Substitution and Area Between Curves

Part 2: Area Between Two Curves
Upper curve
\[ y = f(x) \]

Lower curve
\[ y = g(x) \]
Area Between Two Curves

If $f$ and $g$ are continuous with $f(x) \geq g(x)$ on $[a, b]$, then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$ and on the right by the line $x = b$ is

$$A = \int_a^b [f(x) - g(x)] \, dx.$$ 

We call $A$ the area of the region between $y = f(x)$ and $y = g(x)$ from $a$ to $b$. 

J. Gonzalez-Zugasti, University of Massachusetts - Lowell
Example 1

Find the area of the region between \( y = x + 6 \) and \( y = x^2 \) from 0 to 2.

Solution:
To determine the top function, the bottom function, and the limits of integration, it is often helpful to make a sketch.
Example 1 (continued)

\[ A = \int_{0}^{2} [(x + 6) - x^2] \, dx \]

\[ = \left( \frac{1}{2} x^2 + 6x - \frac{1}{3} x^3 \right) \bigg|_{0}^{2} \]

\[ = \left( \frac{1}{2} \cdot 2^2 + 6 \cdot 2 - \frac{1}{3} \cdot 2^3 \right) \]

\[ - \left( \frac{1}{2} \cdot 0^2 + 6 \cdot 0 - \frac{1}{3} \cdot 0^3 \right) \]

\[ = \frac{34}{3} \]
Example 2

Find the area of the region enclosed between $y = x + 6$ and $y = x^2$.

Solution:
First, find where the two curves meet:

\[
x^2 = x + 6
\]

\[
x^2 - x - 6 = 0
\]

\[
(x - 3)(x + 2) = 0
\]

\[
x = 3, x = -2
\]
Example 2 (continued)

\[ A = \int_{-2}^{3} [(x + 6) - x^2] \, dx \]

\[ = \left[ \left( \frac{1}{2} x^2 + 6x - \frac{1}{3} x^3 \right) \right]_{-2}^{3} \]

\[ = \left( \frac{1}{2} \cdot 3^2 + 6 \cdot 3 - \frac{1}{3} \cdot 3^3 \right) \]

\[ - \left( \frac{1}{2} \cdot (-2)^2 + 6 \cdot (-2) - \frac{1}{3} \cdot (-2)^3 \right) = \frac{125}{6} \]
Example 3

Find the area of the region enclosed by the curves $y = \sqrt{x}$, $y = -x + 6$ and $y = 1$.

Solution:
First sketch the curves, clearly labeling the intersection points.
Example 3 (continued)

Notice that we need to break this up into two integrals:

\[ A_1 = \int_{1}^{4} [\sqrt{x} - 1] \, dx = \cdots = \frac{5}{3} \]

\[ A_2 = \int_{4}^{5} [(-x + 6) - 1] \, dx \]

\[ = \cdots = \frac{1}{2} \]

Answer:

\[ A = A_1 + A_2 = \frac{5}{3} + \frac{1}{2} = \frac{13}{6} \]
Sometimes you need to find the area of a region bounded above and below by horizontal lines and bounded on the left and right by the graphs of two functions of $y$. 
Example 4

Repeat Example 3 by integrating with respect to $y$.

(Find the area of the region enclosed by the curves $y = \sqrt{x}$, $y = -x + 6$ and $y = 1$.)
Solution:

\[ y = -x + 6 \Rightarrow x = -y + 6 \]
\[ y = \sqrt{x} \Rightarrow x = y^2 \]

\[ A = \int_{1}^{2} [(-y + 6) - y^2] \, dy \]
\[ = \cdots = \frac{13}{6} \]

This was much easier since we did not need to calculate two different integrals to find the area.
Example 5

Find the area of the region enclosed by the curves $y = x^4 - 4x^2 + 4$ and $y = x^2$.

Solution:
Clearly, $y = x^2$ is a parabola with vertex (0,0) that opens upwards.

\[
y = x^4 - 4x^2 + 4 \\
= (x^2 - 2)^2 \\
= (x - \sqrt{2})^2 (x + \sqrt{2})^2
\]
So the only $x$-intercepts are $(-\sqrt{2}, 0)$ and $(\sqrt{2}, 0)$.

Since $y = x^4 - 4x^2 + 4$ is a 4th degree polynomial with a positive leading coefficient, we know that the shape of the curve will look like a rounded “W”.

Since the graph only crosses the $x$-axis at two points, they must be the bottoms of the “W”.
But the question is, where do the two curves intersect?

\[ x^4 - 4x^2 + 4 = x^2 \]
\[ x^4 - 5x^2 + 4 = 0 \]
\[ (x^2 - 4)(x^2 - 1) = 0 \]
\[ (x - 2)(x + 2)(x - 1)(x + 1) = 0 \]

So the curves intersect when \( x = 2, x = -2, x = 1 \) and \( x = -1 \).
Example 5 (continued)
Example 5 (continued)

\[ A_1 = \int_{-2}^{-1} \left[ x^2 - (x^4 - 4x^2 + 4) \right] dx \]

\[ = \int_{-2}^{-1} [-x^4 + 5x^2 - 4] \, dx \]

\[ = \left. \left( -\frac{1}{5}x^5 + \frac{1}{3}x^3 - 4x \right) \right|_{-2}^{-1} \]

\[ = \left( -\frac{1}{5}(-1)^5 + \frac{1}{3}(-1)^3 - 4(-1) \right) \]
\[ - \left( -\frac{1}{5}(-2)^5 + \frac{1}{3}(-2)^3 - 4(-2) \right) \]

\[ = \left( \frac{1}{5} - \frac{5}{3} + 4 \right) - \left( \frac{32}{5} - \frac{40}{3} + 8 \right) \]
Example 5 (continued)

\[A_2 = \int_{-1}^{1} [(x^4 - 4x^2 + 4) - x^2] \, dx = \int_{-1}^{1} [x^4 - 5x^2 + 4] \, dx\]

\[= \left( \frac{1}{5} x^5 - 5 \cdot \frac{1}{3} x^3 + 4x \right) \bigg|_{-1}^{1}\]

\[= \left( \frac{1}{5} \cdot 1^5 - 5 \cdot \frac{1}{3} \cdot 1^3 + 4 \cdot 1 \right) - \left( \frac{1}{5} (-1)^5 - 5 \cdot \frac{1}{3} (-1)^3 + 4(-1) \right)\]

\[= \left( \frac{1}{5} - \frac{5}{3} + 4 \right) - \left( -\frac{1}{5} + \frac{5}{3} - 4 \right)\]

\[A_3 = \int_{1}^{2} [x^2 - (x^4 - 4x^2 + 4)] \, dx = \int_{1}^{2} [-x^4 + 5x^2 - 4] \, dx\]

\[= \left( -\frac{1}{5} x^5 + 5 \cdot \frac{1}{3} x^3 - 4x \right) \bigg|_{1}^{2}\]

\[= \left( -\frac{1}{5} \cdot 2^5 + 5 \cdot \frac{1}{3} \cdot 2^3 - 4 \cdot 2 \right) - \left( -\frac{1}{5} \cdot 1^5 + 5 \cdot \frac{1}{3} \cdot 1^3 - 4 \cdot 1 \right)\]

\[= \left( -\frac{32}{5} + \frac{40}{3} - 8 \right) - \left( -\frac{1}{5} + \frac{5}{3} - 4 \right)\]
Example 5 (continued)

\[ A = A_1 + A_2 + A_3 \]
\[ = \left[ \left( \frac{1}{5} - \frac{5}{3} + 4 \right) - \left( \frac{32}{5} - \frac{40}{3} + 8 \right) \right] \]
\[ + \left[ \left( \frac{1}{5} - \frac{5}{3} + 4 \right) - \left( -\frac{1}{5} + \frac{5}{3} - 4 \right) \right] \]
\[ + \left[ \left( -\frac{32}{5} + \frac{40}{3} - 8 \right) - \left( -\frac{1}{5} + \frac{5}{3} - 4 \right) \right] \]
\[ = \frac{60}{5} + \frac{60}{3} = -12 + 20 = 8 \]
3 out of 2 people have trouble with fractions.