On the Sternfeld–Levin counterexamples to a conjecture of Chogoshvili–Pontrjagin

Fredric D. Ancel\textsuperscript{a,}\textsuperscript{*}, Tadeusz Dobrowolski\textsuperscript{b,1}

\textsuperscript{a} Department of Mathematical Sciences, University of Wisconsin–Milwaukee, Milwaukee, WI 53201, USA
\textsuperscript{b} Department of Mathematics, Pittsburg State University, Pittsburg, KS 66762-5880, USA

Received 15 May 1996

Abstract

An inconclusive proof in a 1937 paper by G. Chogoshvili spawned an interesting dimension-theoretic conjecture which we call the Chogoshvili–Pontrjagin Conjecture. In 1991, Y. Sternfeld found an ingenious counterexample to this conjecture which he and M. Levin greatly generalized in 1995. In this note we point out a previously unobserved property of the Sternfeld–Levin examples, and we reinterpret their significance in light of this property. Also, we present a version of the Levin–Sternfeld proof which is more “topological” and less “lattice-theoretic” than the original.

\textsuperscript{*} Corresponding author. E-mail: ancel@csd.uwm.edu.
\textsuperscript{1} E-mail: tdobrowo@mail.pittstate.edu.

Keywords: Chogoshvili–Pontrjagin Conjecture; Hereditarily indecomposable continuum; Atom; Atomic map

AMS classification: 54F45

1. Introduction

For $X, Y \subset \mathbb{R}^m$, we say that $X$ can be removed from $Y$ if for every $\varepsilon > 0$, there is a map $f: X \to \mathbb{R}^m$ such that $f(X) \cap Y = \emptyset$ and $\|x - f(x)\| < \varepsilon$ for every $x \in X$. A subset of $\mathbb{R}^m$ is codimension-$k$ if its dimension is $m - k$. In the 1928 paper [1], P.S. Alexandroff established:

**Theorem 0.** If a compact subset $X$ of $\mathbb{R}^m$ can be removed from every codimension-$(k + 1)$ closed subpolyhedron of $\mathbb{R}^m$, then $\dim(X) \leq k$. 
The 1937 paper [4] of G. Chogoshvili asserted that the hypothesis of Theorem 0 could be weakened. It purported to prove that if a subset \( X \) of \( \mathbb{R}^m \) can be removed from every codimension-(\( k + 1 \)) affine subspace of \( \mathbb{R}^m \), then \( \dim(X) \leq k \). (An affine subspace of \( \mathbb{R}^m \) is a subset of the form \( V + x = \{ v + x : v \in V \} \) where \( V \) is a vector subspace of \( \mathbb{R}^m \).) The first indication of a flaw in Chogoshvili’s argument came in the 1953 paper [15] of K. Sitnikov (or see [8, pp. 113–115]) which gave an example of a noncompact 2-dimensional subspace of \( \mathbb{R}^3 \) that can be removed from every codimension-2 closed subpolyhedron of \( \mathbb{R}^3 \), showing that Chogoshvili’s claim is false without a compactness hypothesis. However, in about 1989, R. Daverman, A. Dranishnikov and R. Pol independently observed that even with an additional compactness hypothesis, Chogoshvili’s proof has a fundamental gap (see [9]). This led to the following formulation:

**Chogoshvili Conjecture.** If \( X \) is a compact subset of \( \mathbb{R}^m \) that can be removed from every codimension-(\( k + 1 \)) affine subspace of \( \mathbb{R}^m \), then \( \dim(X) \leq k \).

Chogoshvili’s paper [4] contains a remark attributed to L.S. Pontrjagin to the effect that Chogoshvili’s argument (were it correct) would actually entail a stronger result:

**Chogoshvili–Pontrjagin Conjecture.** If \( X \) is a compact subset of \( \mathbb{R}^m \) that can be removed from every codimension-(\( k + 1 \)) coordinate subspace of \( \mathbb{R}^m \), then \( \dim(X) \leq k \). (A coordinate subspace of \( \mathbb{R}^m \) is a subset of the form \( V + x = \{ v + x : v \in V \} \) where \( V \) is a vector subspace of \( \mathbb{R}^m \) that is determined by coordinate axes.)

It is known that the Chogoshvili–Pontrjagin Conjecture is true for compacta in \( \mathbb{R}^m \) for the values \( k = 0, m - 2 \) and \( m - 1 \) [13]. Hence, it is true for all compacta in \( \mathbb{R}^3 \). It is also known that compact 2-dimensional ANRs cannot be counterexamples to the Chogoshvili–Pontrjagin Conjecture [6]. However, in 1991, Y. Sternfeld constructed a fascinating family of counterexamples to the Chogoshvili–Pontrjagin Conjecture [16]. He showed that for each \( n \geq 2 \), there is an \( n \)-dimensional compactum that can be embedded in a Euclidean space so that it can be removed from every codimension-2 coordinate subspace. In 1995, M. Levin and Sternfeld broadened Sternfeld’s original example [14]. They showed that, for \( n \geq 3 \), every \( n \)-dimensional compactum can be embedded in a Euclidean space so that it can be removed from every codimension-3 coordinate subspace. We will refer to the 1991 examples of Sternfeld and the 1995 examples of Levin and Sternfeld together as the *Sternfeld–Levin examples*. Also in 1995, A. Dranishnikov constructed ingenious counterexamples to the original Chogoshvili Conjecture [7]. He found that, for each \( m \geq 4 \) and \( 2 \leq k \leq m - 2 \), there is a \( k \)-dimensional compactum in \( \mathbb{R}^m \) that can be removed from every codimension-\( k \) affine subspace.

We will now formulate a conjecture that is weaker than the Chogoshvili–Pontrjagin Conjecture which we will show is also disproved by the Sternfeld–Levin examples. Let \( \prod_{a \in A} E_a \) be a finite Cartesian product of Euclidean spaces. For \( B \subset A \), let

\[
\pi_B : \prod_{a \in A} E_a \to \prod_{a \in B} E_a
\]

We will now formulate a conjecture that is weaker than the Chogoshvili–Pontrjagin Conjecture which we will show is also disproved by the Sternfeld–Levin examples. Let \( \prod_{a \in A} E_a \) be a finite Cartesian product of Euclidean spaces. For \( B \subset A \), let
denote the natural projection: $\pi_B((x_a)_{a \in A}) = (x_a)_{a \in B}$. If $B$ is a subset of $A$ with $r$ elements and $Q$ is a closed subpolyhedron of $\prod_{a \in B} E_a$, then a closed subpolyhedron of $\prod_{a \in A} E_a$ of the form $\pi_{-B}^{-1}(Q)$ is called $r$-projected. Note that if $Q$ is a codimension-$k$ closed subpolyhedron of $\prod_{a \in B} E_a$, then $\pi_{-B}^{-1}(Q)$ is codimension-$k$ in $\prod_{a \in A} E_a$. With this terminology, we state:

**Projected Polyhedron Conjecture.** For each $k \geq 1$, there is an $r \geq 1$ with the following property: if $X$ is any compact subset of a finite Cartesian product of Euclidean spaces $\prod_{a \in A} E_a$ such that $X$ can be removed from every $r$-projected codimension-$(k + 1)$ closed subpolyhedron of $\prod_{a \in A} E_a$, then $\dim(X) \leq k$.

The power of this conjecture would be felt when the cardinality of $A$ is much greater than $r$. For then the dimension of $X$ would be bounded by removing it from subpolyhedra arising from small subproducts of $\prod_{a \in A} E_a$. This conjecture is clearly stronger than Theorem 0. On the other hand, it is weaker than the Chogoshvili–Pontrjagin Conjecture because, once coordinate axes are specified in each $E_a$, every codimension-$(k + 1)$ coordinate subspace is apparently a $(k + 1)$-projected codimension-$(k + 1)$ closed subpolyhedron.

The Sternfeld–Levin examples disprove the Projected Polyhedron Conjecture. Thus, the Sternfeld–Levin examples not only can be removed from a large class of codimension-$(k + 1)$ affine subspaces (the coordinate subspaces) without bounding its dimension by $k$; they can also be removed from an even larger class of codimension-$(k + 1)$ closed subpolyhedra (the $r$-projected subpolyhedra) without bounding its dimension. In light of this observation, we reinterpret the significance of the Sternfeld–Levin examples. We take the point of view that these examples are not directly germane to the issues separating the Chogoshvili Conjecture from Theorem 0. They do not illuminate the geometric distinction between removing a compactum from codimension-$(k + 1)$ closed subpolyhedra and removing it from codimension-$(k + 1)$ affine subspaces. Instead, the examples illustrate the difference between being able to remove a compactum from all codimension-$(k + 1)$ closed subpolyhedra versus being able to remove it from the large but restricted class of $r$-projected codimension-$(k + 1)$ closed subpolyhedra which depends on a particular Cartesian factorization of the ambient Euclidean space.

The construction of the Sternfeld–Levin examples relies essentially on the notion of a *hereditarily indecomposable continuum* or *atom*. Recall that a continuum is a compact connected metrizable space, a continuum is *indecomposable* if it is not the union of two proper subcontinua, and a continuum is *hereditarily indecomposable* if each of its subcontinua is indecomposable. An *atom* is a hereditarily indecomposable continuum. The existence of $n$-dimensional atoms for each $n \geq 1$ is established in [2].

Our main theorems imply a slightly stronger result than the negation of the Projected Polyhedron Conjecture. Their statements require additional terminology. Let $\prod_{a \in A} E_a$ be a finite Cartesian product of spaces each of which is homeomorphic to a Euclidean space. By a *topological subpolyhedron* of $\prod_{a \in A} E_a$, we mean a subset which becomes a subpolyhedron under some identification of each $E_a$ with a Euclidean space. Now
the expression "r-projected codimension-\((k + 1)\) closed topological subpolyhedron of \(\prod_{a \in A} E_a\)" has meaning. If \(A\) is a finite set, let \(|A|\) denote the number of elements in \(A\). If \(p \geq q \geq 0\), let \(\binom{p}{q} = \frac{p!}{q!(p-q)!}\). If \(t\) is a real number, let \([t]\) denote the greatest integer \(\leq t\). Our principal theorems are our versions of the main results of [16] and [14].

**Theorem 1** (The Stemfeld construction). Let \(n \geq 1\). Then for every \(n\)-dimensional atom \(X\), for each \(s \geq n\), there is an embedding of \(X\) into a finite Cartesian product of Euclidean spaces \(\prod_{a \in A} E_a\) where \(|A| = \binom{s}{n-1}\), \(\dim(E_a) \leq 4(s-n) + 3\), and the following condition is satisfied: If \(k\) and \(r\) are positive integers such that \(r \leq k\left[(s-1)/(n-1)\right]\), then \(X\) can be removed from every \(r\)-projected codimension-\((k + 1)\) closed topological subpolyhedron of \(\prod_{a \in A} E_a\).

**Theorem 2** (The Stemfeld–Levin construction). Let \(n \geq 1\). Then for every \(n\)-dimensional compactum \(X\), for each \(s \geq n\), there is an embedding of \(X\) into a finite Cartesian product of Euclidean spaces \(\prod_{a \in A} E_a\) where \(|A| = \binom{s}{n-1}\), \(\dim(E_a) \leq 6(s-n) + 5\), and the following condition is satisfied: If \(k\) and \(r\) are positive integers such that \(r \leq k\left[(s-1)/(n-1)\right]\), then \(X\) can be removed from every \(r\)-projected codimension-\((2k + 1)\) closed topological subpolyhedron of \(\prod_{a \in A} E_a\).

We will present a proof which simultaneously establishes Theorems 1 and 2. Our proofs are more "topological" and less "lattice-theoretic" variations on the original arguments, and they draw slightly stronger conclusions in that we remove \(X\) from \(r\)-projected subpolyhedra rather than just coordinate subspaces.

Observe that the hypothesis of Theorem 2 is more general than that of Theorem 1: Theorem 2 considers all finite-dimensional compacta, while Theorem 1 considers only atoms. However this gain in generality comes at a price: the codimension of the projected polyhedra from which \(X\) can be removed jumps from \(k + 1\) in Theorem 1 to \(2k + 1\) in Theorem 2.

Both Theorem 1 and Theorem 2 provide counterexamples to the Projected Polyhedron and Chogoshvili-Pontrjagin Conjectures, though Theorem 1 does this more efficiently. For instance, if \(k\) and \(r\) are any positive integers, \(n > k\), and \(s \geq n\) is chosen so large that \([(s-1)/(n-1)] \geq r/k\), then Theorem 1 implies the following counterexample to the Projected Polyhedron Conjecture:

**Corollary 3.** For all positive integers \(k\) and \(r\), for each integer \(n > k\), there is an embedding of an \(n\)-dimensional compactum \(X\) in a finite Cartesian product of Euclidean spaces \(\prod_{a \in A} E_a\) such that \(X\) can be removed from every \(r\)-projected codimension-\((k + 1)\) closed topological subpolyhedron of \(\prod_{a \in A} E_a\).

Since every codimension-\((k + 1)\) coordinate subspace of a finite Cartesian product of Euclidean spaces is a \((k + 1)\)-projected codimension-\((k + 1)\) closed subpolyhedron, Corollary 3 implies the following counterexample to the Chogoshvili-Pontrjagin Conjecture:
Corollary 4. For all positive integers $n > k$, there is an embedding of an $n$-dimensional compactum $X$ in a finite Cartesian product of Euclidean spaces $\prod_{a \in A} E_a$ such that $X$ can be removed from every codimension-$(k + 1)$ coordinate subspace of $\prod_{a \in A} E_a$.

The lowest dimensions in which Theorem 1 produces interesting examples are $n = 2$ and $k = 1$. If we take $s = 2$, so that $|A| = 2$ and $\dim(E_a) = 3$ for $a \in A$, then Theorem 1 produces a 2-dimensional atom in $\mathbb{R}^3 \times \mathbb{R}^3$ which can be removed from every 1-projected codimension-2 closed subpolyhedron. Since every codimension-2 coordinate subspace is a 2-projected codimension-2 closed subpolyhedron, we see that if we take $s = 3$, so that $|A| = 3$ and $\dim(E_a) = 7$ for each $a \in A$, then Theorem 1 provides a 2-dimensional atom in $\mathbb{R}^6$ which can be removed from every codimension-2 coordinate subspace.

2. The proofs of Theorems 1 and 2

Because these proofs are technically complicated, we preface them with a short discussion to highlight the essential goal of the argument. Suppose $X$ is an $n$-dimensional atom (as in Theorem 1) or more generally an $n$-dimensional compactum (as in Theorem 2). Choose $s \geq n$ and set $r_1 = [(s - 1)/(n - 1)]$. Our principal goal will be to embed $X$ in a finite Cartesian product of Euclidean spaces $\prod_{a \in A} E_a$ with the following property: If $B \subset A$ and $|B| \leq r_1$, then the restriction $\pi_B|X : X \to \prod_{a \in B} E_a$ of the natural projection $\pi_B : \prod_{a \in A} E_a \to \prod_{a \in B} E_a$ factors through a 2-dimensional compactum $Z$ (i.e., $\pi_B|X : X \to \prod_{a \in B} E_a$ is the composition of maps $X \to Z$ and $Z \to \prod_{a \in B} E_a$). Moreover, when $X$ is an atom, a 1-dimensional $Z$ can be found. This goal is accomplished via the sequence of Lemmas 5–10 below. Once this goal is attained, even more is true: If $B \subset A$ and $|B| < kr_1$ for some positive integer $k$, then $\pi_B|X : X \to \prod_{a \in B} E_a$ factors through a compactum $Z$ which is the Cartesian product of $k$ 2-dimensional compacta. Thus $\dim(Z) \leq 2k$. Then the map $Z \to \prod_{a \in B} E_a$ can be moved off any codimension-$(2k + 1)$ closed subpolyhedron of $\prod_{a \in B} E_a$ by “transversality”. (See Lemma 11.) It follows that $X$ can be removed from any $(kr_1)$-projected codimension-$(2k + 1)$ closed subpolyhedron of $\prod_{a \in A} E_a$. Furthermore, when $X$ is an atom, $\dim(Z) \leq k$ and $X$ can be removed from any $(kr_1)$-projected codimension-$(k + 1)$ closed subpolyhedron of $\prod_{a \in A} E_a$. These are the conclusions of Theorems 1 and 2.

Properties of monotone and light maps play crucial roles in the subsequent proofs. We review these notions. A map is monotone if each point inverse is connected. A map is light if each point inverse is 0-dimensional. The monotone-light factorization theorem [5, Theorem 3, p. 18] says that any map $\varphi : X \to Y$ between compact metric spaces can be factored as $\varphi = \lambda \circ \mu$ where $\mu : X \to Z$ is a monotone map, $Z$ is a compact metric space and $\lambda : Z \to Y$ is a light map. If $X$ and $Y$ are topological spaces, let $\text{Map}(X, Y)$ denote the space of all maps from $X$ to $Y$ with the compact-open topology, and let $\mathcal{L}(X, Y)$ denote the subspace of all light maps from $X$ to $Y$. Hurewicz proved that if $X$ is an $n$-dimensional compactum, then $\mathcal{L}(X, \mathbb{R}^n)$ is a dense $G_\delta$ subset of $\text{Map}(X, \mathbb{R}^n)$. (See [11, Assertion IX, p. 77].)
Certain basic properties of atoms are used in the following proofs. Clearly an subcontinuum of an atom is an atom. If two subcontinua of an atom intersect, then one must contain the other because their union is an atom. Consequently, any collection of subcontinua of an atom with a point in common must be nested. We define a map to be atomic if each point inverse is an atom. The Ph.D. thesis [3] constructs an atomic map from $\mathbb{R}^m - \{0\}$ to $\mathbb{R}$ for each $m \geq 1$.

Finally, we record an observation which is used repeatedly below. If 

$$
\varphi = (\varphi_i)_{i \in a} : X \to \prod_{i \in a} Y_i
$$

is a map to a product, then

$$
\varphi^{-1}(\varphi(x)) = \bigcap_{i \in a} \varphi_i^{-1}(\varphi_i(x))
$$

for each $x \in X$.

We now begin the sequence of lemmas that lead to proofs of Theorems 1 and 2.

Let $X$ be an $n$-dimensional compactum and let $s \geq n$. Lemma 5 allows us to decompose $X$ into atoms in a controlled way.

**Lemma 5.** There is an atomic map $\alpha : X \to T$ where $T$ is a 1-dimensional compactum.

**Proof.** First we obtain a map $\gamma : X \to \mathbb{R}$ such that for each $r \in \mathbb{R}$, each component of $\gamma^{-1}(r)$ is an atom. Embed $X$ in $\mathbb{R}^{2n+1} - \{0\}$. The thesis [3] provides an atomic map 

$$
\varphi : \mathbb{R}^{2n+1} - \{0\} \to \mathbb{R}.
$$

Set $\gamma = \varphi|X : X \to \mathbb{R}$. For $r \in \mathbb{R}$, each component of $\gamma^{-1}(r)$ is an atom because it is a subcontinuum of the atom $\varphi^{-1}(r)$. Alternatively, the existence of $\gamma$ is implied by Theorem 1.8 of [12]. (Moreover, the proofs in [12] are independent of [3].)

The previously mentioned monotone-light factorization theorem implies $\gamma = \beta \circ \alpha$ where $\alpha : X \to T$ is a monotone map, $T$ is a compact metric space, and $\beta : T \to \mathbb{R}$ is a light map. For each $t \in T$, $\alpha^{-1}(t)$ is an atom because it is a subcontinuum of a component of $\gamma^{-1}(\beta(t))$ which is an atom. Thus, $\alpha$ is an atomic map. Furthermore, according to a theorem of Hurewicz [8, Theorem 1.12.4, p. 136],

$$
\dim(T) \leq \dim(\mathbb{R}) + \max\{\dim(\beta^{-1}(r)) : r \in \mathbb{R}\}.
$$

Hence, $\dim(T) \leq 1$. □

Lemma 6 is the heart of the Sternfeld–Levin constructions. Our proof is simply a reexposition of Sternfeld’s clever argument in [16] and its generalization in [14].

**Lemma 6.** For $1 \leq i \leq s$, there are atomic maps $\mu_i : X \to Z_i$ onto 2-dimensional compacta $Z_i$ with the following two properties.

(a) For $1 \leq i \leq s$ and $x \in X$, $\mu_i^{-1}(\mu_i(x)) \subset \alpha^{-1}(\alpha(x))$. (Here, $\alpha$ is the map from Lemma 5.)
For every \( b \in \{1, 2, \ldots, s\} \) with \( |b| = n \), the product map

\[
\mu_b = (\mu_i)_{i \in b} : X \to \prod_{i \in b} Z_i
\]

is an embedding.

Moreover, when \( X \) is an atom, all \( Z_i \) can be assumed to be 1-dimensional.

**Proof.** Set \( B = \{ b \in \{1, 2, \ldots, s\} : |b| = n \} \). Let \( b \in B \). Then \( b = \{b_1, b_2, \ldots, b_n\} \) where \( 1 \leq b_1 < b_2 < \cdots < b_n \leq s \). Let \( \pi_b : \mathbb{R}^s \to \mathbb{R}^n \) denote the natural projection \( \pi_b((x_1, x_2, \ldots, x_s)) = (x_{b_1}, x_{b_2}, \ldots, x_{b_n}) \) for \((x_1, x_2, \ldots, x_s) \in \mathbb{R}^s\). Define

\[
\tilde{\pi}_b : \text{Map}(X, \mathbb{R}^s) \to \text{Map}(X, \mathbb{R}^n)
\]

by \( \tilde{\pi}_b(f) = \pi_b \circ f \) for \( f \in \text{Map}(X, \mathbb{R}^s) \). It is easily argued that \( \tilde{\pi}_b \) is an open map. Since \( \mathcal{L}(X, \mathbb{R}^n) \) is a dense \( G_\delta \) subset of \( \text{Map}(X, \mathbb{R}^n) \), it follows that \( \tilde{\pi}_b^{-1}(\mathcal{L}(X, \mathbb{R}^n)) \) is a dense \( G_\delta \) subset of \( \text{Map}(X, \mathbb{R}^s) \).

Since \( \text{Map}(X, \mathbb{R}^s) \) has the Baire property, it follows that \( \bigcap_{b \in B} \tilde{\pi}_b^{-1}(\mathcal{L}(X, \mathbb{R}^n)) \) is a dense \( G_\delta \) subset of \( \text{Map}(X, \mathbb{R}^s) \). Choose \( \varphi \in \bigcap_{b \in B} \tilde{\pi}_b^{-1}(\mathcal{L}(X, \mathbb{R}^n)) \). Hence, for each \( b \in B \), \( \pi_b \circ \varphi : X \to \mathbb{R}^n \) is a light map.

Since \( \varphi : X \to \mathbb{R}^s \), then there are maps \( \varphi_i : X \to \mathbb{R} \) for \( 1 \leq i \leq s \) such that \( \varphi = (\varphi_1, \varphi_2, \ldots, \varphi_s) \). Lemma 5 provides an atomic map \( \alpha : X \to T \) such that \( \dim(T) \leq 1 \). For \( 1 \leq i \leq s \), consider the map \( \psi_i = (\varphi_i, \alpha) : X \to \mathbb{R} \times T \). We apply the monotone-light factorization theorem to \( \psi_i \) to obtain a factorization \( \psi_i = \lambda_i \circ \mu_i \), where \( \mu_i : X \to Z_i \) is a monotone map, \( Z_i \) is a compact metric space, and \( \lambda_i : Z_i \to \mathbb{R} \times T \) is a light map. According to a previously cited theorem of Hurewicz,

\[
\dim(Z_i) \leq \dim(\mathbb{R} \times T) + \max \{ \dim(\lambda_i^{-1}(p)) : p \in \mathbb{R} \times T \}.
\]

Hence, \( \dim(Z_i) \leq 2 \) for \( 1 \leq i \leq s \).

Let \( 1 \leq i \leq s \) and \( x \in X \). Then

\[
\psi_i^{-1}(\varphi_i(x)) = \varphi_i^{-1}(\varphi_i(x)) \cap \alpha^{-1}(\alpha(x)) \quad \text{and} \quad \psi_i^{-1}(\varphi_i(x)) = \mu_i^{-1}(\lambda_i^{-1}(\mu_i(x))) \supset \mu_i^{-1}(\mu_i(x)).
\]

So \( \mu_i^{-1}(\mu_i(x)) \subset \alpha^{-1}(\alpha(x)) \). Since \( \mu_i^{-1}(\mu_i(x)) \) is a subcontinuum of the atom \( \alpha^{-1}(\alpha(x)) \), then \( \mu_i^{-1}(\mu_i(x)) \) is an atom. Thus \( \mu_i \) is an atomic map.

Let \( b \in B \). We will now prove that

\[
\mu_b = (\mu_i)_{i \in b} : X \to \prod_{i \in b} Z_i
\]

is a monotone map. Let \( x \in X \). The collection \( \{\mu_i^{-1}(\mu_i(x)) : i \in b\} \) must be nested because its elements are subcontinua of the atom \( \alpha^{-1}(\alpha(x)) \) with the point \( x \) in common. Since \( \mu_b^{-1}(\mu_b(x)) = \bigcap_{i \in b} \mu_i^{-1}(\mu_i(x)) \), then \( \mu_b^{-1}(\mu_b(x)) = \mu_i^{-1}(\mu_i(x)) \) for some \( i \in b \). We conclude that \( \mu_b \) is monotone.

Next we prove that \( \mu_b \) is a light map. Consider the product map

\[
\psi_b = (\psi_i)_{i \in b} : X \to (\mathbb{R} \times T)^n.
\]
Let $x \in X$. We compute $\psi_b^{-1}(\psi_b(x))$.

$$
\psi_b^{-1}(\psi_b(x)) = \bigcap_{i \in b} \psi_i^{-1}(\psi_i(x)) = \left( \bigcap_{i \in b} \varphi_i^{-1}(\varphi_i(x)) \right) \cap \alpha^{-1}(\alpha(x))
$$

So $\psi_b^{-1}(\psi_b(x)) \subseteq (\pi_b \circ \varphi)^{-1}(\pi_b \circ \varphi(x))$. On the other hand, since

$$
\psi_b = (\lambda_i \circ \mu_i)_{i \in b} = \left( \prod_{i \in b} \lambda_i \right) \circ (\mu_i)_{i \in b} = \Lambda \circ \mu_b
$$

where $\Lambda = (\prod_{i \in b} \lambda_i)$, then

$$
\psi_b^{-1}(\psi_b(x)) = \mu_b^{-1}(\Lambda^{-1}(\Lambda(\mu_b(x)))) \supset \mu_b^{-1}(\mu_b(x)).
$$

Thus, $\mu_b^{-1}(\mu_b(x)) \subseteq (\pi_b \circ \varphi)^{-1}(\pi_b \circ \varphi(x))$. Since $\pi_b \circ \varphi$ is a light map, we conclude that $\mu_b$ is light.

Since $\mu_b$ is monotone and light, it is an embedding.

In the case that $X$ is an atom, we can take $T$ to be a singleton and $\alpha : X \to T$ to be a constant map. Now the preceding computation of $\dim(Z_i)$ yields $\dim(Z_i) \leq 1$. $\square$

Now set $A = \{a \subseteq \{1, 2, \ldots, s\} : |a| = s - n + 1\}$. Then

$$
|A| = \binom{s}{s - n + 1} = \binom{s}{n - 1}.
$$

Let $a \in A$. For each $x \in X$, set $C_a(x) = \bigcup_{i \in a} \mu_i^{-1}(\mu_i(x))$. Define

$$
D_a = \{C_a(x) : x \in X\}.
$$

Lemma 7 establishes that each $D_a$ is an upper semicontinuous decomposition of $X$.

The remarkable fact that distinct elements of $D_a$ are disjoint is absolutely essential to the completion of the Sternfeld–Levin construction, and guaranteeing this fact seems to be the principal role of atoms in the entire argument. Sternfeld presents the proof of Lemma 7 from a lattice-theoretic perspective, whereas our proof is direct and naive.

**Lemma 7.** For each $a \in A$, $D_a$ is an upper semicontinuous decomposition of $X$.

**Proof.** Let $a \in A$. For each $x \in X$, the collection $\{\mu_i^{-1}(\mu_i(x)) : i \in a\}$ must be nested because its elements are subcontinua of the atom $\alpha^{-1}(\alpha(x))$ with the point $x$ in common. Hence, for each $x \in X$, there is an $i \in a$ such that $C_a(x) = \mu_i^{-1}(\mu_i(x))$. Furthermore, $C_a(x) \subseteq \alpha^{-1}(\alpha(x))$.

Next we prove that $D_a$ partitions $X$. $D_a$ covers $X$ because $x \in C_a(x)$ for each $x \in X$. Suppose $x, y \in X$ and $C_a(x) \cap C_a(y) \neq \emptyset$. Then $\alpha^{-1}(\alpha(x)) \cap \alpha^{-1}(\alpha(y)) \neq \emptyset$. Since the point inverses of $\alpha$ partition $X$, then $\alpha^{-1}(\alpha(x)) = \alpha^{-1}(\alpha(y))$. Thus, $C_a(x)$ and $C_a(y)$ are intersecting subcontinua of the atom $\alpha^{-1}(\alpha(x))$. So we may assume that $C_a(x) \subseteq C_a(y)$. Since $C_a(y) = \mu_i^{-1}(\mu_i(y))$ for some $i \in a$. Also $\mu_i^{-1}(\mu_i(x)) \subseteq C_a(x)$. Hence, $\mu_i^{-1}(\mu_i(x)) \subseteq C_a(x)$. Since the point inverses of $\mu$ partition $X$, then $\mu_i^{-1}(\mu_i(x)) = \mu_i^{-1}(\mu_i(y))$. Consequently, $C_a(x) \supset C_a(y)$. We conclude that $C_a(x) = C_a(y)$.
Finally we prove that $D_a$ is upper semicontinuous. Let $U$ be an open subset of $X$. It suffices to prove that the set $V = \bigcup\{C \in D_a: C \subset U\}$ is an open subset of $X$. For each $i \in a$, set $V_i = \bigcup\{\mu_i^{-1}(\mu_i(x)): x \in X \text{ and } \mu_i^{-1}(\mu_i(x)) \subset U\}$. For each $i \in a$, $V_i$ is an open subset of $X$ because the point inverses of $\mu_i$ form an upper semicontinuous decomposition of $X$. We will prove that $V = \bigcap_{i \in a} V_i$. Since $D_a$ partitions $X$, it follows that $x \in V$ if and only if $C_a(x) \subset U$. Since $C_a(x) = \bigcup_{i \in a} \mu_i^{-1}(\mu_i(x))$, we have that $C_a(x) \subset U$ if and only if $\mu_i^{-1}(\mu_i(x)) \subset U$ for each $i \in a$. Since the point inverses of $\mu_i$ partition $X$, it follows that $\mu_i^{-1}(\mu_i(x)) \subset U$ if and only if $x \in V_i$. Combining these remarks, we conclude that $x \in V$ if and only if $x \in V_i$ for each $i \in a$. Thus, $V = \bigcap_{i \in a} V_i$. So $V$ is open.

For each $a \in A$, set $Y_a = X/D_a$ and let $f_a: X \to Y_a$ denote the quotient map. Then each $Y_a$ is a compact metric space. Let $f = (f_a)_{a \in A}: X \to \prod_{a \in A} Y_a$ denote the product map.

**Lemma 8.** $f: X \to \prod_{a \in A} Y_a$ is an embedding.

**Proof.** This proof depends on the fact (established in Lemma 6) that all $\mu_b$ are embeddings and on the choice of the number $s - n + 1$ in the definition of the set $A$.

First we establish the assertion that if $a \subset \{1, 2, \ldots, s\}$, $|a| \geq n$ and $x \in X$, then $\bigcap_{i \in a} \mu_i^{-1}(\mu_i(x)) = \{x\}$. In this situation, let $b \subset a$ such that $|b| = n$. Since $x \in \mu_i^{-1}(\mu_i(x))$ for each $i \in a$, then

$$\{x\} \subset \bigcap_{i \in a} \mu_i^{-1}(\mu_i(x)) \subset \bigcap_{i \in b} \mu_i^{-1}(\mu_i(x)) = \mu_b^{-1}(\mu_b(x)).$$

Lemma 6 implies that $\mu_b^{-1}(\mu_b(x)) = \{x\}$. Our assertion follows.

Let $x \in X$. It suffices to prove that $f^{-1}(f(x)) = \{x\}$.

$$f^{-1}(f(x)) = \bigcap_{a \in A} f_a^{-1}(f_a(x)).$$

For each $a \in A$, there is an $i_a \in a$ such that $f_a^{-1}(f_a(x)) = C_a(x) = \mu_{i_a}^{-1}(\mu_{i_a}(x))$.

Hence, $f^{-1}(f(x)) = \bigcap_{a \in A} \mu_{i_a}^{-1}(\mu_{i_a}(x))$. Now observe that if $|\{i_a: a \in A\}| \geq n$, then the assertion of the preceding paragraph implies $f^{-1}(f(x)) = \{x\}$.

We prove $|\{i_a: a \in A\}| \geq n$ by contradiction. Suppose $|\{i_a: a \in A\}| \leq n - 1$. Then

$$|\{1, 2, \ldots, s\} - \{i_a: a \in A\}| \geq s - n + 1.$$  

So there is a $b \in A$ such that $b \cap \{i_a: a \in A\} = \emptyset$. This contradicts the fact that $i_b \in b$. □

**Lemma 9.** For each $a \in A$, $\dim(Y_a) \leq 3(s - n) + 2$. Moreover, if $X$ is an atom, then $\dim(Y_a) \leq 2(s - n) + 1$ for each $a \in A$.  

Proof. (Here we are again expounding Sternfeld’s proof from [16].) Recall that for each
\(y \in Y_a\), \(f_a^{-1}(y) = C_a(x) = \mu_i^{-1}(\mu_i(x))\) for some \(x \in X\) and some \(i \in a\). For \(1 \leq i \leq s\), set
\[M_i = \{y \in Y_a : f_a^{-1}(y) = \mu_i^{-1}(\mu_i(x))\ \text{for some} \ x \in X\}.
It follows that \(Y_a = \bigcup_{i \in a} M_i\). Hence, by [8, Theorem 1.5.10, p. 45],
\[
\dim(Y_a) \leq \sum_{i \in a} \dim(M_i) + |a| - 1.
\]
\(f_a|f_a^{-1}(M_i)\) has the same set of point inverses as \(\mu_i|f_a^{-1}(M_i) : f_a^{-1}(M_i) \to Z_i\). Hence, 
\(M_i = f_a(f_a^{-1}(M_i))\) is homeomorphic to \(\mu_i(f_a^{-1}(M_i)) \subset Z_i\). Therefore, \(\dim(M_i) \leq \dim(Z_i) \leq 2\) by Lemma 6. Hence,
\[
\dim(Y_a) \leq 3|a| - 1 = 3(s - n + 1) - 1 = 3(s - n) + 2.
\]
When \(X\) is an atom, \(\dim(M_i) \leq \dim(Z_i) \leq 1\) and, consequently,
\[
\dim(Y_a) \leq 2|a| - 1 = 2(s - n + 1) - 1 = 2(s - n) + 1.
\]
Lemma 9 implies that for each \(a \in A\), there is an embedding \(e_a : Y_a \to E_a\) where \(E_a\) is a Euclidean space of dimension \(\leq 2(3(s - n) + 2) + 1 = 6(s - n) + 5\). Furthermore, when \(X\) is an atom, then we can assume \(\dim(E_a) \leq 2(2(s - n) + 1) + 1 = 4(s - n) + 3\). Set
\[
e = \prod_{a \in A} e_a : \prod_{a \in A} Y_a \to \prod_{a \in A} E_a.
\]
e is an embedding. So \(e \circ f : X \to \prod_{a \in A} E_a\) is an embedding. Identify \(X\) with the subset 
\(e \circ f(X)\) of \(\prod_{a \in A} E_a\). Then \(\text{id}_X = e \circ f\). Also for \(B \subset A\), let \(\pi_B : \prod_{a \in A} E_a \to \prod_{a \in B} E_a\) denote the natural projection.

The \(k = 1\) case of the next lemma is the result that we identified previously as our principal goal.

Lemma 10. If \(B \subset A\) and \(k\) is a positive integer such that \(|B| \leq k\lfloor(s - 1)/(n - 1)\rfloor\),
then there is a \(c \in \{1, 2, \ldots, s\}\) with \(|c| \leq k\) such that \(\pi_B|X : X \to \prod_{a \in B} E_a\) factors
as the composition of the map \(\mu_c : X \to \prod_{i \in c} Z_i\) and a map from \(\prod_{i \in c} Z_i\) to \(\prod_{a \in B} E_a\).

Proof. We first prove the \(k = 1\) case and then deduce the general case from it. By way
of motivation, we briefly explain why factorization occurs in the \(k = 1\) case. In this
situation \(\bigcap B \neq \emptyset\) simply because of the constraints on the size of \(B\) and the size of its
elements. Hence, we can fix an \(i \in \bigcap B\). Then for each \(a \in B\),
\[
\mu_i^{-1}(\mu_i(x)) \subset C_a(x) = f_a^{-1}(f_a(x))
\]
for each \(x \in X\). Hence, \(\mu_i^{-1}(\mu_i(x)) \subset \bigcap_{a \in B} f_a^{-1}(f_a(x)) \subset (\pi_H|X)^{-1}(\pi_H(x))\) for each
\(x \in X\). Consequently, \(\pi_B|X\) factors as the composition of \(\mu_i : X \to Z_i\) and a map from
\(Z_i\) to \(\prod_{a \in B} E_a\).
Here are the details. For each \( B \subset A \), let \( f_B = \left( f_a \right)_{a \in B} : X \to \prod_{a \in B} Y_a \), and note that \( \pi_B | X - \pi_B \circ e \circ f - \left( \prod_{a \in B} e_a \right) \circ f_B \). Hence, it suffices to prove that if \( B \subset A \) and \( k \) is a positive integer such that \( |B| \leq k\left( (s-1)/(n-1) \right) \), then there is a \( c \subset \{1, 2, \ldots, s\} \) with \( |c| \leq k \) such that \( f_B : X \to \prod_{a \in B} Y_a \) factors as the composition of the map \( \mu_c : X \to \prod_{i \in c} Z_i \) and a map from \( \prod_{i \in c} Z_i \) to \( \prod_{a \in B} Y_a \).

Begin by considering the case \( k = 1 \): let \( B \subset A \) such that \( |B| \leq \left( (s-1)/(n-1) \right) \). We first prove \( n_B \neq 0 \). For suppose \( n_B = 0 \). Then \( s = \{1, 2, \ldots, s\} - n_B = \{1, 2, \ldots, s\} - 0 = \{1, 2, \ldots, s\} - 0 = \{1, 2, \ldots, s\} - 0 = (n-1)\left( (s-1)/(n-1) \right) = s-1 \), a contradiction.

To complete the \( k = 1 \) case, let \( i \in \bigcap B \). Let \( x \in X \). For each \( a \in B \), since \( i \in a \), then \( \mu_i^{-1}(\mu_i(x)) \subset C_a(x) = f_a^{-1}(f_a(x)) \). Hence

\[
\mu_i^{-1}(\mu_i(x)) \subset \bigcap_{a \in B} f_a^{-1}(f_a(x)) = f_B^{-1}(f_B(x)).
\]

Thus, each point inverse of \( \mu_i \) is contained in a point inverse of \( f_B \). Since \( \mu_i \) is a surjective closed map, it follows that \( f_B \) factors as the composition of \( \mu_i \) and a map from \( Z_i \) to \( \prod_{a \in B} Y_a \).

We now consider the general case: \( B \subset A \) and \( k \) is a positive integer such that \( |B| \leq k\left( (s-1)/(n-1) \right) \). We decompose \( B \) into a disjoint union \( B = B_1 \cup B_2 \cup \cdots \cup B_k \) where \( |B_j| \leq \left( (s-1)/(n-1) \right) \) for \( 1 \leq j \leq k \). Then for \( 1 \leq j \leq k \), the \( k = 1 \) case of this lemma provides an \( i_j \in \{1, 2, \ldots, s\} \) such that \( f_B \) factors as the composition of the map \( \mu_{i_j} : X \to Z_{i_j} \) and a map from \( Z_{i_j} \) to \( \prod_{a \in B_j} Y_a \). Set \( c = \{i_j : 1 \leq j \leq k\} \). It follows that \( f_B = (f_B)_{1 \leq j \leq k} \) factors as the composition of the map

\[
(\mu_{i_j})_{1 \leq j \leq k} = (\mu_{i_j})_{i \in c} = \mu_c : X \to \prod_{i \in c} Z_i
\]

and a map from \( \prod_{i \in c} Z_i = \prod_{1 \leq j \leq k} Z_{i_j} \) to \( \prod_{1 \leq j \leq k} \prod_{a \in B_j} Y_a = \prod_{a \in B} Y_a \). \( \square \)

To complete the proofs of Theorems 1 and 2: let \( k \) be a positive integer, let \( B \subset A \) such that \( |B| \leq k\left( (s-1)/(n-1) \right) \), and let \( Q \) be a codimension-(2\( k+1 \)) closed topological subpolyhedron of \( \prod_{a \in B} E_a \). We must prove that \( X \) can be removed from \( \pi_B^{-1}(Q) \). (In the case that \( X \) is an atom, we take \( Q \) to be codimension-(\( k+1 \)).) Identify each \( E_a \) with a Euclidean space so that \( \prod_{a \in B} E_a \) and \( \prod_{a \in A} E_a \) become Euclidean spaces and \( Q \) becomes a subpolyhedron of \( \prod_{a \in B} E_a \).

Lemma 10 provides a subset \( c \) of \( \{1, 2, \ldots, s\} \) and a map \( \zeta : \prod_{i \in c} Z_i \to \prod_{a \in B} E_a \) such that \( |c| \leq k \) and \( \pi_B \restriction X = \zeta \circ \mu_c \). Since \( \text{dim}(Z_i) \leq 2 \) for each \( i \in c \) by Lemma 6, then \( \text{dim}(\prod_{i \in c} Z_i) \leq 2|c| \leq 2k \) by [8, Theorem 1.5.16, p. 46]. (When \( X \) is an atom, \( \text{dim}(Z_i) \leq 1 \) for \( i \in c \) and, therefore, \( \text{dim}(\prod_{i \in c} Z_i) \leq k \).)
The following lemma formalizes a well known dimension-theoretic technique. (A more general result appears in [10, Theorem 6.1, p. 164, and Remark 6.3, p. 168.])

Lemma 11. If \( \zeta : Z \to E \) is a map from a compact \( k \)-dimensional metric space to a Euclidean space, then for every \( \varepsilon > 0 \), there is a compact \( k \)-dimensional polyhedron \( P \) and maps \( \psi : Z \to P \) and \( \chi : P \to E \) such that \( |\zeta(z) - \chi \circ \psi(z)| \) for every \( z \in Z \).

Proof. There is a map \( \psi : Z \to |K| \) where \( K \) is a \( k \)-dimensional finite simplicial complex such that \( \text{diam}(\zeta(\psi^{-1}(\text{star}(v, K)))) < \varepsilon \) for each vertex \( v \) of \( K \). (\( K \) is the “nerve” of an appropriate open cover of \( Z \). See [8, Theorem 1.10.16, p. 110].) Map each vertex \( v \) of \( K \) to a point of \( \zeta(\psi^{-1}(\text{star}(v, K))) \) and extend this map affinely over each simplex of \( K \) to obtain a map \( \chi : |K| \to E \). Now consider a point \( z \in Z \). \( \psi(z) \) belongs to some simplex \( \sigma \) of \( K \). The preceding choices guarantee that \( |\zeta(z) - \chi(v)| < \varepsilon \) for every vertex \( v \) of \( \sigma \).

Since \( \chi \circ \psi(z) \in \chi(\sigma) = \text{the convex hull of } \{ \chi(v) \mid v \text{ is a vertex of } \sigma \} \), it follows easily that \( |\zeta(z) - \chi \circ \psi(z)| < \varepsilon \). \( \Box \)

We apply Lemma 11 to the map \( \zeta : \prod_{i \in c} Z_i \to \prod_{a \in B} E_a \). Let \( \varepsilon > 0 \). Since \( \text{dim}(\prod_{i \in c} Z_i) < 2k \), then Lemma 11 provides a compact \( 2k \)-dimensional polyhedron \( P \) and maps \( \psi : \prod_{i \in c} Z_i \to P \) and \( \chi : P \to \prod_{a \in B} E_a \) such that \( \zeta \) and \( \chi \circ \psi \) differ by \( < \varepsilon \). (When \( X \) is an atom, \( \text{dim}(\prod_{i \in c} Z_i) \leq k \) and, hence, \( \text{dim}(P) \leq k \).) We perturb \( \chi \) slightly to make it transverse to \( Q \). Then \( \chi(P) \cap Q = \emptyset \), because \( \text{dim}(P) = 2k \) and \( Q \) is codimension-(\( 2k + 1 \)) in \( \prod_{a \in B} E_a \). (Similarly, \( \chi(P) \cap Q = \emptyset \) when \( X \) is an atom, because \( \text{dim}(P) = k \) and \( Q \) is codimension-(\( k + 1 \)) in \( \prod_{a \in B} E_a \).) It follows that the map \( \pi_B \mid X = \zeta \circ \mu_c \) differs by \( < \varepsilon \) from the map \( \chi \circ \psi \circ \mu_c : X \to \prod_{a \in B} E_a \) and \( \chi \circ \psi \circ \mu_c(X) \cap Q = \emptyset \). Define the map \( \xi : X \to \prod_{a \in A} E_a \) by \( \xi = (\chi \circ \psi \circ \mu_c, \pi_{A-B} \mid X) \).

Hence, the map \( \text{id}_X = (\pi_B \mid X, \pi_{A-B} \mid X) : X \to \prod_{a \in A} E_a \) differs by less than \( \varepsilon \) from \( \xi \) and \( \xi(X) \cap \pi_B^{-1}(Q) \subset \pi_B^{-1}(\chi \circ \psi \circ \mu_c(X) \cap Q) = \emptyset \). We conclude that \( X \) can be removed from \( \pi_B^{-1}(Q) \). \( \Box \)

References