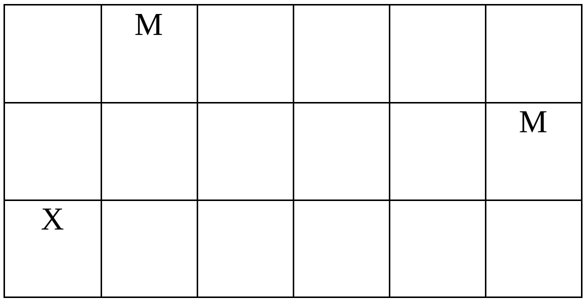
e. If the rectangular Chomp! board is finite, can either the first or the second player force a win?
From Zermelo's Theorem we know that there exists a winning strategy; however, we do not know which player has the winning strategy. We know that the removal of the upper right piece is either a winning or a losing strategy. If the first player removes this piece and it has a winning strategy then there exists a winning strategy in P for player one. However, if this does not have a winning strategy then player two does not have to lose. That means, player two can choose a piece that has a winning strategy in P. But player one could have created the same board by choosing this piece in the first move. Hence, there exists a winning strategy for player one. Therefore, player one can force a win in any finite rectangular Chomp! board (Halmos, 213).

Alterations: 3 x n Chomp! boards

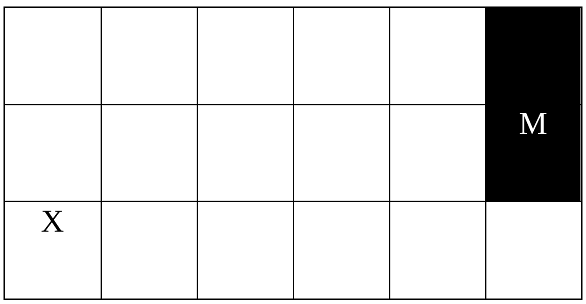
We want to determine the winning strategy for 3 x *n* Chomp! boards where *n* is a natural number. We will first look at all of the known strategies and look for patterns to develop a a winning strategy for the 3 x *n* Chomp! boards. Below is a table that explains the notation inside each 3 x n Chomp! board.
Table 1:

<table>
<tr><td>Symbols:</td><td>Losing strategy from original questions:</td><td>How the other player wins:</td><td>Example:</td></tr>
<tr><td>1 x n</td><td>By (a):</td><td>Choose the tile to the right of the termination piece and remove n – 1 pieces</td><td>If player one removes a then player two can choose 2 and win the game.
<table>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td>1 x n</td><td></td><td></td><td></td></tr>
<tr><td>X</td><td>2</td><td></td><td></td></tr>
</table></td></tr>
<tr><td>2 x n</td><td>By (c)</td><td>Create a board where the second row or column is one piece less than the first row or column.</td><td>If player one chooses the piece labeled c then player 2 will choose the tile labeled 2 and win the game.
<table>
<tr><td>2 x n</td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td>2</td></tr>
<tr><td>X</td><td></td><td></td><td></td></tr>
</table></td></tr>
<tr><td>SQ</td><td>By (b)</td><td>Chose the piece in the second row and second column, creating a board where C = R.</td><td>If player one chooses the piece labeled b then player two will choose the piece labeled 2 and win.
<table>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td>2</td><td></td><td></td></tr>
<tr><td>X</td><td></td><td></td><td>SQ</td></tr>
</table></td></tr>
<tr><td>M</td><td>By (c)</td><td>Create the first move in the 2 x n case choosing to remove all other rows or columns depending on the M piece choosen.</td><td>If player one chooses the piece labeled M then player two will choose the piece labeled 2 and win.
<table>
<tr><td></td><td>M</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td>X</td><td></td><td>2</td><td></td></tr>
</table></td></tr>
<tr><td>U.L.</td><td>By (b)</td><td>Make the number of pieces in the first column equal to the number of pieces in the first row.</td><td>If player one chooses the piece labeled U.L. then player two will choose the tile labeled 2 and win.
<table>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td>U.L.</td><td></td><td></td></tr>
<tr><td>X</td><td></td><td></td><td>2</td></tr>
</table></td></tr>
</table>

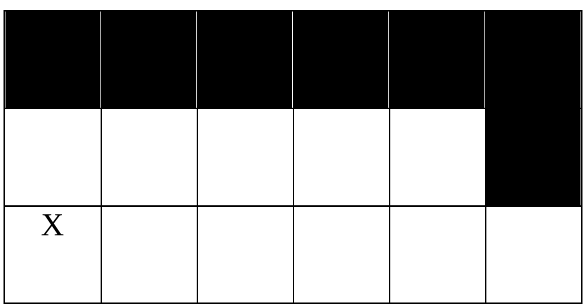
Pieces that are labeled with an "M" represent the first move in a 2 x n board. For example, below is a 3 x 6 Chomp! board labeled with only M's.



Case one: if player one chooses the following M piece:

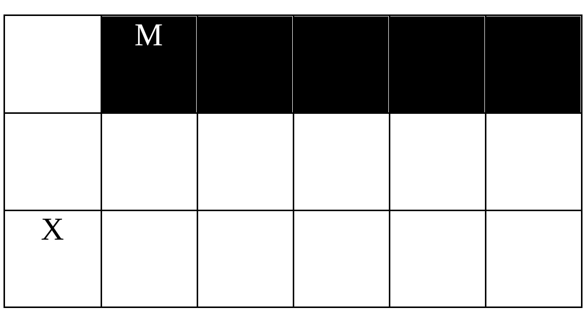


Then player two will remove the third row to create a 2 x n board where the second row is one less piece then the first row.

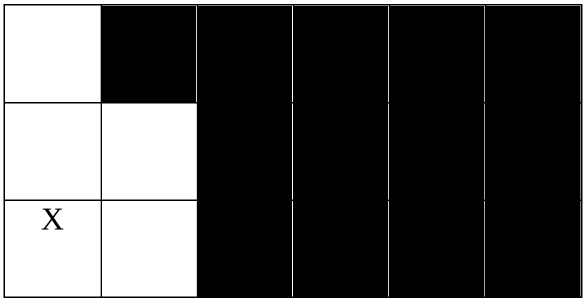


Player two will then use the strategy in the 2 x n winning strategy to win the game.

Case two: player one chooses the following "M" piece:

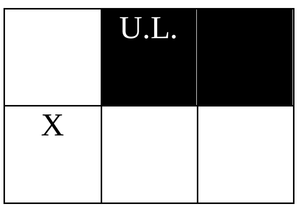


Then player two will remove all but two columns and create a 3 x 2 Chomp! board.



Now player two will use the strategy from the 2 x n case to win the game.

If the pieces that are denoted with a U.L. are removed, they create a board where the number of pieces in the first column, C, does not equal the number of pieces in the first row, R. So, R ≠ C. As we know from the square game above if we can create a R = C board then we have a winning strategy. Here is an example of a 2 x 3 board.



If U.L. is chosen then C = 1 ≠ 2 = R. Therefore the opponent will choose to remove 1 piece from the first row and create a board where C = 1 = R. Therefore, players will not want to choose a piece labeled U.L..

3 x 4 Chomp! Board Example:

Below is a 3 x 4 Chomp! board with the known losing strategies are labeled:

2 x n	M		
1 x n	U.L.		M
X	n x 1	n x 2	SQ

If player one chooses the piece labeled 1 then player two is forced to choose a piece with a losing strategy; thus losing the game.

2 x n	M		
1 x n	U.L.	1	
X	n x 1	n x 2	SQ

3 x 5 Chomp! Board Example:

Below is a 3 x 5 Chomp! board with the known losing strategies labeled:

2 x n	M			
1 x n	U.L.	3 x 4		M
X	n x 1	n x 2	SQ	3 x 4

The 3 x 4 pieces represent the pieces that if chosen create a 3 x 4 board which we know the winning strategy from above. So, if player one chooses one of the pieces labeled 3 x 4 then player two will chose the other one and win the game.

To show that player one wants to remove the piece labeled 1 for their first move we must show that if player 2 chooses either piece 2a or 2b they are forced to lose.

2 x n	M	2a	1	
1 x n	U.L.	3 x 4	2b	M
X	n x 1	n x 2	SQ	3 x 4

Case one: let player two choose 2a.

2 x n	M	2a		
1 x n	U.L.	3 x 4	2b	M
X	n x 1	n x 2	SQ	3 x 4

Now player one will choose 2b and force player two to choose a tile with a known losing strategy; thus losing the game.

2 x n	M	2a		
1 x n	U.L.	3 x 4		
X	n x 1	n x 2	SQ	3 x 4

Case two: let player two chose 2b.

2 x n	M	2a		
1 x n	U.L.	3 x 4	2b	
X	n x 1	n x 2	SQ	3 x 4

Then, player two will choose the piece labeled 2a and force player one to choose a piece with a known losing strategy; thus losing the game.

2 x n	M	2a		
1 x n	U.L.	3 x 4		
X	n x 1	n x 2	SQ	3 x 4

Solving the General 3 x n Chomp! Problems:

As you can see in the appendices, the general trend of winning strategies are labeled in this 3 x 8 Chomp! board.

	3 x 2		3 x 5	3 x 7			
3 x 1	3 x 3	3 x 4	3 x 6	3 x 8			
X	3 x 1	3 x 2	3 x 3	3 x 4	3 x 5	3 x 6	3 x 7

There seems to a cyclic nature of the winning strategies where the winning strategy alternates between the second row and third row; however, we notice that it deviates from this pattern from 3 x 3 to 3 x 4 where the winning strategy for both 3 x 3 and 3 x 4 boards are located on the second row. From the other examples, there does not seem to be any more deviates from the pattern; however, we cannot be sure. Doron Zeilberger (2001) developed a program to find positions in P, or the winning positions, to win the 3 x *n* cases. Here are some of the examples:

N	Move	P
$(n_1$,0, 0)	Remove n_1– 1 pieces	(1,0,0)
(1,1,1)	Choose the pieces in row 2, this will remove the tile in both row 2 and row 3	(1,0,0)
$(n_1, n_2$, 0)	Case one:$n_1 = n_2$. Remove one piece from row 2. Case two: $n_1 > n_2$+ 1 . So we know there exists some c st n_2 + 1 + c = n_1 . Hence we should remove c pieces from n_1.	$(n_1, n_1 - 1$, 0)
(2,2,2)	Remove one piece from row 3	(2,2,1)
$(n_1$, 1, 1)	Remove n_1 – 3 pieces from row one to create an even “L”	(3, 1, 1)
(3,3,3)	Remove 2 pieces from row 2, which will also remove two pieces from row 3 and create an even “L” or C = R.	(3, 1, 1)
(4,4,4)	Choose the piece in the second row and third column to two pieces from both the second and third rows.	(4,2,2)
(5,5,5)	Remove two pieces from the third row.	(5,5,3)
(6,6,6)	Choose the piece in the second row and fourth column to three pieces from both the second and third rows.	(6,3,3)
(7,7,7)	Remove two pieces from the third row.	(7,7,5)
(8,8,8)	Choose the piece in the second row and fifth column to four pieces from both the second and third rows.	(8,4,4)

Also, we do not know if these are all the ordered triples in N therefore we can't show that there is always a move from N to P.

Conclusion

Aviezri S. Fraenkel (1996) described why games between two players with perfect information that must result in a win or a loss are hard. These games take a large procession of moves and many decisions to find out if a piece has a winning or a losing strategy. These games are cyclic in nature because to show that a piece has a winning strategy then we must show that if player one chooses any of the other pieces that they would lose and to show that a piece is a losing strategy we must show that there exists another piece that has a winning strategy and so on and so forth. We can see this in the 3 x 5 case and all the cases in the appendices. Players also have more and more pieces that are possible winning strategies the bigger the Chomp! board. Possible further researcher includes 4 x *n* Chomp! boards and *n* x *n* +1 Chomp! boards. We know the winning strategy for 4 x 1, 4 x 2, 4 x 3, and 4 x 4 case; the 4 x *n* Chomp! boards may have a pattern that we can see through a computer program like the 3 x *n* case. On the other hand, the *n* x *n* +1 Chomp! boards have more obstacles including that redistribution of winning and losing strategies depending on the previous move made. If we can generalize the types of moves that are possible to make in a game of the size *n x (n+1)* then we may be able to generalize the outcomes. By making these generalizations we may be able to find a winning strategy.

References

Alexander, J. McKenzie, "Evolutionary Game Theory", *The Stanford Encyclopedia of Philosophy* (Fall 2009 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/fall2009/entries/game-evolutionary/>.

Gale, D., A curious Nim-type game, Appendix 1 in: Tracking the Automatic Ant, Springer, New York, 1998.

Halmos, Paul R. Problems for Mathematicians, Young and Old. The Mathematical Association of America, 1991. Print.

Nowakowski, Richard J. Games of No Chance. New York, New York: Cambridge University Press, 1998. Print.

Polak, Ben. Backward Induction: Chess, Strategies, and Credible Threats. New Haven, n.d. Video.

Rapoport, Anatol. Two-Person Game Theory. The University of Michigan Press, 1966. Print.

Appendix

3 x 6 Chomp! Board:

Here is the 3 x 6 Chomp! board with the losing strategies labeled:

2 x n	M				
1 x n	U.L.	3 x 4			M
X	n x 1	n x 2	SQ	3 x 4	

So, we want to make a first move such that all the remaining pieces have a losing strategy. Let player one choose the piece labeled 1.

2 x n	M	2a			
1 x n	U.L.	3 x 4	1		
X	n x 1	n x 2	SQ	3 x 4	2b

Now player two can choose either 2a or 2b. Let player 2 choose the piece labeled 2a.

2 x n	M	2a			
1 x n	U.L.	3 x 4			
X	n x 1	n x 2	SQ	3 x 4	2b

Then player one will choose the piece labeled 2b and force player two to choose the a tile with a losing strategy.

2 x n	M				
1 x n	U.L.	3 x 4			
X	n x 1	n x 2	SQ	3 x 4	2b

We can see that if player two choose the piece labeled 2b in the second move, then player one would have responded by choosing the piece labeled 2a and again forcing player two to choose a piece with a losing strategy; thus losing the game.
Hence, player one can force a win in the 3 x 5 game by choosing tile 1.

3 x 7 Chomp! Board:
Here is an example of a 3 x 7 Chomp! board with all of the losing strategies labeled.

2 x n	M					
1 x n	U.L.	3 x 4				M
X	n x 1	n x 2	SQ	3 x 4		

So, we want to make a first move such that all the other pieces have a losing strategy. Let player one make the first move by choosing the piece labeled one.

2 x n	M	2a	2b	1		
1 x n	U.L.	3 x 4	2c	2d	2e	M
X	n x 1	n x 2	SQ	3 x 4	2f	2g

Now we want to show that 2a through 2g all have losing strategies.

Case one: player two chooses the piece labeled 2a.

2 x n	M	2a				
1 x n	U.L.	3 x 4	2c	2d	2e	M
X	n x 1	n x 2	SQ	3 x 4	2f	2g

In response player one will choose 2e.

2 x n	M					

1 x n	U.L.	3 x 4	2c	2d	2e	
X	n x 1	n x 2	SQ	3 x 4	2f	2g

Then player two can only change the number of pieces without a strategy in row two or row one. So, player one will mirror player two's move in the opposite row and keep the number of pieces without a strategy in row two equal to the number of pieces without a strategy in row one. Until there are only pieces that have losing strategies remain for player two to choose. For example, let player two choose 2c.

2 x n	M					
1 x n	U.L.	3 x 4	2c			
X	n x 1	n x 2	SQ	3 x 4	2f	2g

Then player one will choose 2f and force player two to lose.

2 x n	M					
1 x n	U.L.	3 x 4				
X	n x 1	n x 2	SQ	3 x 4	2f	

Hence, 2a has a losing strategy.

Case two: Player two chooses the piece labeled 2b.

2 x n	M	2a	2b			

1 x n	U.L.	3 x 4	2c	2d	2e	M
X	n x 1	n x 2	SQ	3 x 4	2f	2g

Then player one will choose 2f.

2 x n	M	2a				
1 x n	U.L.	3 x 4	2c	2d		
X	n x 1	n x 2	SQ	3 x 4	2f	

No matter which tile player two chooses 2a, 2c, or 2d player two will be forced to choose a losing piece.

Assume player two chooses 2a, then player one will choose 2c and leave a board with only pieces with losing strategies to choose from.

2 x n	M	2a				
1 x n	U.L.	3 x 4	2c			
X	n x 1	n x 2	SQ	3 x 4		

Assume player two chooses 2c, then player one will choose 2a and leave a board with only pieces with losing strategies to choose from; forcing player two to lose much like the case above.

Assume player two chooses 2d, then player one will choose 2 x n and leave a 2 x n board where the number of pieces in the first row is one more then the number of pieces in the second row. We know that player two will lose this game.

2 x n						
1 x n	U.L.	3 x 4	2c	2d		

X	n x 1	n x 2	SQ	3 x 4		

Hence, the piece 2b has a losing strategy.

Case three: player two chooses piece 2c.

2 x n	M	2a				
1 x n	U.L.	3 x 4	2c			
X	n x 1	n x 2	SQ	3 x 4	2f	2g

Then plyer one will choose 2g and create the board we saw in the 3 x 6 case which we know player two loses.

2 x n	M	2a				
1 x n	U.L.	3 x 4				
X	n x 1	n x 2	SQ	3 x 4	2f	2g

Hence, 2c has a losing strategy.

Case Four: player two chooses the piece labeled 2d.

2 x n	M	2a	2b			
1 x n	U.L.	3 x 4	2c	2d		
X	n x 1	n x 2	SQ	3 x 4	M	2g

Then player one will choose the piece labeled 2a.

2 x n	M	2a				

1 x n	U.L.	3 x 4	2c			
X	n x 1	n x 2	SQ	3 x 4	M	2g

No matter if player two chooses 2c or 2g player one will choose the other and leave a board of only pieces with losing strategies for player two to choose from; thus losing the game.

Hence, the piece labeled 2d has a losing strategy.

Case five: player two chooses 2e.

2 x n	M	2a	2b			
1 x n	U.L.	3 x 4	2c	2d	2e	
X	n x 1	n x 2	SQ	3 x 4	2f	M

Then player one will choose 2a and create the following board:

2 x n	M	2a				
1 x n	U.L.	3 x 4	2c	2d		
X	n x 1	n x 2	SQ	3 x 4	2f	M

We know that player two will lose from case one. Hence the piece labeled 2e has a losing strategy.

Case six: player two chooses 2f.

2 x n	M	2a	2b			

1 x n	U.L.	3 x 4	2c	2d		
X	n x 1	n x 2	SQ	3 x 4	2f	

Then player two will choose the piece labeled 2b creates the following board:

2 x n	M	2a	2b			
1 x n	U.L.	3 x 4	2c	2d		
X	n x 1	n x 2	SQ	3 x 4		

We know from case two that player two loses this game, thus the piece labeled 2f has a losing strategy.

Case Seven: player two chooses the piece labeled 2g.

2 x n	M	2a	2b			
1 x n	U.L.	3 x 4	2c	2d	2e	
X	n x 1	n x 2	SQ	3 x 4	2f	2g

Then player one will choose the piece labeled 2c and creates the following board:

2 x n	M	2a				
1 x n	U.L.	3 x 4	2c			
X	n x 1	n x 2	SQ	3 x 4	2f	

We know that player two loses game from case three. Hence 2g has a losing strategy.

Therefore, choosing the piece labeled one is the winning strategy for player one in the 3 x 7 Chomp! board.

2 x n	M	2a	2b	1		
1 x n	U.L.	3 x 4	2c	2d	2e	M
X	n x 1	n x 2	SQ	3 x 4	2f	2g

3 x 8 Chomp! board:

Here is the example of the 3 x 8 Chomp! board with all of the known losing strategies labeled.

2 x n	M		3 x 5				
1 x n	U.L.	3 x 4	3 x 6				M
X	n x 1	n x 2	SQ	3 x 4	3 x 5	3 x 6	

We want to show that removing the tile labeled 1 is the winning strategy for player one.

2 x n	M	2a	3 x 5				
1 x n	U.L.	3 x 4	3 x 6	1			
X	n x 1	n x 2	SQ	3 x 4	3 x 5	3 x 6	2b

Case one: player two chooses the piece labeled 2a.

2 x n	M	2a					
1 x n	U.L.	3 x 4	3 x 6				
X	n x 1	n x 2	SQ	3 x 4	M	3 x 6	2b

Player one will choose the piece labeled 2b and create the following board:

2 x n	M						
1 x n	U.L.	3 x 4	3 x 6				
X	n x 1	n x 2	SQ	3 x 4	M	3 x 6	2b

We are left with a board that only has pieces with a losing strategy and player two will be forced to choose a piece with a losing strategy; thus, losing the game. On the other hand if player two chooses 2b then player one will choose 2a creating the same board; thus player two will lose. Therefore, removing the piece labeled one is the winning strategy for player one on the 3 x 8 board.