The Definite Integral

Part 1
Riemann Sums

Let $f(x)$ be any function defined on a closed interval $[a, b]$ and $P = \{x_0, x_1, x_2, x_3, \ldots, x_{n-1}, x_n\}$ be a partition of the interval. Then the Riemann sum is

\[ S_P = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + \cdots + f(c_n)\Delta x_n \]

\[ = \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k \]

If we increase $n$ in such a way that $\|P\| \to 0$, then the width of every rectangle tends to zero.
Riemann Sums

\[
x = a \quad x_1 \quad x_2 \quad \ldots \quad x_{k-1} \quad x_k \quad x_{n-1} \quad x_n = b
\]

\[
(c_1, f(c_1)) \quad (c_2, f(c_2)) \quad \ldots \quad (c_k, f(c_k)) \quad \ldots \quad (c_n, f(c_n))
\]
Area Under a Curve

If the function $f(x)$ is continuous and non-negative on $[a, b]$, then the area under the curve $y = f(x)$ over the interval $[a, b]$ is

$$A = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$$

where $c_1, c_2, c_3, \cdots, c_n$ are arbitrary points in successive subintervals.
Definite Integral

If the function \( f(x) \) is defined on \([a, b]\), then the definite integral of \( f \) from \( a \) to \( b \) is

\[
\int_{a}^{b} f(x) \, dx = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k
\]

provided the limit exists.
\[ \int_{a}^{b} f(x) \, dx \]

- \( \int \) is the **integral sign**
- \( dx \) is the **variable of integration**
- \( a \) is the **lower limit of integration**
- \( b \) is the **upper limit of integration**
- \( f(x) \) is the **integrand**
Example 1

Express

\[ \lim_{\|P\| \to 0} \sum_{k=1}^{n} 4c_k(1 - 3c_k)\Delta x_k \]

where \( P \) is a partition of \([-3,3]\) as a definite integral.
Example 1 (continued)

\[ \lim_{\|P\| \to 0} \sum_{k=1}^{n} 4c_k(1 - 3c_k)\Delta x_k \]

Solution:

\[ \int_{a}^{b} f(x) \, dx = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k \]

We have

\[ f(c_k) = 4c_k(1 - 3c_k) \]

and \([a, b] = [-3,3]\)

Therefore

\[ f(x) = 4x(1 - 3x) \]

Answer:

\[ \lim_{\|P\| \to 0} \sum_{k=1}^{n} 4c_k(1 - 3c_k)\Delta x_k = \int_{-3}^{3} 4x(1 - 3x) \, dx \]
Areas Under a Curve, Riemann Sums, and Definite Integrals

If the function $f(x)$ is continuous and non-negative on $[a, b]$, then

$$A = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k = \int_{a}^{b} f(x) \, dx$$
Example 2

Graph the integrand and use areas to evaluate
\[ \int_{-5}^{5} \sqrt{25 - x^2} \, dx. \]

Solution:

In general, \( y = \sqrt{r^2 - x^2} \) is the upper semi-circle of radius \( r \) centered at the origin.
Example 2 (continued)

\[ r = 5 \]

Since the area of a circle of radius \( r \) is \( \pi r^2 \), the area of a semi-circle is \( \frac{1}{2} \pi r^2 \).

Answer:
\[
\int_{-5}^{5} \sqrt{25 - x^2} \, dx = \frac{1}{2} \pi (5)^2
\]
\[
= \frac{25\pi}{2}
\]
Acute angle