PHY 342 HW Ch.4d

Do problem 4.34 and 4.26[bonus], plus the following.

q4.11
A CO_2 molecule is free to rotate in 3D space about a perpendicular axis (fixed) through the center (carbon atom). Let a be the C-O bond length and m the mass of the oxygen atom. (a) Write down the classical kinetic energy in terms its angular momentum and its rotational inertia. (b) Using the Hamiltonian above, explain (no math analysis necessary) that the allowed energies of this quantum rotor are (rotational levels of molecules)

$$E_l = \frac{l(l+1)\hbar^2}{4ma^2}, \quad l = 0, 1, 2, ...$$

(c) What are the eigenfunctions of this system? What is the degeneracy for a given l?

q4.12
The wave function of a hydrogen atom is given by $\Psi = 2\psi_{200} - i\psi_{21,-1} + 2i\psi_{321}$. (a) Let $\Phi = A\Psi$. If Φ is normalized, what is A? Does Φ and Ψ represent different quantum states? (b) An operator is defined as $\hat{Q} = qL_z$ where q is a constant. If a measurement of $\hat{Q}$ is taken, what are its possible values? (c) Find $\hat{Q}|\Psi\rangle$, and the expectation value $\langle\hat{Q}\rangle$.

q4.13
(a) Find the eigenvalues and eigenvectors of S_y of a spin-1/2 particle. (b) Suppose a measurement of S_z is made of the particle in an eigenstate of S_y (e.g., $+\hbar/2$), what is the probability of getting $+\hbar/2$? $-\hbar/2$? (c) Explain the results above per emphasized reading on pp.176-177.

q4.14
A spin-1/2 particle is in a state

$$\chi = C\begin{pmatrix} 2 \\ -i \end{pmatrix}.$$

(a) Find the normalization constant C. (b) Calculate the expectation values of S_x, S_y, S_z. (c) Find the uncertainties $\Delta_{S_x}, \Delta_{S_y}, \Delta_{S_z}$. (d) Show that S_x, S_y, S_z satisfy the uncertainty principle given by Eqn. (3.62) with $A = S_x, B = S_y$.

q4.15 [bonus]
(a) Plot the polar surfaces of the angular distribution, i.e., $|Y_{lm}|^2 \sin\theta$, using Python (or other packages you are familiar with). Try $l = 3, 4$, and $m = 0, 1, ..., l$.
(b)[optional, but really cool] Explain to at least one person (family, friends etc) what Figures 4.5 and 4.6 mean. Write down the key points you conveyed and what they asked of you or commented on.

q4.16 [bonus]

The angular momentum ladder operators in spherical coordinates are

$$L_{\pm} = \pm \hbar e^{\pm i\varphi} \left(\frac{\partial}{\partial \theta} \pm i \cot \theta \frac{\partial}{\partial \varphi} \right).$$

Let the common eigenfunctions of L^2, L_z be $f_{lm} = P_{lm}(\theta) e^{im\varphi}$, so the highest "rung" on the ladder is $f_{ll}(\theta, \varphi) = P_{ll}(\theta) e^{il\varphi}$. Using the fact that $\partial f_{ll}/\partial \varphi = il f_{ll}$ and $L_+ f_{ll} = 0$, write down the differential equation satisfied by $P_{ll}(\theta)$, and show that its solution is $P_{ll} = C \sin^l \theta$, and

$$f_{ll} = C \sin^l \theta e^{il\varphi} \quad \text{(this is } Y_{ll}\text{)}.$$

Check against Table 4.3 (p.139) in the text to verify that, apart from the constant C, it agrees with the first few spherical harmonics, such as $Y_{00}, Y_{11}, Y_{22}, ...$

Apply L_- to Y_{22} to obtain Y_{21}. Don't bother with the normalization constant.