Sigma Notation and Limits of Finite Sums

Part 1: Sigma Notation
Sigma Notation

\[ \sum_{k=a}^{b} f(k) = f(a) + f(a + 1) + f(a + 2) + \cdots + f(b - 1) + f(b) \]

- \( \Sigma \) is the Greek letter capital sigma
- \( k \) is the index of summation
- \( a \) is the lower limit of summation
- \( b \) is the upper limit of summation
Examples of Sigma Notation

\[
\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2
\]

\[
\sum_{k=4}^{8} k^3 = 4^3 + 5^3 + 6^3 + 7^3 + 8^3
\]

\[
\sum_{k=1}^{5} 2k = 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + 2 \cdot 4 + 2 \cdot 5
\]

\[
= 2 + 4 + 6 + 8 + 10
\]
Examples of Sigma Notation (continued)

\[
\sum_{k=0}^{5} (2k + 1) = 1 + 3 + 5 + 7 + 9 + 11
\]

\[
\sum_{k=0}^{5} (-1)^k (2k + 1) = 1 − 3 + 5 − 7 + 9 − 11
\]

\[
\sum_{k=-3}^{1} k^3 = (-3)^3 + (-2)^3 + (-1)^3 + 0^3 + 1^3
\]

\[
\sum_{k=1}^{3} k \sin \left( \frac{k\pi}{5} \right) = \sin \left( \frac{\pi}{5} \right) + 2 \sin \left( \frac{2\pi}{5} \right) + 3 \sin \left( \frac{3\pi}{5} \right)
\]
Examples of Sigma Notation (continued)

\[ \sum_{k=2}^{2} k^3 = 2^3 \]

\[ \sum_{k=4}^{8} 2 = 2 + 2 + 2 + 2 + 2 + 2 = 10 \]

\[ \sum_{k=1}^{5} 2k = \sum_{k=0}^{4} (2k + 2) = \sum_{k=2}^{6} (2k - 2) = 2 + 4 + 6 + 8 + 10 \]
Example 1

Express

\[ \sum_{k=3}^{7} 5^{k-2} \]

in sigma notation so that the lower limit of summation is 0 rather than 3.
Example 1 (continued)

\[ \sum_{k=3}^{7} 5^{k-2} \]

Solution:
Let \( j = k - 3 \) \( \Rightarrow \) \( k = j + 3 \). Then we have:

<table>
<thead>
<tr>
<th>( k )</th>
<th>( j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
</tr>
</tbody>
</table>

So, as \( k \) goes from 3 to 7, \( j \) goes from 0 to 4.
Example 1 (continued)

\[ \sum_{k=3}^{7} 5^{k-2} \]

Answer:

\[ \sum_{k=3}^{7} 5^{k-2} = \sum_{j=0}^{4} 5^{(j+3)-2} = \sum_{j=0}^{4} 5^{j+1} = \sum_{k=0}^{4} 5^{k+1} \]

\[ \sum_{k=3}^{7} 5^{k-2} = \sum_{k=0}^{4} 5^{k+1} \]
General Sum

To represent a general sum, use letters with subscripts:

\[ \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n \]
Algebraic Properties of Sigma Notation

\[ \sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \]

\[ \sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k \]

\[ \sum_{k=1}^{n} c \cdot a_k = c \sum_{k=1}^{n} a_k \]

\[ \sum_{k=1}^{n} c = n \cdot c \]
Important formulas

\[\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + (n - 1) + n = \frac{n(n + 1)}{2}\]

\[\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + (n - 1)^2 + n^2 = \frac{n(n + 1)(2n + 1)}{6}\]

\[\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + (n - 1)^3 + n^3 = \left[\frac{n(n + 1)}{2}\right]^2\]
Example 2

Evaluate

$$\sum_{k=1}^{30} k(k + 1)$$

Solution:
Using $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$ we get:

$$\sum_{k=1}^{30} k(k + 1) = \sum_{k=1}^{30} (k^2 + k) = \sum_{k=1}^{30} k^2 + \sum_{k=1}^{30} k$$
Example 2 (continued)

\[ \sum_{k=1}^{30} k(k + 1) \]

Next, using \( \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \) and \( \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \) we get:

\[
\begin{align*}
\sum_{k=1}^{30} k^2 + \sum_{k=1}^{30} k &= \frac{30(30 + 1)(2(30) + 1)}{6} + \frac{30(30 + 1)}{2} \\
&= 9920
\end{align*}
\]

Answer:

\[ \sum_{k=1}^{30} k(k + 1) = 9920 \]
$\sqrt{-1} \ 2^{3} \ \sum \ \pi$

and it was delicious