Substitution and Area Between Curves

Part 1
Theorem

If \( u = g(x) \) and \( \frac{du}{dx} = g'(x) \) is continuous on \([a, b]\) and if \( f \) is continuous on \( R(g) \) (= the range of \( g \)), then

\[
\int_{a}^{b} \left( f(u) \frac{du}{dx} \right) dx = \int_{g(a)}^{g(b)} f(u) \, du
\]
Let \( F'(x) = f(x) \). Then

\[
\frac{d}{dx} \left[ F(g(x)) \right] = F'(g(x)) \cdot g'(x)
\]

\[
= F'(u) \cdot \frac{du}{dx}
\]

\[
= f(u) \frac{du}{dx}
\]

So the left hand side of the equation on the Theorem becomes:
\[
\int_a^b \left( f(u) \frac{du}{dx} \right) dx = \int_a^b \frac{d}{dx} [F(g(x))] \, dx \\
= F(g(x)) \bigg|_a^b = F(g(b)) - F(g(a)) \\
= F(u) \bigg|_{g(a)}^{g(b)} \\
= \int_{g(a)}^{g(b)} f(u) \, du,
\]
proving our theorem.
Example 1

Evaluate

\[ \int_{0}^{2} 2x(x^2 + 1)^3 \, dx \]

Solution:

We will use substitution to solve this.

Let \( u = x^2 + 1 \)

\[ du = 2x \, dx \]
Example 1 (continued)

When we make the substitution, we are no longer in “$x$” land – we have moved to “$u$” land, so we need to change every part of the original integral to be in terms of $u$ – including the limits of integration.

\[ u = x^2 + 1 \]
\[ du = 2x \, dx \]

Upper limit of integration:
\[ x = 2 \Rightarrow u = 2^2 + 1 = 5 \]

Lower limit of integration:
\[ x = 0 \Rightarrow u = 0^2 + 1 = 1 \]
Example 1 (continued)

\[
\int_{0}^{2} 2x(x^2 + 1)^3 \, dx = \int_{0}^{2} (x^2 + 1)^3 \cdot 2x \, dx
\]

\[
= \int_{0}^{1} u^3 \, du = \frac{1}{4} u^4 \bigg|_{1}^{5}
\]

\[
= \frac{1}{4} \cdot 5^4 - \frac{1}{4} \cdot 1^4 = 156
\]

Notice that since we changed the limits of integration, we did not need to replace \( u \) with its equivalent in \( x \).
Example 2

Evaluate

\[ \int_{0}^{\pi/4} \cos(\pi - x) \, dx \]

Solution:

\[
\begin{align*}
  u &= \pi - x \\
  du &= -dx \Rightarrow -du = dx \\
  x = \frac{\pi}{4} &\Rightarrow u = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \\
  x = 0 &\Rightarrow u = \pi - 0 = \pi
\end{align*}
\]
Example 2 (continued)

\[
\int_{0}^{\pi/4} \cos(\pi - x) \, dx = \int_{\pi}^{3\pi/4} \cos(u) \, (-du)
\]

\[
= \int_{3\pi/4}^{\pi} \cos(u) \, du = \sin(u) \bigg|_{3\pi/4}^{\pi}
\]

\[
= \sin(\pi) - \sin\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}
\]
Theorem

- If $f$ is even, then
  \[ \int_{-a}^{a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx \]

- If $f$ is odd, then
  \[ \int_{-a}^{a} f(x) \, dx = 0 \]
Example 3

\[ \int_{-\pi/4}^{\pi/4} \cos(x) \, dx \]

\[ = 2 \int_{0}^{\pi/4} \cos(x) \, dx = 2\sin(x) \bigg|_{0}^{\pi/4} \]

\[ = 2 \sin\left(\frac{\pi}{4}\right) - 2 \sin(0) = \sqrt{2} \]

\[ \int_{-\pi/4}^{\pi/4} \sin(x) \, dx \]

\[ = 0 \]
Home On the Range