Areas of Surfaces of Revolution
Surface Area

Let $f$ be a smooth, nonnegative function on an interval $[a, b]$.

Problem:
Find the area of the surface generated by revolving the curve $y = f(x)$ about the $x$-axis.
Let $f$ be a nonnegative, smooth function on $[a, b]$,

and

$P = \{a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b\}$

be a partition of $[a, b]$.

A slice of the surface generated by revolving the curve about the $x$-axis is like a **frustum** (the portion of a solid that lies between two parallel planes cutting it) of a cone.
Surface Area

The lateral area of the frustum can be obtained from the formula

\[ \pi \left( f(x_{k-1}) + f(x_k) \right) \cdot l \]

where \( l \) is the slant height (that is, \( l \) is the distance between the points \((x_{k-1}, f(x_{k-1}))\) and \((x_k, f(x_k))\)).
Surface Area

\[ S_k \approx \pi \left( f(x_{k-1}) + f(x_k) \right) \cdot l \]
\[ = \pi \left( f(x_{k-1}) + f(x_k) \right) \sqrt{(\Delta x_k)^2 + \left( f(x_k) - f(x_{k-1}) \right)^2} \]

By the Mean-Value Theorem, there is a point \( c_k \) between \( x_{k-1} \) and \( x_k \) such that
\[ \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(c_k) \]

or
\[ f(x_k) - f(x_{k-1}) = f'(c_k) \Delta x_k \]

This gives us
\[ S_k \approx \pi \left( f(x_{k-1}) + f(x_k) \right) \sqrt{(\Delta x_k)^2 + \left( f'(c_k) \Delta x_k \right)^2} \]
\[ = \pi \left( f(x_{k-1}) + f(x_k) \right) \sqrt{1 + (f'(c_k))^2} \cdot \Delta x_k \]
Now

\[ \frac{1}{2} \left( f(x_{k-1}) + f(x_k) \right) \]

is between \( f(x_{k-1}) \) and \( f(x_k) \).

By the Intermediate Value Theorem, we know that there exists a \( d_k \) in \([x_{k-1}, x_k]\) such that

\[ \frac{1}{2} \left( f(x_{k-1}) + f(x_k) \right) = f(d_k) \]
Surface Area

\[ S_k \approx \pi \left( f(x_{k-1}) + f(x_k) \right) \sqrt{1 + \left( f'(c_k) \right)^2} \cdot \Delta x_k \]
\[ = 2\pi f(d_k) \sqrt{1 + \left( f'(c_k) \right)^2} \cdot \Delta x_k \]

That means that the total surface area, \( S \), is approximately

\[ S = \sum_{k=1}^{n} S_k \approx \sum_{k=1}^{n} 2\pi f(d_k) \sqrt{1 + \left( f'(c_k) \right)^2} \cdot \Delta x_k \]
Surface Area

We expect that

\[ S = \lim_{\|P\| \to 0} \sum_{k=1}^{n} 2\pi f(d_k)\sqrt{1 + (f'(c_k))^2} \cdot \Delta x_k \]

If \( c_k = d_k \), then this would be the definite integral

\[ \int_{a}^{b} 2\pi f(x)\sqrt{1 + (f'(x))^2} \, dx \]

It can be proved (not by us now) that the limit is indeed the definite integral even if \( c_k \neq d_k \).
Surface Area Definition

Let \( f \) be a nonnegative, smooth function on \([a, b]\). Then the surface area \( S \) generated by revolving the portion of the curve \( y = f(x) \) between \( x = a \) and \( x = b \) about the \( x \)-axis is

\[
S = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx
\]

Let \( x = g(y) \) be a nonnegative, smooth function on \([c, d]\). Then the surface area \( S \) generated by revolving the portion of the curve \( x = g(y) \) between \( y = c \) and \( y = d \) about the \( y \)-axis is

\[
S = \int_c^d 2\pi g(y) \sqrt{1 + (g'(y))^2} \, dy
\]
Example 1

Find the surface area generated by revolving the curve

\[ y = \sqrt{1 - x^2}, \quad 0 \leq x \leq \frac{1}{2} \]

about the x-axis.

Solution:
The graph of the curve is the upper semi-circle of radius 1 centered at the origin.
Example 1 (continued)

\[ y = \sqrt{1 - x^2} \]
\[ \frac{dy}{dx} = \frac{-x}{\sqrt{1 - x^2}} \]

\[ S = \int_{a}^{b} 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx \]

\[ = \int_{0}^{1/2} 2\pi \sqrt{1 - x^2} \sqrt{1 + \left( \frac{-x}{\sqrt{1 - x^2}} \right)^2} \, dx \]

\[ = \int_{0}^{1/2} 2\pi \sqrt{1 - x^2} \sqrt{1 + \frac{x^2}{1 - x^2}} \, dx \]

\[ = \int_{0}^{1/2} 2\pi \sqrt{1 - x^2} \frac{1}{\sqrt{1 - x^2}} \, dx \]

\[ = \int_{0}^{1/2} 2\pi \, dx \]

\[ = \cdots = \pi \]

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Example 2

Find the surface area generated by revolving the curve

\[ y = \sqrt[3]{3x}, \quad 0 \leq y \leq 2 \]

about the \( y \)-axis.

Solution:

\[ y = \sqrt[3]{3x} \Rightarrow x = \frac{1}{3}y^3 \]

\[ \frac{dx}{dy} = y^2 \]
Example 2 (continued)

\[ S = \int_{c}^{d} 2\pi g(y)\sqrt{1 + (g'(y))^2} \, dy \]

\[ = \int_{0}^{2} 2\pi \left( \frac{1}{3} y^3 \right) \sqrt{1 + (y^2)^2} \, dy \]

\[ = \int_{0}^{2} \frac{2\pi}{3} y^3 \sqrt{1 + y^4} \, dy \]

\[
\begin{align*}
  u &= 1 + y^4 \\
  du &= 4y^3 \, dy \Rightarrow \frac{1}{4} \, du = y^3 \, dy \\
  y = 2 &\Rightarrow u = 17 \\
  y = 0 &\Rightarrow u = 1
\end{align*}
\]
Example 2 (continued)

\[ S = \int_0^2 \frac{2\pi}{3} y^3 \sqrt{1 + y^4} \, dy \]

\[ = \int_1^{17} \frac{2\pi}{3} \sqrt{u} \cdot \frac{1}{4} \, du \]

\[ = \cdots = \frac{\pi}{9} (17\sqrt{17} - 1) \]
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Nobody's ordering it.
With all due respect, sir - I suggest we give it an easier-to-pronounce name.

http://thecomicninja.wordpress.com/tag/math/