Frieze and Wallpaper Symmetry Groups Classification under Affine and Perspective Distortion

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ABSTRACT

In this paper we study classification of 2D repeated patterns in terms of their respective symmetry groups — the well-known seven Frieze groups and 17 wallpaper groups. Computer algorithms for Frieze and wallpaper symmetry group classification are developed in Euclidean as well as affine spaces. Several symmetry invariants of these groups under affine transformations are analyzed in detail and used to extend the Euclidean group classification algorithm for patterns that are distorted under affine transformations. Experimental results on computer generated images and photos of natural scenes are presented. Precise classification of 2D repeated patterns in terms of their symmetry groups provides a computational means for image indexing, image matching, object recognition, and motion recovery.

This is a report of an on-going research effort. Existing prbolems and future work are discussed.

1 Motivation

Any subset S of the Euclidean space R^n is associated with a **symmetry group** — all the rigid transformations that keep S setwise invariant [2, 9, 21]. A symmetry group is thus one of the important descriptors of any geometric entity. In particular, the symmetry group of a repeated pattern in a plane belongs to a finite possible set. These are the seven *Frieze groups* [14, 29] for 2D patterns repeated along one dimension, and the seventeen *wallpaper groups* [26] for patterns extended by two linearly independent translational generators. These groups are also called one and two dimensional Crystallographic groups [12].

Frieze and wallpaper groups were discovered and studied more than one hundred years ago: In 1831 Hessel first classified the 32 three-dimensional point groups (finite subgroups of the orthogonal group O(3)) which correspond to the three-dimensional crystal classes [13]. All of the symmetrical networks of points which can have crystallographic symmetry were found geometrically by Frankenheim in 1835 [8]. In the late 19th century Fedorov, Schoenflies, and Barlow classified the 17 wallpaper groups (two-dimensional crystallographic groups) and the 320 three-dimensional crystallographic groups [5, 7, 6, 27, 1]. Dutch artist Escher (1898-1972) used these groups in creating his intriguing interlocking repeated patterns. Yet we have not seen existing computational algorithms to classify these groups automatically for given repeated 2D patterns, nor are we aware of any existing applications of these groups in computer vision.

Using wallpaper groups as a basic scheme for indexing repeated 2D image patterns is especially meaningful since there is only a finite number of such groups regardless of the infinite possible varieties of different 2D repeated patterns. If one can determine the symmetry group of a given 2D repeated pattern then one can quickly sort out a large number of images into a finite number of buckets, 7 Frieze groups or 17 wallpaper groups. Further treatment of image patterns can be guided by their symmetry groups, for example, only those patterns who have the same symmetry groups can possibly match each other — a necessary condition. Furthermore, the symmetry group of a pattern is an effective descriptor because it is independent of the size, absolute color, lighting and density of the pattern.

In this paper, we report our effort on developing an algorithm to identify the symmetry group of a given 2D repeated pattern under Euclidean, affine transformations. The problem we are trying to solve in this paper is not to detect a repeated pattern from a complicated scene as some authors have done [18, 24]. Instead, given a repeated 2D pattern in a plane our goal is to classify which one of the seven Frieze groups or 17 wallpaper groups the pattern belongs to. The outcome of our work is a mathematical classification (in terms of symmetry groups) of repeated patterns into finitely countable classes.

Our approach is intensity based, it is not limited to recognizing the exact patterns in the picture (pattern contours), but captures instead the underlying grid structure of a pattern. Taking advantage of how exactly these symmetry groups behave under various transformations, our algorithm looks for a selected set of possible invariant symmetries. By verifying their existence in a logical manner, it is possible to determine the symmetry group (from a distorted pattern) of a pattern before its deformation.

Section 2 briefly describes some related work. In Section 3 the structures of Frieze and wallpaper groups are given respectively. A simple wallpaper group classification algorithm in Euclidean space is also described. Section 4 studies wallpaper groups under non-rigid transformations. Using several invariant symmetry subgroups, an algorithm for wallpaper group classification under affine distortions is presented. Experiment results are shown in Section 5. Section 6 concludes the paper.

2 Related Work

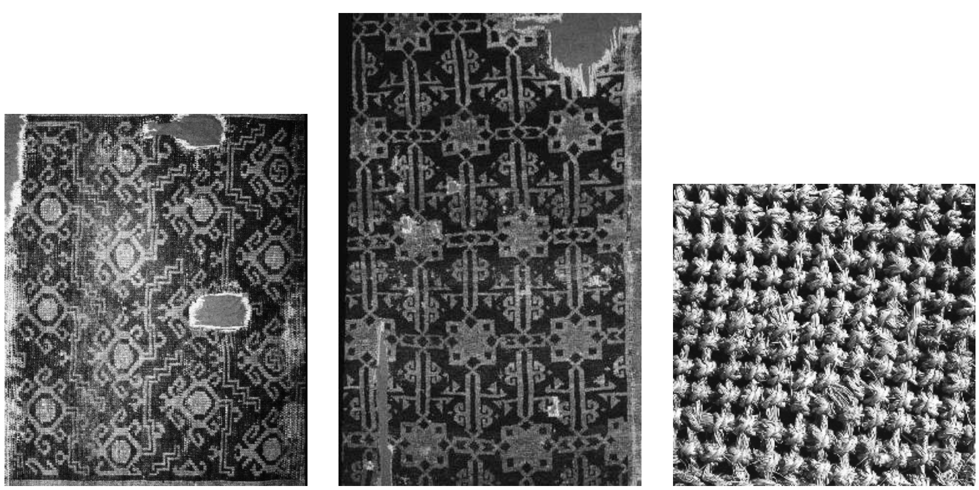
Humans have an acute ability for symmetry perception. While the automation of such powerful insights is less than obvious, many researchers have made an effort on developing symmetry detection algorithms to find symmetry axis in Euclidean space or to find skewed symmetry axes under affine or perspective skewing.

The term *skewed symmetry* was first coined in [16], meaning planar shape mirror symmetry (bilateral symmetry) viewed obliquely. Analysis of skewed symmetry, has attracted several researchers [10, 22, 31, 30], where the goal is to automatically determine the axis of the skewed symmetry.

Work has also been done on finding repeated patterns in real images [18, 24, 25, 19]. Leung and Malik examine overlapping windows in the original image to find ones with the right ratio of image 'energy' according to the second moment of the window. The two eigenvalues of the second moment matrix tell whether the pattern repeats in one or two directions. Selected windows are matched with neighboring patches, and the output of their algorithm is a list of basic elements in the repeated pattern, neighboring patches that match well with these elements, and the affine transformation relating them. More recently, Schaffalitzky and Zisserman [24, 25] describe a procedure to find repeated patterns in a real image. Their method uses image features such as edges, corners or closed contours to find repeating elements. Then a graph among the elements is constructed and enriched by a set of optimized affine transformation parameters. The output is a grouping of the repeating elements (by translations) in a real image. Although each of these 'repeated pattern' finders has its limits and no quantitative evaluations are given, the results on real images are impressive.

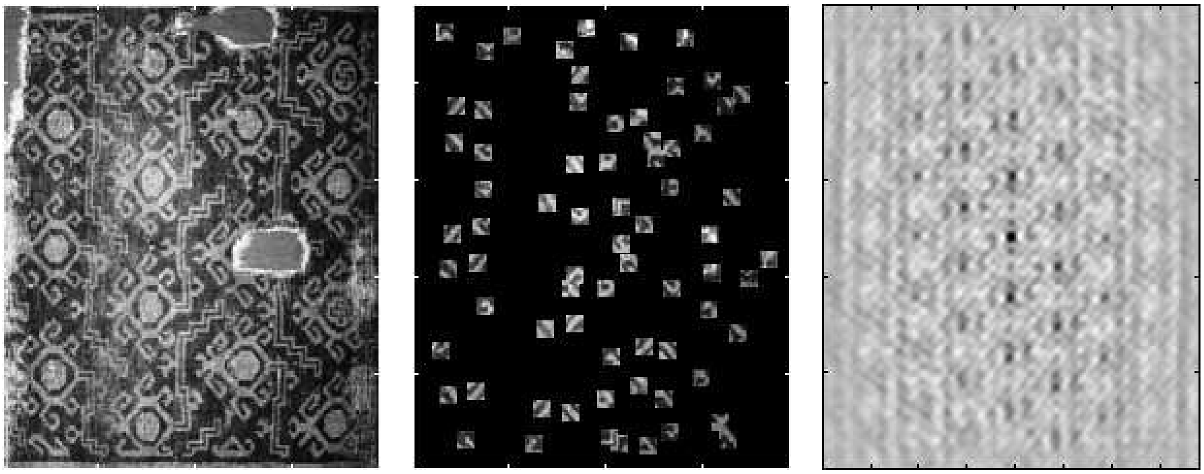
In summary, there are two typical approaches to detecting repetition of a 2D pattern. The first is to extract a sparse set of features and to hypothesize links (translations) between them based on visual similarity or conformance to a particular parametric model. This approach is exemplified by [18] and [24]. The benefits of this method are the ability to detect small regions of repeating pattern within a larger image, and the ability to group pattern elements despite local surface deformations (such as the folds of a shirt). The drawback is the need for a pattern with distinct corner features. The more traditional image processing approach to detecting pattern repetition is to use autocorrelation [19] or the Fourier transform [23]. These approaches work for any intensity image (not just ones with strong corners). The drawback is the assumption that a single repeated pattern occupies a large portion of the image. Thus this approach is more relevant to analyzing patterns that have already been segmented in some other way.

Our approach attempts to draw on the strengths of both methods. Specifically, a sparse set of "interesting" image patches is extracted. Figure 2b shows patches extracted for the rug in Figure 1a. Each interesting intensity patch is correlated over a local neighborhood (in these examples the neighborhood is the whole image), to yield a correlation score for each pixel displacement from the original feature position. This is repeated for all of the extracted intensity patches, and the correlation images are accumulated into a single correlation score vs. displacement image. The accumulated score image for Figure 1a is shown in Figure 2c. Our goal is to precisely classify the repeated pattern in terms of its wallpaper symmetry group.



(a) (b) (c)

Figure 1: Examples of imperfect, real-world patterns.



(A) (B) (C)

Figure 2: A) Rug image. B) Extracted "interesting" feature patches. (C) Accumulated correlation score vs pixel displacement image.

3 Frieze and Wallpaper Groups Classification

Any repeated pattern on a 2D plane has an associated non-trivial symmetry group, which has to be one of the seven Frieze groups or the 17 wallpaper groups [26]. Interested readers can find a comprehensive introduction on 2D patterns and their symmetry groups in [11]. A simple proof for the existence of 17 wallpaper groups can also be found in [28]. Our goal is to find the right symmetry group given a 2D pattern repeating along one or two linearly independent directions.

Mathematically speaking, the Frieze and wallpaper groups are defined for 2D repeated patterns that cover the whole 2D plane. In practice, we analyze a repeated pattern bounded in a finite area. In this paper we discuss repeated patterns P of finite boundaries. We use the concept of symmetry groups G of P with the assumption that G is the symmetry group of an infinite repeated pattern for which P is a finite patch. The central idea of the classification algorithms in the following text carry the same spirit, that is to determine a global symmetry group using local cues.

3.1 Frieze Groups

Assuming a one dimensional repeated pattern (a Frieze pattern) is placed horizontally, there are only seven possible patterns in terms of their symmetries (Figure 3). Each of the seven Frieze groups contains the following isometries:

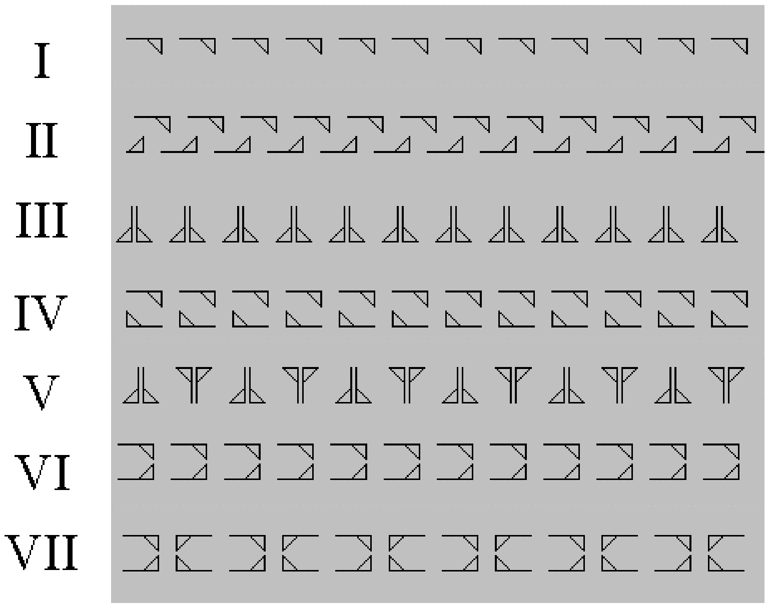


Figure 3: The seven Frieze patterns which have distinct Frieze symmetry groups

- I: one dimensional translation only
- II: translation and nontrivial glide-reflection
- III: translation and vertical reflection
- IV: translation and 2-fold rotations
- V: translations, 2-fold rotations, vertical reflections, nontrivial glide-reflections
- VI: translations, horizontal reflections
- VII: translations, rotations, horizontal reflections and vertical reflections

See Table 1 for a list of different symmetries existing in each of the seven Frieze groups.

Figure 4 displays a simple algorithm to classify a given Frieze pattern into one of the seven Frieze groups, where each rectangular box is a test for the existence of a particular kind of symmetry.

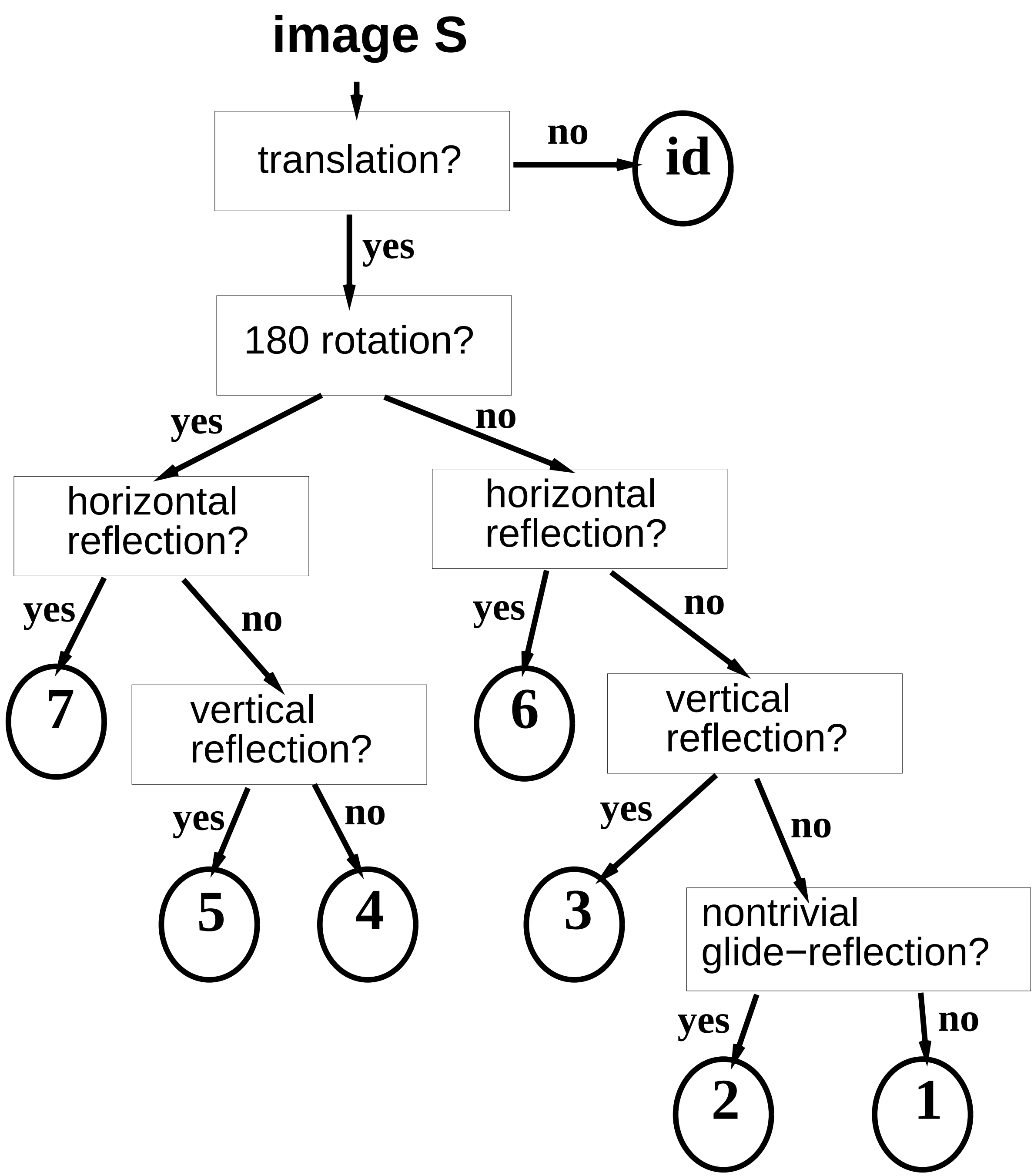


Figure 4: The algorithm to find the seven Frieze groups

Table 1: Recognition Chart for Frieze Patterns

Group Type	translation	180^0 rotation	Horizontal reflection	Vertical reflection	Nontrivial glide-reflection
I	yes	no	no	no	no
II	yes	no	no	no	yes
III	yes	no	no	yes	no
IV	yes	yes	no	no	no
V	yes	yes	no	yes	yes
VI	yes	no	yes	no	no
VII	yes	yes	yes	yes	no

3.2 Wallpaper Groups

In wallpaper patterns, the two linearly independent translations of minimum length are the two basic generators of each group, and they construct a lattice for the group. The smallest region which can be translated by the two generators to cover the whole 2D space is called the **generating region**. Figure 5 shows a diagram from [26] which depicts precisely each generating region of the 17 wallpaper groups, their translation generators, and the rotation, reflection and glide-reflection symmetries. Here *glide-reflection* means symmetries that are composed of a translation along the reflection axis followed-by a reflection about the axis. There are five types of lattices of the 17 wallpaper groups. They are: (1) parallelogram (two groups: $p1, p2$), (2) rectangular (five groups: pm, pg, pmm, pmg, pgg), (3) rhombic (two groups:cm, cmm), (4) square (three groups:$p4, p4m, p4g$) and (5) hexagonal (five groups:$p3, p3ml, p3lm, p6, p6m$).

These five lattices are in general parallelograms. If the angles between the edges are 90^o then the lattice is rectangular. When the two edges have the same length they become rhombic. Square and hexagonal lattices are two special cases of rhombic when the angles between a pair of adjacent edges are 90^o and 60^o respectively. Thus even if a lattice of a repeated wallpaper pattern is explicitly given, there are still multiple sets of possibilities for its symmetry group. For example, a pattern has a square lattice does not mean it has to have one of the symmetry groups under square lattices: $p4, p4m, p4g$, its symmetry group could be, say, $p2$. In another word, having squared lattice is a necessary but not a sufficient condition for the corresponding pattern to have a symmetry group from $p4, p4m$ or $p4g$.

Using a guide map such as the one shown in Figure 5, a human user can follow the symbols to identify the existence of certain symmetries and then determine what is the symmetry group of a

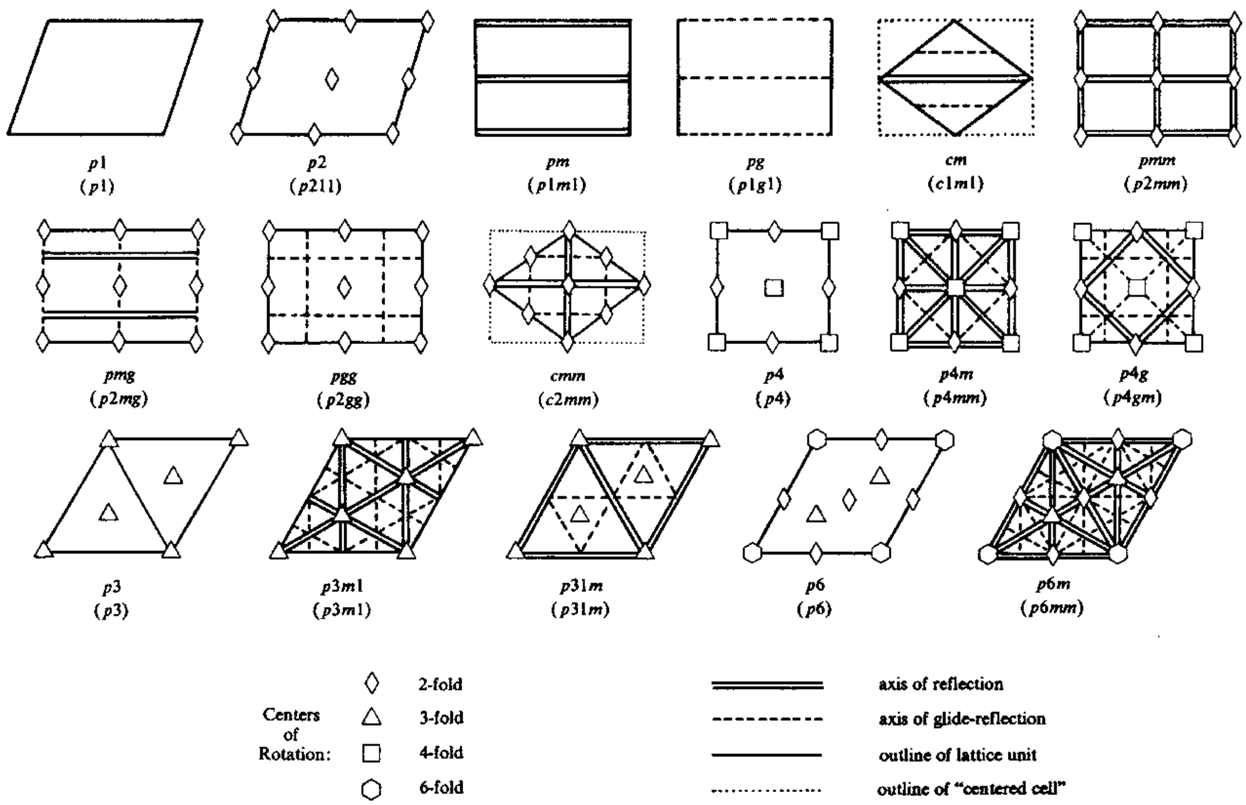


Figure 5: The generating regions for the 17 Wallpaper groups (from [26])

given pattern. When patterns become complicated this is not always an easy task. Using computers to recognize the symmetry of a given pattern also has its shares of problems. The first one is to recognize the underlying lattice structure (see Section 5), and the second is to computationally verify the existence of rotation and reflection symmetries.

Here we describe a symmetry group classification algorithm in Euclidean space. Assume the lattice type is already computed (given by translation vectors T1, T2) by measuring the angles of the corners and lengths of the edges of the lattices. We borrow the labelings for wallpaper groups from [26] which are shown in Figure 5. An T1-reflection (T2-reflection) means a reflection about an edge of the unit lattice parallel with T1 (T2).

Euclidean Algorithm:

1. If the lattice is a parallelogram lattice (two possible groups: $p1, p2$), test 2-fold rotation. If the answer is yes G is $p2$ otherwise G is $p1$.

2. If the lattice is a rectangle lattice there are five plus two (seven) possible groups: pm, pg, pmm, pmg, pgg and $p1, p2$. First test 2-fold rotation. If 2-fold rotation exists (four possible groups: pmm, pmg, pgg, $p2$) test T1-reflection and T2-reflection. If neither is a glide-reflection G is pmm, if one of them is a glide-reflection G is pmg, otherwise G is pgg. If no reflection exists, then the G is $p2$. If 2-fold rotation does not exist (three possible groups: $pm, pg, p1$), check T1-reflection (or T2-reflection). If it is a glide-reflection G is pg otherwise G is pm. If no reflection symmetry then it is $p1$.

3. If the lattice is a rhombic lattice, test diagonal reflection symmetry. If it does not exist, go to step 1. Otherwise there are two possible groups: cm, cmm, test 2-fold rotation. If the symmetry exists G is cmm otherwise cm.

4. If the lattice is a square lattice, test 4-fold rotation symmetry. If 4-fold symmetry does not exist, go to step 3. Otherwise there are three possible groups: $p4, p4m, p4g$, test T1-reflection symmetry. If the symmetry exists G is $p4m$. Otherwise test diagonal-reflection, if the answer is yes G is $p4g$ otherwise $p4$.

5. If the lattice is a hexagonal, first test 3-fold rotation symmetry. If no 3-fold symmetry exists, go to step 3. Otherwise there are five possible groups: $p3, p3ml, p3lm, p6, p6m$, test 2-fold rotation. If 2-fold rotation exists (two possible groups: $p6, p6m$), test T1-reflection. If T1-reflection exists G is $p6m$ otherwise G is $p6$. If there is no 2-fold rotation (three possible groups: $p3, p3ml, p3lm$), test T1-reflection. If the answer is no G is $p3$ otherwise check diagonal-reflection. If the symmetry exists G is $p3ml$ otherwise G is $p3lm$.

In short, given a lattice first test the possible existence of special symmetries (3-fold, 4-fold rotations, diagonal reflection); when the answer is yes, the search is limited in a small set of possible groups; when the answer is no, the classification process proceeds to a more general type of lattice.

4 Beyond Rigid Transformations

When a 2D pattern undergoes a rigid transformation, its symmetry group remains. Strictly speaking, its symmetry group is conjugated by the transformation that acts on the pattern. Since there exists a bijection between the original symmetry group and the conjugated symmetry group, the two groups are considered equivalent (isomorphic). If one imagines a coordinate system fixed on the pattern, when the pattern goes through rigid transformations its translation, rotation, reflection and glide-reflection symmetries remain the same with respect to its local coordinates.

This situation will no longer be true when the pattern undergoes non-rigid transformation. However, certain symmetries of a repeated pattern may survive some constrained non-rigid transformations. In this section we explore these possible symmetry invariants under non-rigid transformations. Our goal is to determine whether we can still classify the symmetry group of a transformed pattern under similarity, affine and perspective distortions.

First of all, let us state precisely when two wallpaper groups are considered as equivalent. Abiding the same standards as stated in [28], two wallpaper groups G, G' with lattices T, T' respectively are equivalent if there is an isomorphism $G \to G'$ which maps the subgroup T onto T'.

4.1 Under Similarity Transformations

Given a 2D pattern S with symmetry group G and its transformed version $S' = k(S)$ under similarity transformation k, one can establish an isomorphism between G and G' by sending the identity in T to the identity in T', and every non-identity element $t \in T$ to $kt \in T'$ where $k \in R$. Here T, T' are lattices of G and G' respectively.

This proves that the symmetry groups of a repeated pattern and the symmetry group of its enlarged or shrunken version are conjugations of each other. Thus the algorithm described in Section 3 can be applied directly to patterns under similarity transformations.

4.2 Under Affine Transformations

When a 2D repeated pattern undergoes an affine transformation, different lattices in general cannot be distinguished from each other. This is because the five lattice structures can be transformed into each other under general affine transformations. However, some symmetries can survive constrained or even general affine transformations. Here we prove the sufficient conditions for the existence of two types of symmetry invariants. In Section 4.2.3 we explore what happens to a reflection symmetry under skewing.

4.2.1 C_2 symmetry subgroup is invariant under Affine Transformations

The C_2 symmetry subgroup is a group of two elements: the identity and 180 degree rotation. It is pointed out without a formal proof in [10] that "*If a planar figure is C_2-symmetric, then this symmetry is preserved under affine transformations ...*". Here we give a one-line proof for this statement.

Let S be a 2D pattern which has a C_2 subgroup such that there exists a non-trivial rotation (180^o rotation) $g \in C_2, g(S) = S$. Note, $g = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}$ is its own inverse, i.e. $gg = gg^{-1} = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$. Now let $S' = A(S)$ where $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is a general affine transformation. We need to prove that $g(S') = S'$, i.e. C_2 remains a symmetry subgroup of S'.
Proof :
$g(S') = g(A(S)) = g(Ag^{-1}g(S)) = (gAg^{-1})(g(S)) = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} (g(S)) = \begin{vmatrix} -a & -b \\ -c & -d \end{vmatrix} \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} (g(S)) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} (g(S)) = A(g(S)) = A(S) = S'$. $\square$

This proof tells us that 2-fold rotational symmetry of a pattern survives any non-singular affine transformation of the pattern. Conversely, it is also true that if a pattern S does not have a 2-fold symmetry, any form of its affine transformations does not have a 2-fold symmetry either.
Proof: if S becomes 2-fold rotational symmetrical after an affine transformation A, i.e. $r(A(S)) = A(S)$, then $r(A^{-1}A(S)) = r(S) = S$. Thus S has a 2-fold rotational symmetry, a contradiction. $\square$

4.2.2 Reflection Symmetries are invariant under Parallel and Perpendicular Scalings

Next we examine reflection symmetries. We are trying to prove the following: a reflection and glide-reflection symmetry remains *if* the pattern undergoes a non-uniform scaling parallel or perpendicular to its reflection axis.

Without loss of generality, let g be a reflection about an axis parallel to the X or Y axis, $g = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}$ or $g = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$. g is a symmetry of S, i.e. $g(S) = S$. Let $K = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}$ be a non-uniform scaling along the X and Y axes, and $S' = K(S)$. Prove $g(S') = S'$.

Proof :

$g(S') = g(K(S)) = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} (S) = \begin{vmatrix} -a & 0 \\ 0 & b \end{vmatrix} (S) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} (S) = K(g(S)) = K(S) = S'$ □

A similar proof can be given for $g = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$.

4.2.3 Reflection Symmetries under Skewing

Here we show that for a reflection symmetry g of a repeated pattern S to remain as a symmetry after a non-trivial skew transformation $A = \begin{vmatrix} 1 & a \\ b & 1 \end{vmatrix}$ (where a, b are not simultaneously zeros), some specific constraints must be satisfied.

Proof :

Without loss of generality, let us assume g is a reflection symmetry of S, i.e. $g(S) = S$, and its reflection axis is X axis, $g = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$. A is a skewing transformation $A = \begin{vmatrix} 1 & a \\ b & 1 \end{vmatrix}$. In order for A to be invertible, $ab \neq 1$, and $A^{-1} = \begin{vmatrix} \frac{1}{1-ab} & -\frac{a}{1-ab} \\ -\frac{b}{1-ab} & \frac{1}{1-ab} \end{vmatrix}$.

Let us assume $g(A(S)) = A(S)$. Then we have $A^{-1}gA(S) = S$.

$$\begin{vmatrix} \frac{1}{1-ab} & -\frac{a}{1-ab} \\ -\frac{b}{1-ab} & \frac{1}{1-ab} \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \begin{vmatrix} 1 & a \\ b & 1 \end{vmatrix} (S) = S$$

$$\Rightarrow \frac{1}{1-ab} \begin{vmatrix} 1 & -a \\ -b & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \begin{vmatrix} 1 & a \\ b & 1 \end{vmatrix} (S) = S$$

$$\Rightarrow \frac{1}{1-ab} \begin{vmatrix} 1 & 2a \\ -2b & 1 \end{vmatrix} (S) = S$$

$$\Rightarrow \frac{1}{1-ab}\begin{vmatrix} 1 & -2a \\ -2b & 1 \end{vmatrix}\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}(S) = S$$
$$\Rightarrow \frac{1}{1-ab}\begin{vmatrix} 1 & -2a \\ -2b & 1 \end{vmatrix}(S) = S.$$

This result states that for a specific pair of $a, b \in R^1, a \neq 0, b \neq 0, ab \neq 1$, a skewing $\begin{vmatrix} 1 & -2a \\ -2b & 1 \end{vmatrix}$ followed by a uniform scaling $\frac{1}{1-ab}$ of S is a symmetry of the repeated pattern S. In general, a pattern S can only be its non-trivially scaled version if S is a pattern of uniform density, therefore S would not be a repeated pattern: a contradiction.

When one of a, b is zero, say $b = 0$, the above result means that the horizontal skewing $\begin{vmatrix} 1 & -2a \\ 0 & 1 \end{vmatrix}$ is a symmetry of the repeated pattern S. This is to say that there exists a particular number a such that for all points x, y on S, $\begin{vmatrix} 1 & -2a \\ 0 & 1 \end{vmatrix}\begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} x - 2ay \\ y \end{vmatrix}$. $\square$

This proof basically says that reflection symmetries do not survive a general affine skewing $A = \begin{vmatrix} 1 & a \\ b & 1 \end{vmatrix}$ where $a \neq 0, b \neq 0, ab \neq 1$. However, a reflection symmetry may remain when one of a, b is zero, as far as the repeated pattern S satisfies the condition that there is a correspondence between point $[x, y]$ and the point $[x + ay, y]$ of S. If S is a pattern composed of stripes parallel with and equal distanced from the X axis, then regardless of the value of a, the reflection symmetry will be preserved. However, in that case S is no longer a repeated pattern. For a repeated pattern S and a given skewing A, this reflection-preserving condition can be tested algorithmly.

4.3 Wallpaper Group Classification for Affine Distorted Patterns

We assume that the lattice structure of an affine distorted repeated pattern is given. The problem is this: given a 2D repeated pattern which has possibly gone through some affine transformation, what is its symmetry group? Immediately we need to answer the question that given a 2D repeated pattern S in Euclidean space, whether it still makes sense to talk about the symmetry group of $A(S)$, where A is a general affine transformation? The answer is yes, this is because *a 2D repeated pattern S with symmetry group G remains to be a 2D repeated pattern $A(S)$ under affine transformation A.*

Proof:

Since S is a repeated pattern, there are two independent translation vectors t_1, t_2 such that for any point $s \in S, \exists s' \in S, s' = t_1^n(s) + t_2^m(s)$ where n, m are integers including zero. All such s's form a 2D grid.

Given the invariant properties of affine transformations: parallelism, segment division ratio, betweenness and midpoints properties [3, 20], when S undergoes an affine transformation A, for any three consecutive grid points a, b, c of the original pattern S parallel with t_1, $\exists t$ such that $A(a), A(b), A(c)$ are collinear, $A(b)$ is in between $A(a)$ and $A(c)$, and $A(b) = t(A(a)), A(c) = t(A(b)) = t^2(A(a))$. Using induction, this can be expanded infinitely for any triplet points along t. Similarly, we can proof this for any three points along t_2 direction in S in t' direction in $A(S)$ where t, t' are linearly independent of each other (provided no degeneration into one line). Therefore $A(S)$ is a 2D pattern that repeats along t, t' respectively. $\square$

Thus $A(S)$ has to have one of the 17 wallpaper groups as its current symmetry group G', though it is not necessarily true that $G = G'$. The second question: Since the five possible lattice types of the 17 wallpaper groups can be transformed into each other freely under affine transformations, under which lattice type should the symmetry group of a given pattern be defined? There are two observations to be made here:

- Although there are five possible lattice types for the 17 distinct wallpaper groups, there are only two basic definite types: square and hexagonal. The justification for this is that square lattice is a special case for parallelogram, rectangular and rhombic lattices, each of the latter types has a non-determined parameter in its shape (adjacent edge ratio or angle), and the deformation of the latter to the former does not destroy any of their corresponding symmetries (this fact can be verified one by one using the proven results in Sections 4.2.1 and 4.2.2).

- In terms of the hierarchy of symmetry groups, we take the stand as stated in [17], a more symmetrical structure is preferred over a less symmetrical one, i.e. a strongly symmetrical structure that a pattern can be affinely deformed into is NOT an accident. Thus if more symmetries present themselves when the lattice of a pattern is turned into a square lattice that fact is to be explicitly acknowledged regardless of the original intentions of the pattern designer.

These two observations lead us to divide the 17 wallpaper groups into two separate sets, those of “square” lattices: groups 1 to 12 and those of hexagonal lattices: groups 13 to 17 (See Table 2, they are in the same order as listed in Figure 5).

The idea of classifying a 2D repeated pattern S is to first perform an affine transformation to “normalize” the detected lattice structure into some standard form then test for the symmetries known to survive that transformation. Using the invariance results from Sections 4.2.1, 4.2.2 and

Table 2: 17 Symmetry Groups and Their Symmetries (adopted from [15]). Here '+' means all rotation centers lie on reflection axes, a nd '*' means not all rotation centers on reflection axes

Symmetry Group	IUC Notation	Lattice type	Rotation orders	Reflection axes
1	p1	parallelogrammatic	none	none
2	p2	parallelogrammatic	2	none
3	pm	rectangle	none	parallel
4	pg	rectangle	none	none
5	cm	rhombus	none	parallel
6	pmm	rectangle	2	90^o
7	pmg	rectangle	2	parallel
8	pgg	rectangle	2	none
9	cmm	rhombus	2	90^o
10	p4	square	4	none
11	p4m	square	4 +	40^o
12	p4g	square	4 *	90^o
13	p3	hexagon	3	none
14	p3m1	hexagon	3 +	30^o
15	p31m	hexagon	3 *	60^o
16	p6	hexagon	6	none
17	p6m	hexagon	6	30^o

4.2.3, we can make a series of logical tests according to which symmetries are preserved after normalizing, and narrow down the range of choices to hopefully one.

In the following, we present three variations of the above idea to construct a symmetry group classification algorithm for a 2D repeated pattern under affine transformation.

4.3.1 A Sequential Approach

Figure 6 displays a symmetry group classification algorithm. Here we divide the 17 wallpaper groups into three separate sets: the groups with hexagonal lattices (groups 13-17), with square lattices (groups 10-12) and the rest (1-9). There is an underlying ranking here that the first set is more symmetrical than the second, and the second is more symmetrical than the last set. This is the reason why the algorithm follows this particular sequential order: first deform the current lattice to a hexagonal one to check whether it is one of the symmetry groups in set one; if not, deform the lattice into a square and check whether the symmetry group is in set two; if not, then the symmetry group has to be in set three.

If a pattern does not have a 4-fold (3-fold) symmetry after deforming to a square (hexagonal) lattice then any of the symmetry groups of a square (hexagonal) lattice must not be the right symmetry group for the pattern. Instead, the symmetry group must be from groups 1 to 9 listed in Table 2. Using 2-fold symmetry test, the hypothesis for the possible group can immediately be narrowed down to either the five which has 2-fold symmetry or the four which does not.

Figure 6 shows only one possible way to go through all the possibilities, which is conceptually clear and computationally effective. In average four or less symmetry tests are needed for determining a unique symmetry group, with a maximum of 5 tests and a minimum of 3.

4.3.2 A Semi-Parallel Approach — a “double” signature

This is a modification of the previous approach due to the consideration that for a more complete characterization of a pattern under affine deformation, both the symmetry group of the pattern under hexagonal lattice and the symmetry group of the pattern under square lattice should be recorded. This is helpful for distinguishing between patterns that has a unique symmetry group invariant of the affine transformation applied (Figure 7) and those patterns that can have non-trivial symmetry groups under both deformations (a pattern of lattice points). Some patterns have a unique symmetry group regardless of how the pattern is deformed under affine transformation. For example, the pattern in Figure 7 has symmetry group $p1$ under both square and hexagonal lattices. On the other hand, some patterns may have an inherent “ambiguity” in terms of their symmetry groups when the pattern itself is deformed, the simplest example is pattern composed

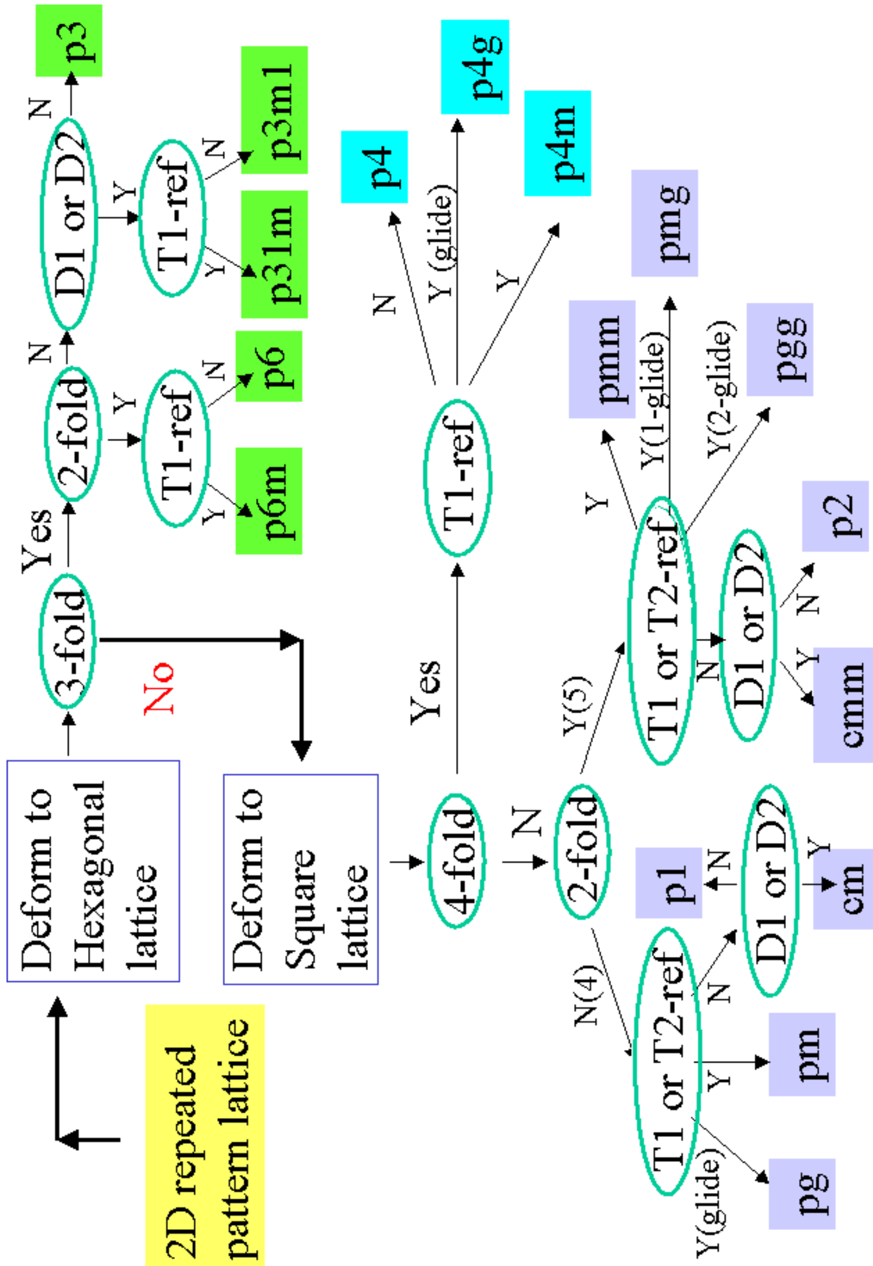


Figure 6: A sequential algorithm for symmetry group classification of 2D repeated pattern under affine transformation: Y(glide) means the reflection symmetry must be a non-trivial glide reflection. Y(n) / N(n) means the test result is positive/negative and n possible symmetry groups need to be further distinguished. n-fold means checking for n-fold rotational symmetry. T1-ref means checking reflection symmetry parallel with generating vector T1. D1 or D2 means reflection symmetries along the diagonal of the lattice.

of lattice points alone. This pattern can have symmetry groups $p2, cmm, pmm, p4m$ or $p6m$ when deformed into parallelogram, rhombic, rectangular, square and hexagonal lattices respectively.

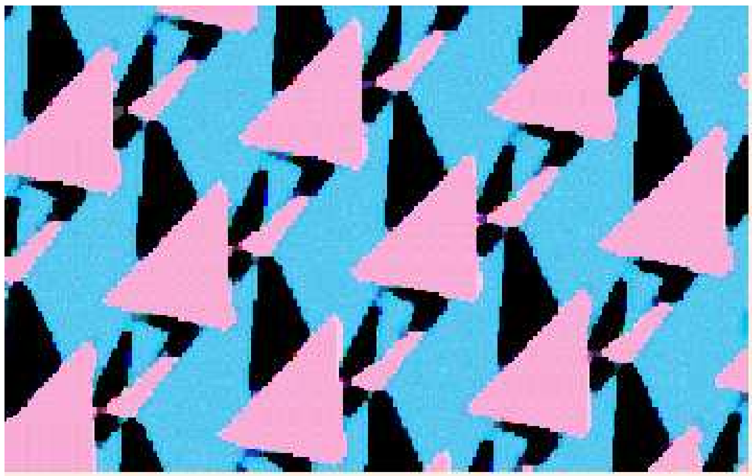


Figure 7: The 2D repeated pattern with symmetry group $p1$, from [15]

The algorithm for this approach is shown in Figure 8. The result from this algorithm is a pair of symmetry groups, one is computed under hexagonal lattice and the other is under square lattice. Now, a given 2D repeated pattern under affine transformation is not necessarily bounded to only one unique symmetry group as in the Euclidean case, rather it is given a double symmetry group 'signature'.

4.3.3 A Migration Diagram Approach

This approach differs from the above two in that instead of using a standard lattice form, it first finds what symmetry group G it is as if in Euclidean space using the algorithm presented in Section 3. Then answer the question "what group could this pattern come from?". For each of the 17 wallpaper groups G, a "migration diagram" of G is constructed. This migration diagram is a graph with wallpaper groups as nodes and arcs relating them showing what other symmetry groups this current pattern with group G can possibly "migrate" to/from when the pattern is affinely modified.

The 17 wallpaper groups can be divided into exclusive sets in different ways. One is to divide them into the set with 2-fold symmetry and the set without. The other way is to divide them into hexagonal lattice type and the "square" lattice type. There are some basic rules we can follow to establish arcs in a migration diagram:

1. the group with 2-fold symmetry can only migrate to groups that have 2-fold symmetry

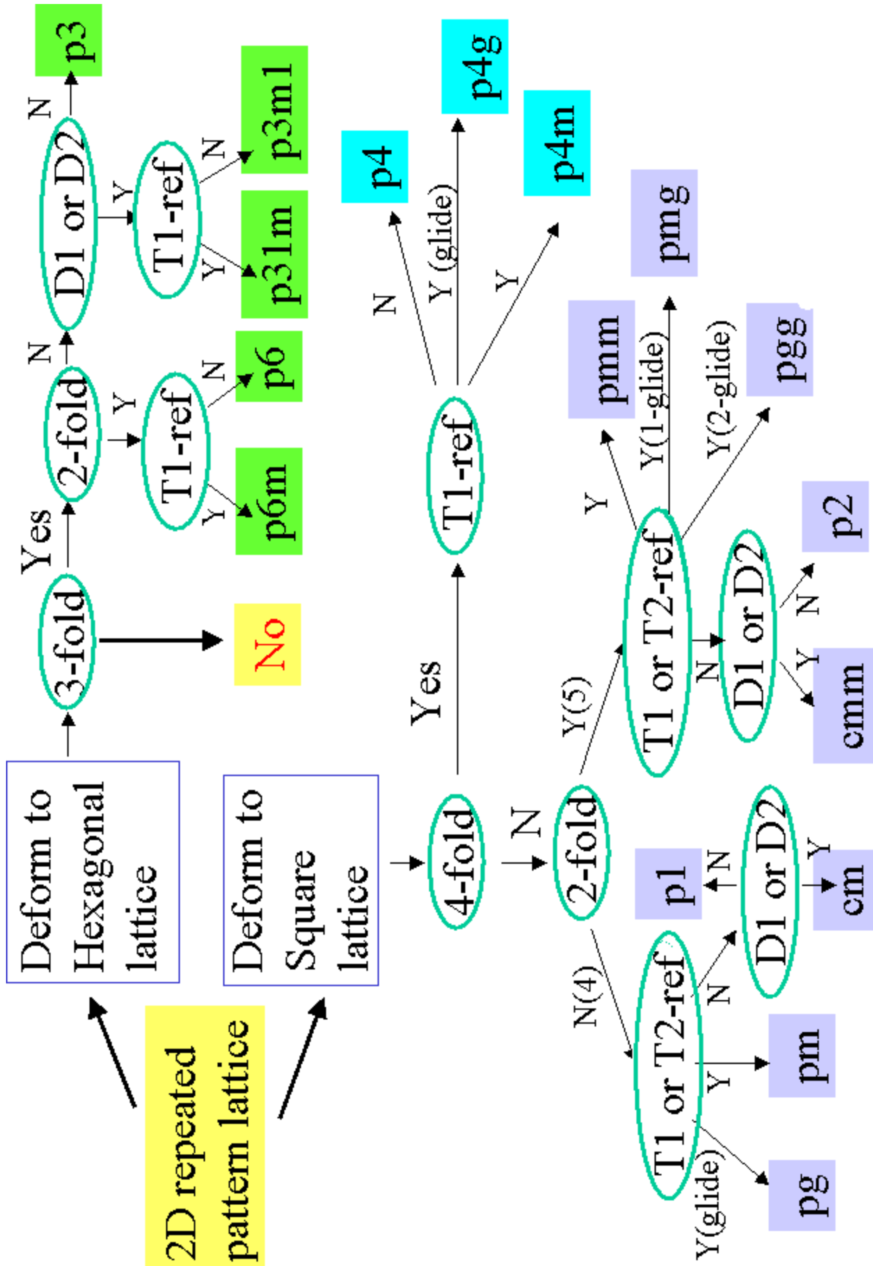


Figure 8: A semi-parallel algorithm for symmetry group classification of 2D repeated pattern under affine transformation: Y(glide) means the reflection symmetry must be a non-trivial glide reflection. Y(n) / N(n) means the test result is positive/negative and n possible symmetry groups need to be further distinguished. n-fold means checking for n-fold rotational symmetry. T1-ref means checking reflection symmetry parallel with generating vector T1. D1 or D2 means reflection symmetries along the diagonal of the lattice.

2. the group with diagonal reflection(s) symmetry can only migrate to groups with diagonal reflection(s) symmetry

4.4 Under Perspective Transformations

At this point, we are not concerned with finding the repeated patterns in a perspective scene. There is previous work on this subject [18, 24]. What we are interested in is when a distorted repeated pattern is found, determining which of the seventeen symmetry groups this pattern possibly belongs to.

Instead of carrying out the classification procedure in perspective space directly, we take a detour into affine space. Using other cues in the image, such as two vanishing points, we construct a transformation to unwarp the projection from perspective to affine by sending the horizon line to infinity [4]. This reduces the perspective transform to an affine one, and the affine algorithm introduced above is used. If we know the camera parameters, the plane can be unwarped to a similarity transformation of the original 2D pattern [4], and we can directly apply our Euclidean algorithm from Section 3.

5 Experimental Results

The examples in this section serve to illustrate the symmetry classification algorithm. The first two are synthetic images taken from a web page about wallpaper groups, maintained by David Joyce at Clark University. We have successfully processed all seventeen of his patterns (See Section 7). Although they could be processed directly since they are generated Euclidean patterns, we perform the general classification routine for affine distorted images in order to illustrate the method.

The first step is to determine the underlying translational lattice structure of the original image, in the form of two independent generating vectors t_1 and t_2. Since we are assuming that the wallpaper pattern has been previously isolated, the lattice points are determined by finding significant peaks in the pattern's autocorrelation surface (Figure 9a-c). If the pattern was embedded in a larger image, a more robust local method such as [24] would need to be employed – here we are only concerned with classifying a previously detected pattern. The lattice of dots is decomposed into two generating vectors by finding the two shortest difference vectors t_1 and t_2 such that the angle between them is between 60 and 90 degrees.

The second step involves transforming the lattice to a square grid, aligned with the horizontal and vertical axes (Figure 9d-f). This is performed by applying an affine transformation to the image and its autocorrelation surface. The transformation used is the unique affine transform leaving the origin (0,0) fixed and taking t_1 to $(L, 0)$ and t_2 to $(0, L)$, where L is the larger of the two generating vectors lengths $||t_1||$ and $||t_2||$.

After transforming to a square lattice, a square generating region (with dimensions $L \times L$) is cropped from the transformed image. This is used as a template for further processing. In particular, rotated and reflected versions of the template are correlated with the transformed image to determine what, if any, type of rotation and reflection symmetry it has. Figure 10 shows a suite of rotated and reflected patterns to be correlated with the transformed image shown in Figure 9d. In the location determined by the highest correlation peak, a match score between the rotated/reflected template and the image is computed as the mean of the absolute difference between corresponding intensity values. The lower the value of this match score, the more likely it is that the image has that particular rotational/reflection symmetry. A threshold value is set by computing the match score between the original square generating region and the image at each node of the generating lattice. This yields a set of "typical" match scores for that pattern – the mean and standard deviation of these scores are used as an adaptive threshold tailored for this pattern. Match scores associated with rotated/reflected templates are compared to this threshold to determine whether that particular symmetry holds.

The table below contains match scores for the rotation and reflection templates shown in Figure 10:

rot180	rot120	rot90	rot60
0.063	0.251	0.275	0.250
H refl	V refl	D1 refl	D2 refl
0.275	0.270	**0.068**	**0.068**

Compared to a reference match score of 0.061 with standard deviation of 0.026 (note, these are computed from absolute differences between intensity values for images that have been scaled to range between 0 and 1 – multiply these numbers by 255 to get corresponding thresholds for 8-bit grey scale images), we find that the pattern has only 2-fold rotational symmetry, and a symmetry under reflection across both diagonals. The pattern is therefore uniquely determined to be from the *cmm* wallpaper symmetry group.

A second example is shown in Figure 11. Similar processing was performed, to yield the lines labeled "square" in the table below:

	rot180	rot120	rot90	rot60
square	**0.040**	0.279	0.296	0.269
hexag	**0.040**	**0.038**	0.310	**0.043**
	H refl	V refl	D1 refl	D2 refl
square	0.272	0.275	0.269	0.268
hexag	0.269	0.271	0.271	0.271

We find that the pattern has two-fold rotation symmetry only, and in particular no additional reflection symmetry, when represented using a square lattice grid. According to the classification algorithm described in the previous section, we next transform the image to a hexagonal lattice structure. This is done by performing the unique affine transformation leaving the origin (0,0) fixed, and mapping t_1 to $(L, 0)$ and t_2 to $(L/2, L * (\sqrt{3}/2))$, where L is a length chosen as before. The lines labeled "hexag" in the table show rotation and reflection results for the hexagonally transformed pattern. We see that now, in addition to two-fold symmetry, the pattern also has 60 and 120 degree rotational symmetry. There are still no reflection symmetries. Based on this information, the pattern is uniquely classified as being from the *p6* wallpaper symmetry group.

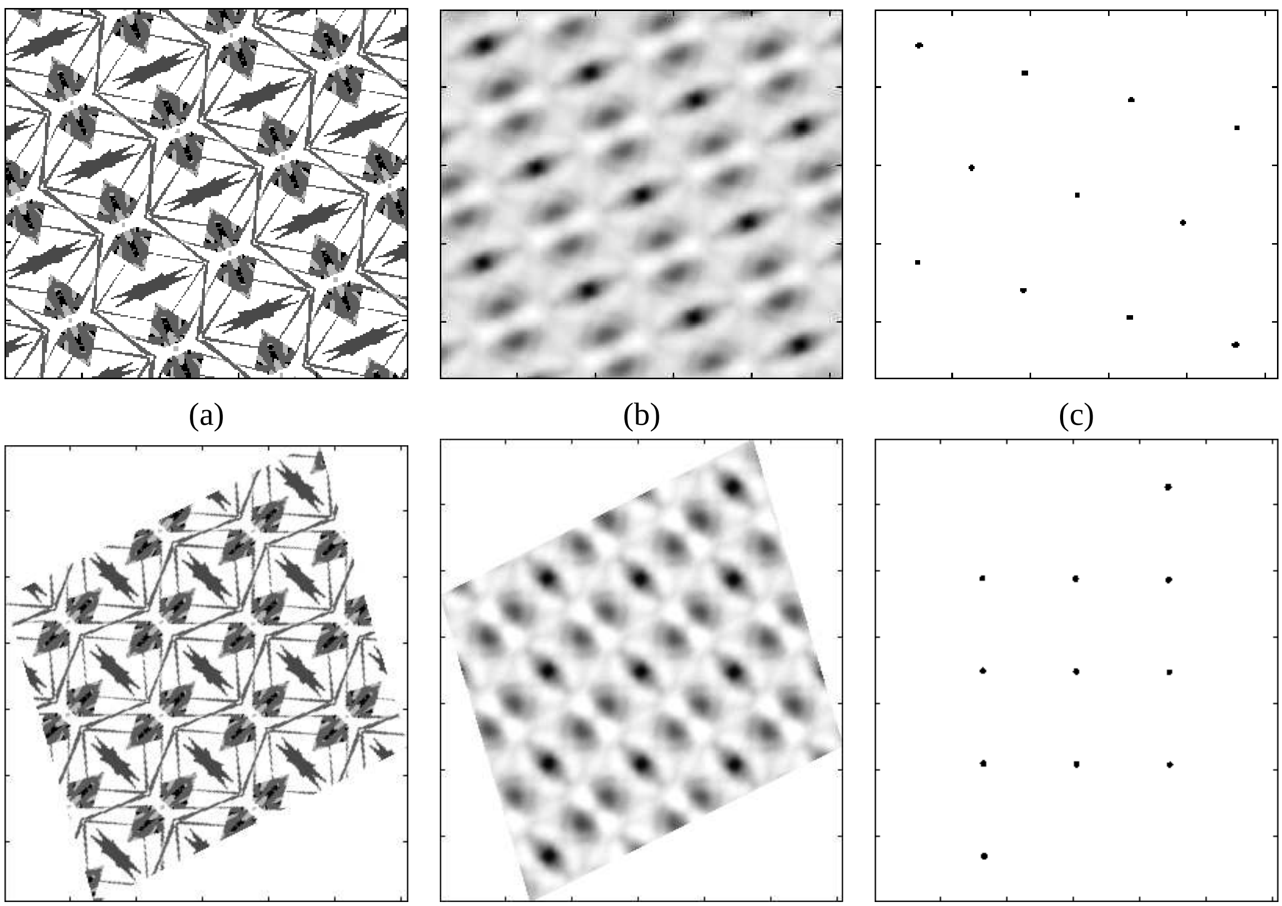
The third example presented here is a real travel photo, taken off the web (Figure 12). Architectural photos are a good source of symmetric patterns, particularly frieze groups, but also occasionally for 2D lattice-work as shown here. For real images, the need to take unknown affine distortions into account is important in order to deal with potential non-isotropic scalings introduced by the digitization/scanning process and the viewing angle. Although unknown pose with respect to a planar surface introduces a projective transform of the pattern, this can be approximated locally by an affine transformation, particularly when the viewing angle is not too oblique to the surface. The doorway lattice was cropped by hand and presented to the wallpaper classification algorithm. Figure 12 shows the lattice pattern determined both before and after transformation to a square grid.

The following table shows the results of reflect/rotation processing on the transformed image:

rot180	rot120	rot90	rot60
0.038	0.168	**0.044**	0.179
H refl	V refl	D1 refl	D2 refl
0.064	**0.034**	**0.038**	**0.045**

As compared to a reference match score of 0.065 with standard deviation 0.035, we see that the pattern has 180 and 90 degree rotational symmetry, and all four reflection symmetries. These results uniquely determine that the pattern is from the *p4m* wallpaper symmetry group.

The forth example is also from a real photo: a building with many windows. This is a case where perspective distortion is approximated as affine distortion. The resulting symmetry group is Pgg. Computation steps and intermediate results are presented in Figure 13.



(a) (b) (c)

(d) (e) (f)

Figure 9: Example of transforming to a square lattice for analysis, using the *cmm* wallpaper group. (a) original image. (b) autocorrelation of image. (c) detected lattice points. From these lattice points, an affine transform to a square grid is computed. (d) transformed image. (e) transformed autocorrelation. (f) transformed lattice points, now a square grid.

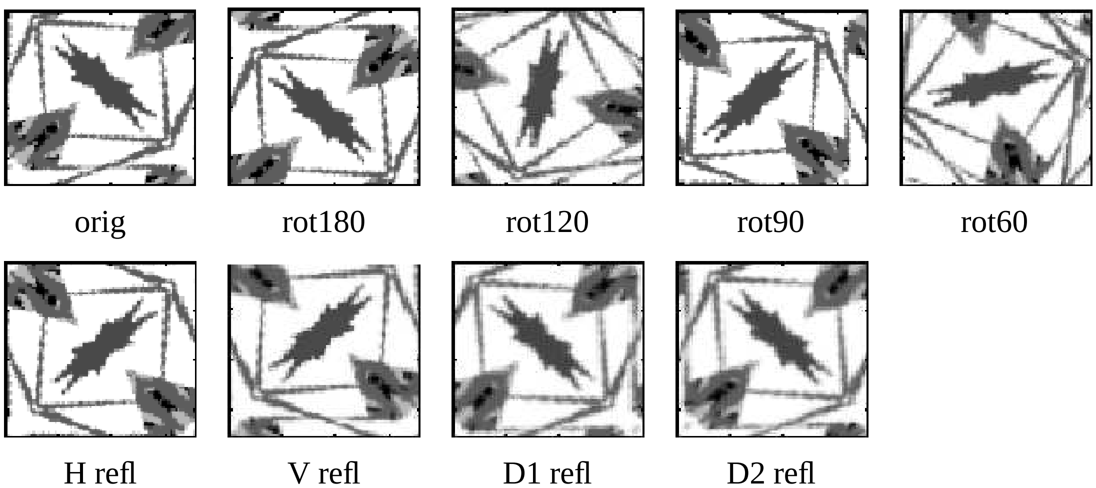
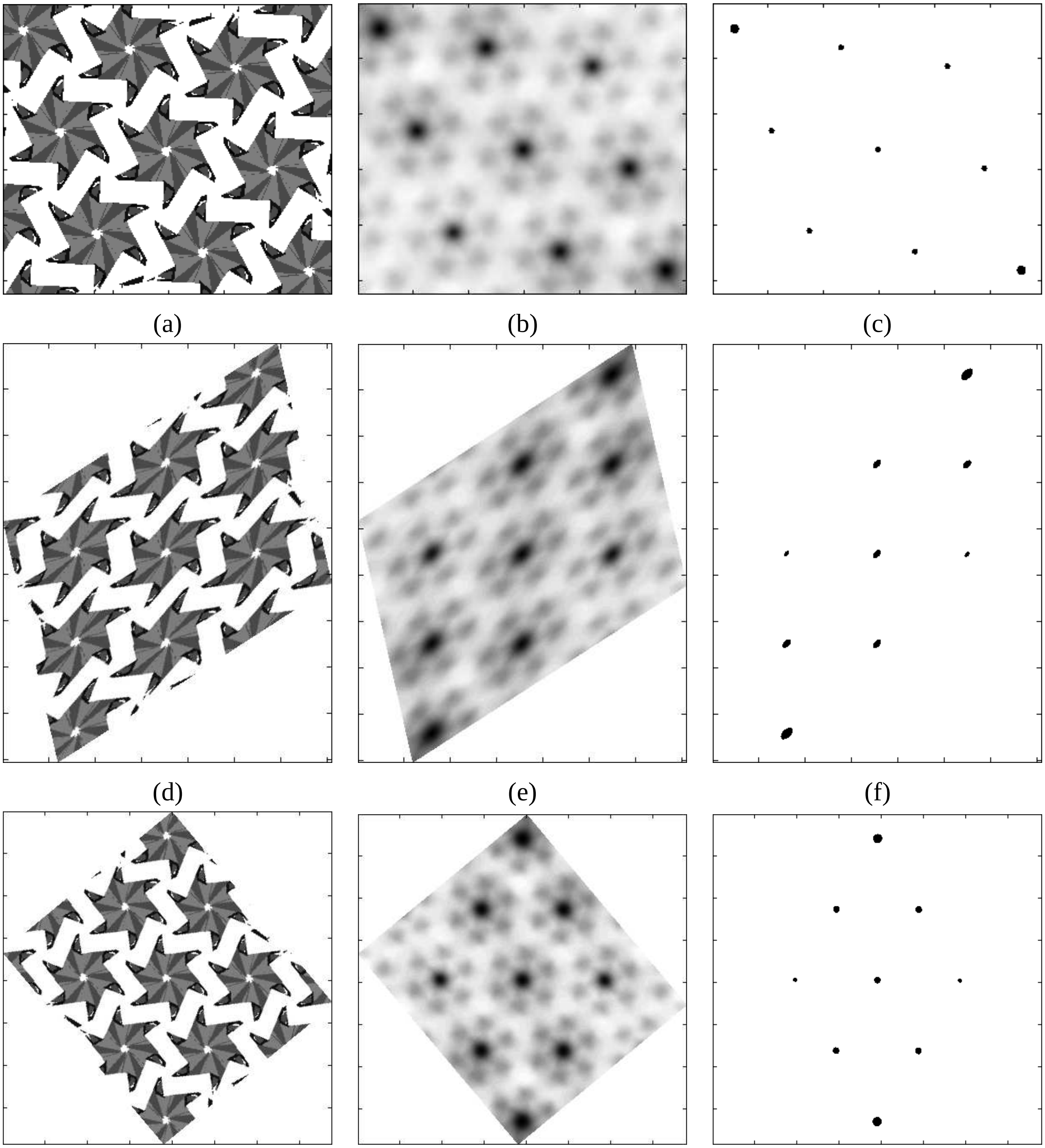


Figure 10: Lattice generating regions, used as templates to determine what types of rotational and reflection symmetry the wallpaper pattern has. Top left is the original generating region template. The next four templates are rotated by 180, 120, 90 and 60 degrees, respectively. The remaining four are reflected horizontally, vertically, and across the two diagonals of the region.



(a) (b) (c)

(d) (e) (f)

(g) (h) (i)

Figure 11: Example of transforming to a square lattice for analysis, using the *p6* wallpaper group. (a) original image. (b) autocorrelation of image. (c) detected lattice points. From these lattice points, an affine transform to a square grid is computed. (d) transformed image. (e) transformed autocorrelation. (f) transformed lattice points, now a square grid. From analysis, it becomes clear that a hexagonal grid is more appropriate. (g) hexagonal transformed image. (h) transformed autocorrelation. (i) transformed lattice points, now a hexagonal grid.

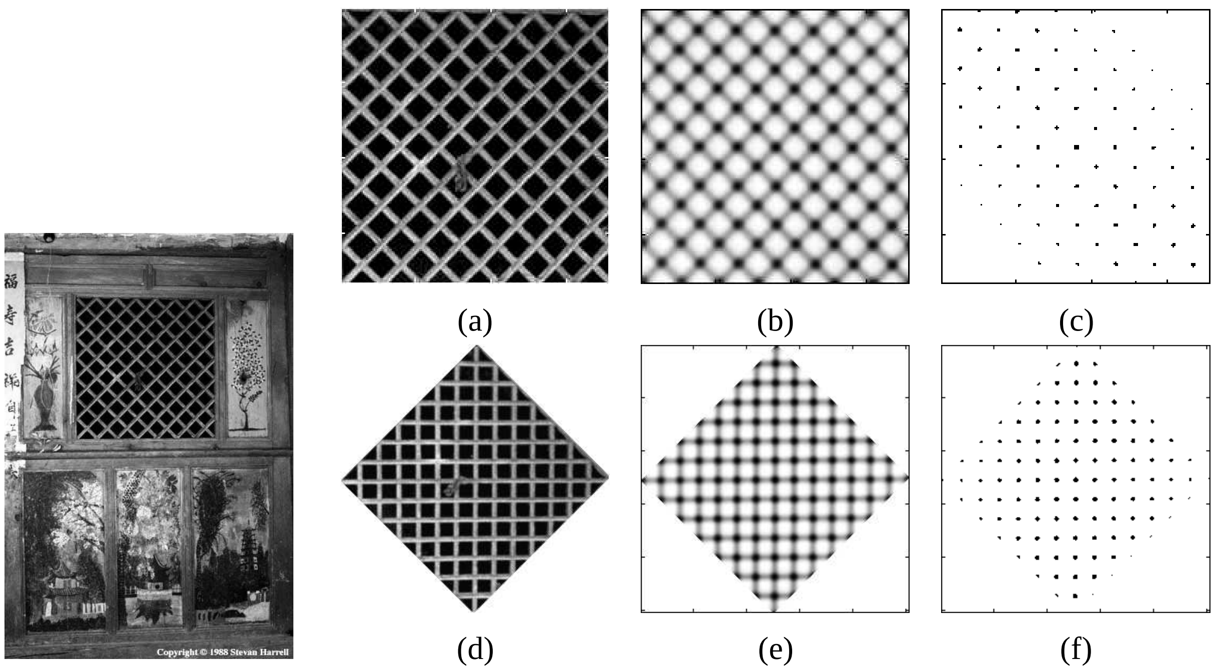


Figure 12: Real image taken off the web. The lattice on the door has *p4* wallpaper symmetry. (a) cropped subimage. (b) autocorrelation. (c) detected lattice points. From these lattice points, an affine transform to a square grid is computed. (d) transformed image. (e) transformed autocorrelation. (f) transformed lattice, now a square grid.

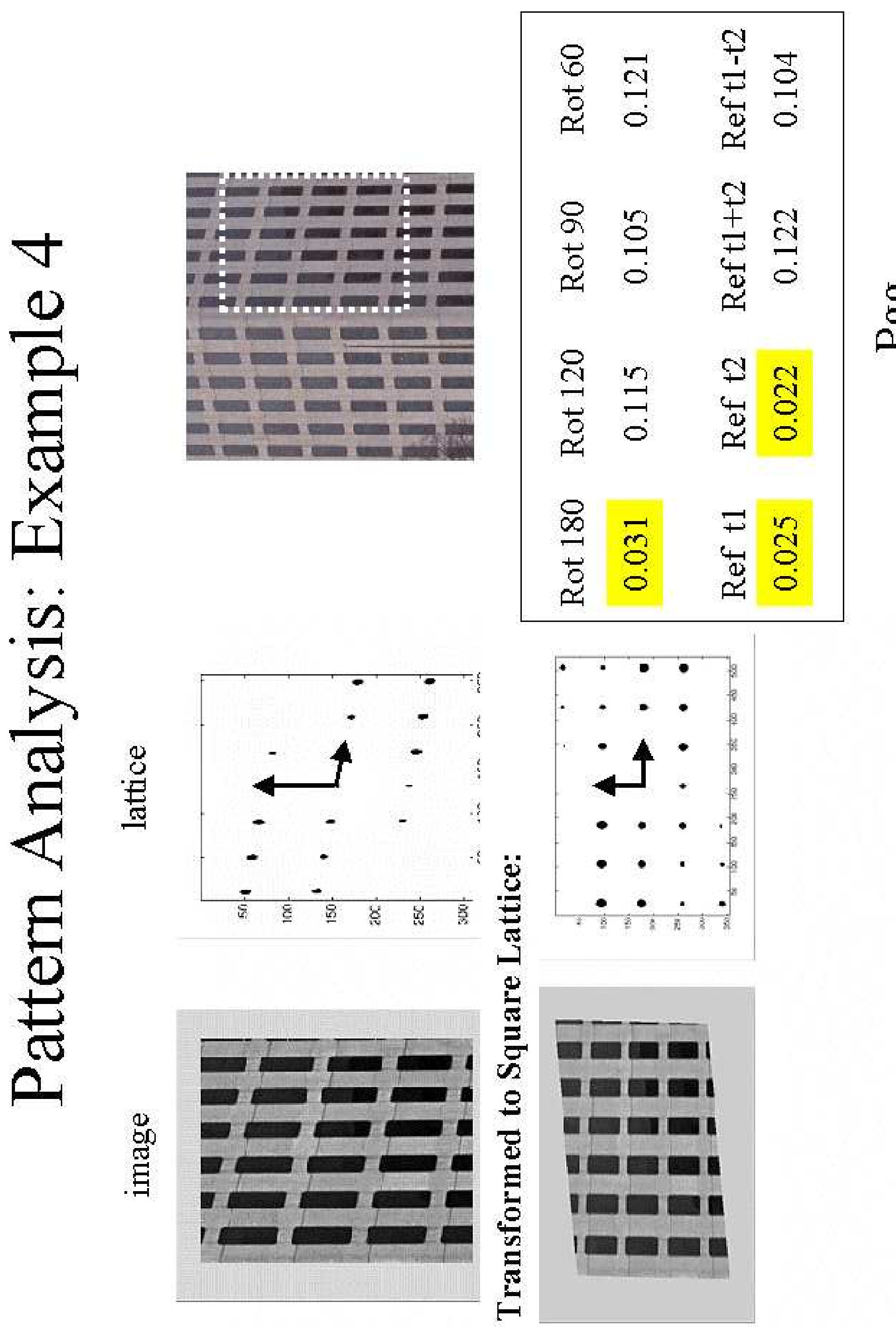


Figure 13: This is an example of treating perspective distortion as affine distortion, using the same classification algorithm to find its symmetry group: pgg.

6 Discussion and Conclusion

We have presented, for the first time, an algorithm for wallpaper symmetry group classification given a 2D repeated pattern. Using the invariant properties of certain symmetry subgroups, we extend the algorithm to work on 2D patterns under affine distortion. Our experiments show promising results.

Once again, we emphasize that in the same spirit as stated in [17], a more symmetrical structure is preferred over a less symmetrical one. This philosophy affects our affine classification algorithm in that a pattern is classified as, say, $p4m$ though it could be the result of a non-uniform scaling of a pmm pattern. In other words, if the pattern has the potential to be a "more symmetrical" pattern, that fact will be acknowledged in our algorithm.

This is a report for an on-going research effort. There is still much remaining work. First of all, a more complete set of justifications for the invariant and, especially, definite-variant symmetries of wallpaper groups under non-rigid transformations need to be included. Secondly, a stress test of the algorithms (both Euclidean and affine) is required, this will be done on a large set of repeated patterns collected from the web, scanned images and digital photos. We shall also report our testing results on Frieze patterns (they are omitted here since they are relatively simpler than the wallpaper patterns). Thirdly, all the migration diagrams will be provided in the near future as the third alternative for pattern classification under affine distortion. Fourthly, quantitative evaluation of the algorithm and comparison of different classification and matching methods in speed and complexity will be carried out. Lastly, application to projectively distorted patterns and other computer vision applications will be explored.

7 Appendix

Here we present the results from processing the 17 wallpaper patterns from [15]. Note, groups $p3m1$ and $p31m$ have been reversed in order to be consistent with the ordering in Figure 5 from [26]. Tables 3 and 4 present results for automatic detection of rotational and reflection symmetry. Each pattern was first analyzed "as is" (no pre-transformation performed), and the results are reported in the row marked "none". The pattern was then warped to have a square lattice and reanalyzed. The results are shown in the row marked "square". Finally, the pattern was warped to have a hexagonal lattice structure before analysis. These results are shown in the row marked "hexag". Each table entry is the score for that rotation or reflection symmetry — those entries marked in bold font pass the acceptance threshold for that pattern. This is an adaptive threshold chosen for each pattern in the manner described earlier.

The original pattern, and lattices found for it under the identity, square, and hexagonal warp transformations, are shown for each pattern in Figures 14 to 30.

Figure 31 and Table 5 show results for a figure that was not processed completely correctly. The pattern is p3. Although the rotation and reflection symmetries are correct for the original pattern, they are incorrect after transforming the pattern to a hexagonal lattice, even though the p3 symmetry group should have a hexagonal lattice to begin with. We hypothesize that the error after warping occurs because the pattern has fine details that not preserved under the bilinear warping transformation. This points out the need to process the image after warping as little as possible. For example, for this particular pattern, the original lattice was very close to hexagonal anyways, so no further warping actually needed to be done. This also points out that our proofs and algorithms can be correct, but still give the wrong results due to low-level image processing problems.

pattern	transform	rot180	rot120	rot90	rot60	T1 refl	T2 refl	D1 refl	D2 refl
p1	none	0.259	0.357	0.264	0.338	0.275	0.270	0.295	0.287
	square	0.254	0.210	0.269	0.321	0.171	0.275	0.329	0.256
	hexag	0.320	0.244	0.347	0.305	0.288	0.233	0.321	0.275
p2	none	**0.031**	0.224	0.282	0.221	0.162	0.250	0.195	0.188
	square	**0.042**	0.245	0.230	0.285	0.196	0.193	0.228	0.227
	hexag	**0.035**	0.264	0.262	0.263	0.173	0.253	0.214	0.212
pm	none	0.271	0.327	0.278	0.345	**0.065**	0.271	0.293	0.290
	square	0.254	0.292	0.258	0.284	**0.037**	0.255	0.256	0.259
	hexag	0.257	0.285	0.292	0.273	0.248	0.270	0.270	0.257
pg	none	0.257	0.283	0.270	0.309	0.256	**0.060**	0.310	0.309
	square	0.232	0.282	0.260	0.267	0.232	**0.093**	0.256	0.259
	hexag	0.251	0.292	0.288	0.301	0.290	0.243	0.315	0.246
cm	none	0.226	0.281	0.268	0.315	0.277	0.276	0.224	**0.097**
	square	0.198	0.284	0.209	0.296	0.210	0.212	0.197	**0.066**
	hexag	0.202	0.247	0.297	0.215	0.226	0.237	0.215	**0.079**
pmm	none	**0.070**	0.321	0.289	0.320	**0.054**	**0.060**	0.302	0.305
	square	**0.063**	0.315	0.274	0.371	**0.047**	**0.062**	0.276	0.275
	hexag	**0.070**	0.342	0.394	0.336	0.343	0.321	0.274	0.276
pmg	none	**0.060**	0.312	0.324	0.297	**0.067**	**0.077**	0.340	0.306
	square	**0.061**	0.280	0.279	0.274	**0.062**	**0.096**	0.279	0.281
	hexag	**0.055**	0.304	0.319	0.296	0.266	0.287	0.282	0.286
pgg	none	**0.047**	0.185	0.218	0.192	**0.028**	**0.034**	0.216	0.178
	square	**0.042**	0.173	0.209	0.189	**0.027**	**0.056**	0.210	0.213
	hexag	**0.044**	0.208	0.246	0.200	0.182	0.158	0.217	0.216

Table 3: Table of results for automatic detection of rotational and reflection symmetry using an example of each wallpaper pattern. For each pattern, it was first analyzed "as is" (no pre-transformation performed), and the results are reported in the row marked "none". The pattern was then warped to have a square lattice and reanalyzed. The results are shown in the row marked "square". Finally, the pattern was warped to have a hexagonal lattice structure before analysis. These results are shown in the row marked "hexag". Each table entry is the score for that rotation or reflection symmetry — those entries marked in bold font pass the acceptance threshold for that pattern.

pattern	transform	rot180	rot120	rot90	rot60	T1 refl	T2 refl	D1 refl	D2 refl
cmm	none	**0.035**	0.289	0.278	0.278	0.268	0.262	**0.100**	**0.088**
	square	**0.045**	0.260	0.271	0.252	0.273	0.278	**0.086**	**0.092**
	hexag	**0.052**	0.266	0.274	0.261	0.264	0.262	**0.104**	**0.100**
p4	none	**0.075**	0.358	**0.078**	0.351	0.220	0.202	0.196	0.203
	square	**0.081**	0.256	**0.079**	0.256	0.186	0.191	0.190	0.193
	hexag	**0.084**	0.320	0.335	0.305	0.319	0.307	0.197	0.198
p4m	none	**0.073**	0.391	**0.040**	0.361	**0.116**	**0.119**	**0.085**	**0.074**
	square	**0.077**	0.290	**0.093**	0.301	**0.089**	**0.092**	**0.078**	**0.073**
	hexag	**0.085**	0.281	0.340	0.277	0.273	0.279	**0.092**	**0.091**
p4g	none	**0.041**	0.309	**0.028**	0.324	**0.040**	**0.038**	**0.039**	**0.031**
	square	**0.047**	0.250	**0.030**	0.319	**0.043**	**0.043**	**0.049**	**0.039**
	hexag	**0.053**	0.325	0.359	0.326	0.314	0.318	**0.055**	**0.049**
p3	none	0.188	**0.027**	0.487	0.199	0.307	0.347	0.294	0.307
	square	0.185	0.332	0.315	0.324	0.317	0.313	0.287	0.296
	hexag	0.183	**0.028**	0.315	0.176	0.282	0.301	0.276	0.287
p3m1	none	0.357	**0.121**	0.479	0.355	0.354	0.356	**0.041**	0.356
	square	0.341	0.334	0.310	0.347	0.311	0.315	**0.055**	0.342
	hexag	0.327	**0.071**	0.353	0.326	0.325	0.326	**0.061**	0.327
p31m	none	0.286	**0.074**	0.325	0.295	**0.081**	**0.091**	0.335	**0.089**
	square	0.247	0.274	0.260	0.273	0.258	0.255	0.276	**0.079**
	hexag	0.271	**0.066**	0.297	0.281	**0.080**	**0.082**	0.274	**0.083**
p6	none	**0.049**	**0.040**	0.244	**0.051**	0.268	0.264	0.262	0.262
	square	**0.051**	0.276	0.282	0.281	0.267	0.276	0.268	0.271
	hexag	**0.052**	**0.065**	0.280	**0.069**	0.276	0.273	0.279	0.277
p6m	none	**0.104**	**0.111**	0.414	**0.117**	**0.131**	**0.144**	**0.085**	**0.098**
	square	**0.110**	0.316	0.299	0.314	0.299	0.295	**0.092**	**0.085**
	hexag	**0.105**	**0.102**	0.334	**0.109**	**0.104**	**0.122**	**0.097**	**0.096**

Table 4: Continuation of the table of results for automatic detection of rotational and reflection symmetry.

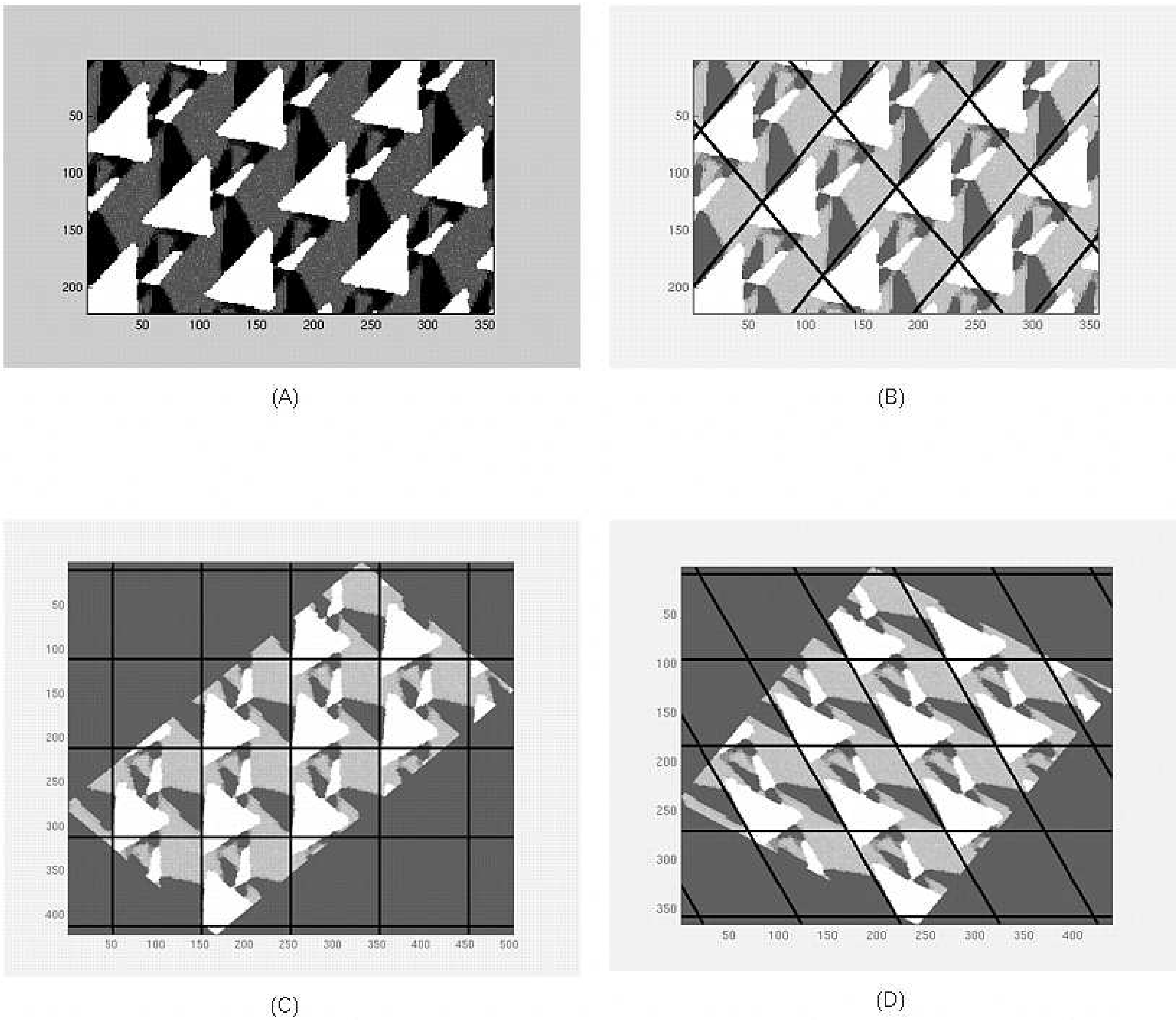


Figure 14: Lattices found for the example of wallpaper group p1. (A) original pattern. (B) Detected lattice superimposed on pattern. The gray level of the pattern has been lightened to allow easy viewing of the superimposed lattice. (C) Lattice detected after warping the pattern into a square grid structure. (D) Detected lattice after warping the pattern into a hexagonal lattice structure.

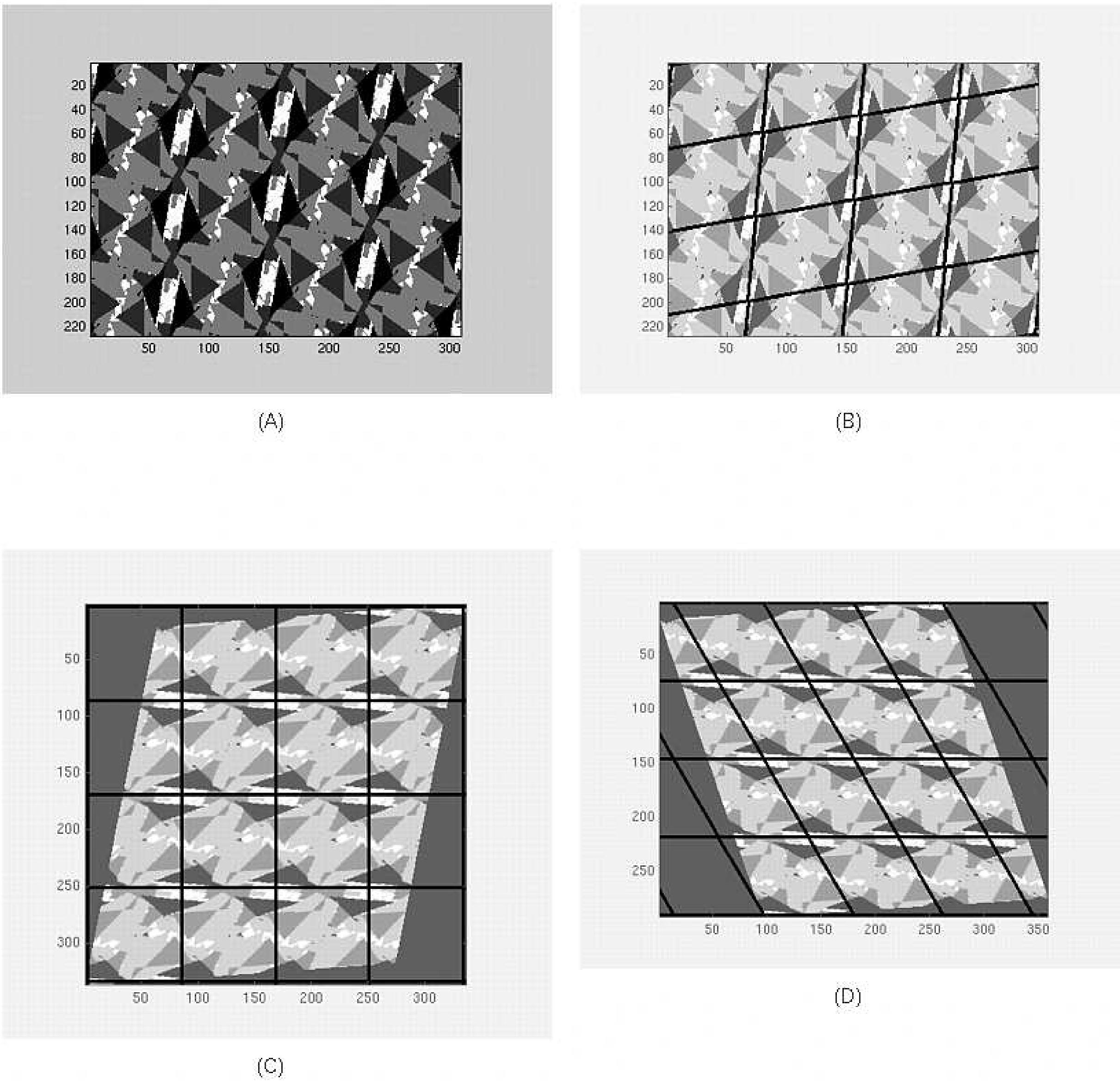


Figure 15: Lattices found for the example of wallpaper group p2. (A) original pattern. (B) Detected lattice superimposed on pattern. The gray level of the pattern has been lightened to allow easy viewing of the superimposed lattice. (C) Lattice detected after warping the pattern into a square grid structure. (D) Detected lattice after warping the pattern into a hexagonal lattice structure.

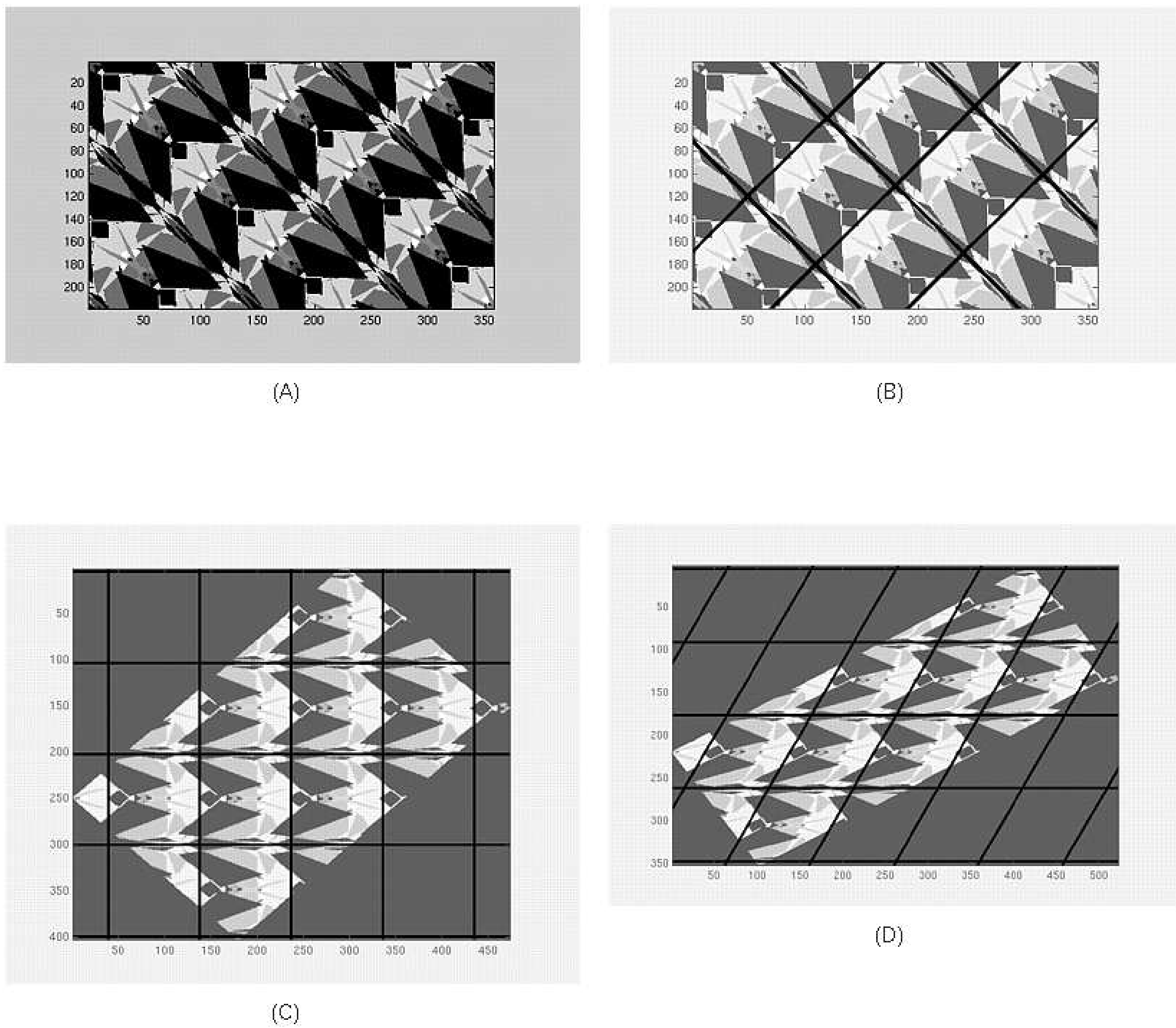


Figure 16: Lattices found for the example of wallpaper group pm. (A) original pattern. (B) Detected lattice superimposed on pattern. The gray level of the pattern has been lightened to allow easy viewing of the superimposed lattice. (C) Lattice detected after warping the pattern into a square grid structure. (D) Detected lattice after warping the pattern into a hexagonal lattice structure.

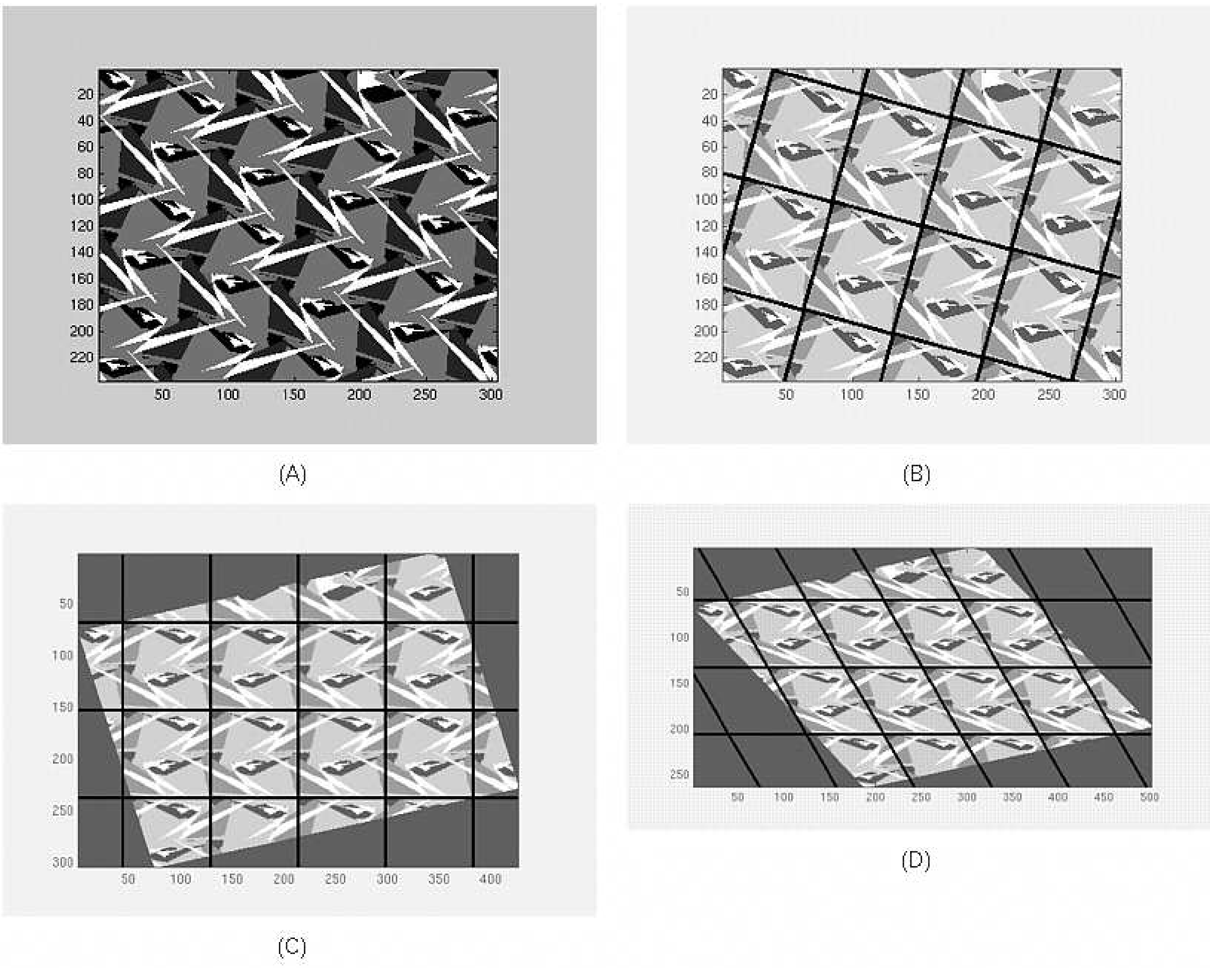


Figure 17: Lattices found for the example of wallpaper group pg. (A) original pattern. (B) Detected lattice superimposed on pattern. The gray level of the pattern has been lightened to allow easy viewing of the superimposed lattice. (C) Lattice detected after warping the pattern into a square grid structure. (D) Detected lattice after warping the pattern into a hexagonal lattice structure.

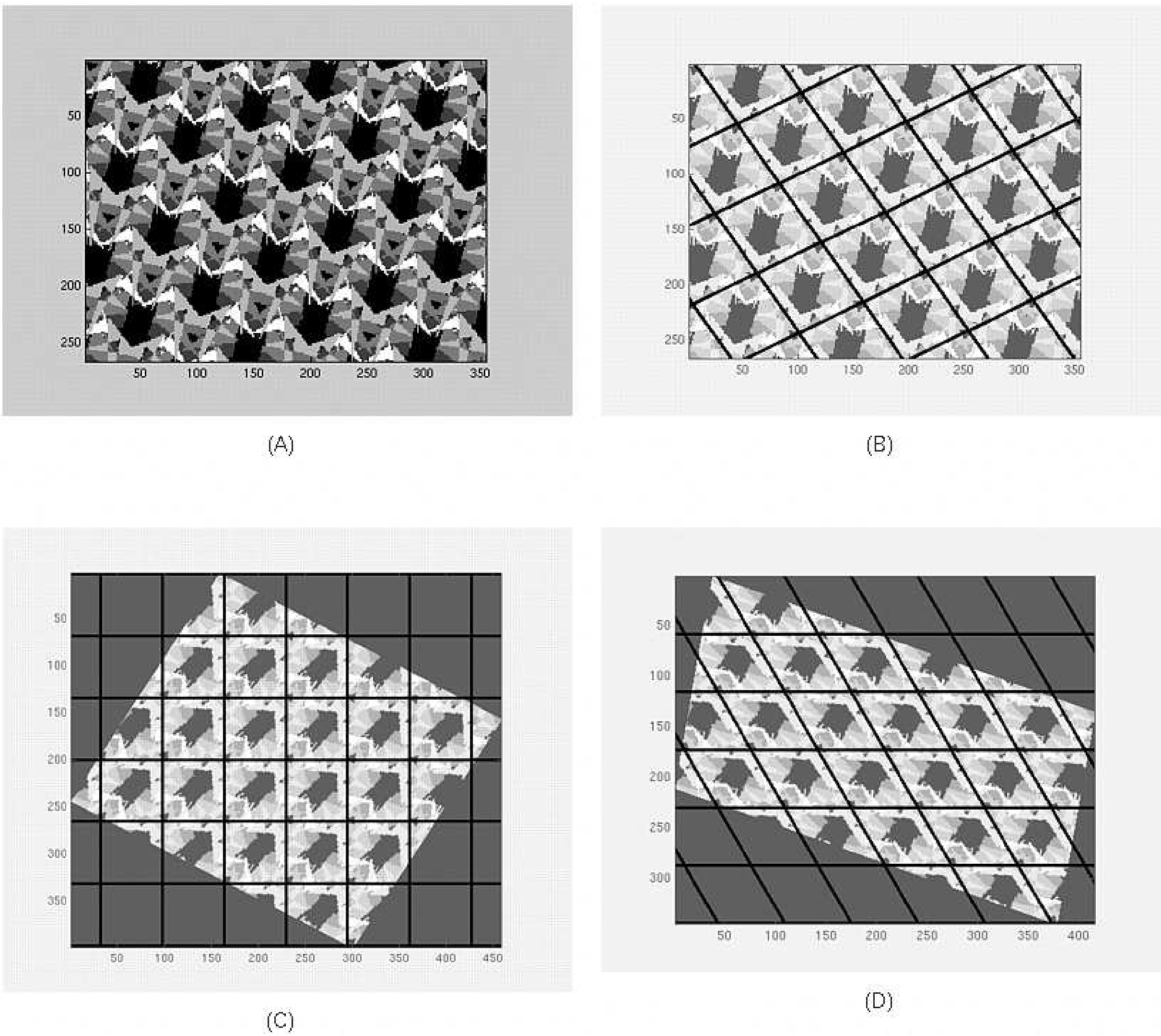


Figure 18: Lattices found for the example of wallpaper group cm. (A) original pattern. (B) Detected lattice superimposed on pattern. The gray level of the pattern has been lightened to allow easy viewing of the superimposed lattice. (C) Lattice detected after warping the pattern into a square grid structure. (D) Detected lattice after warping the pattern into a hexagonal lattice structure.

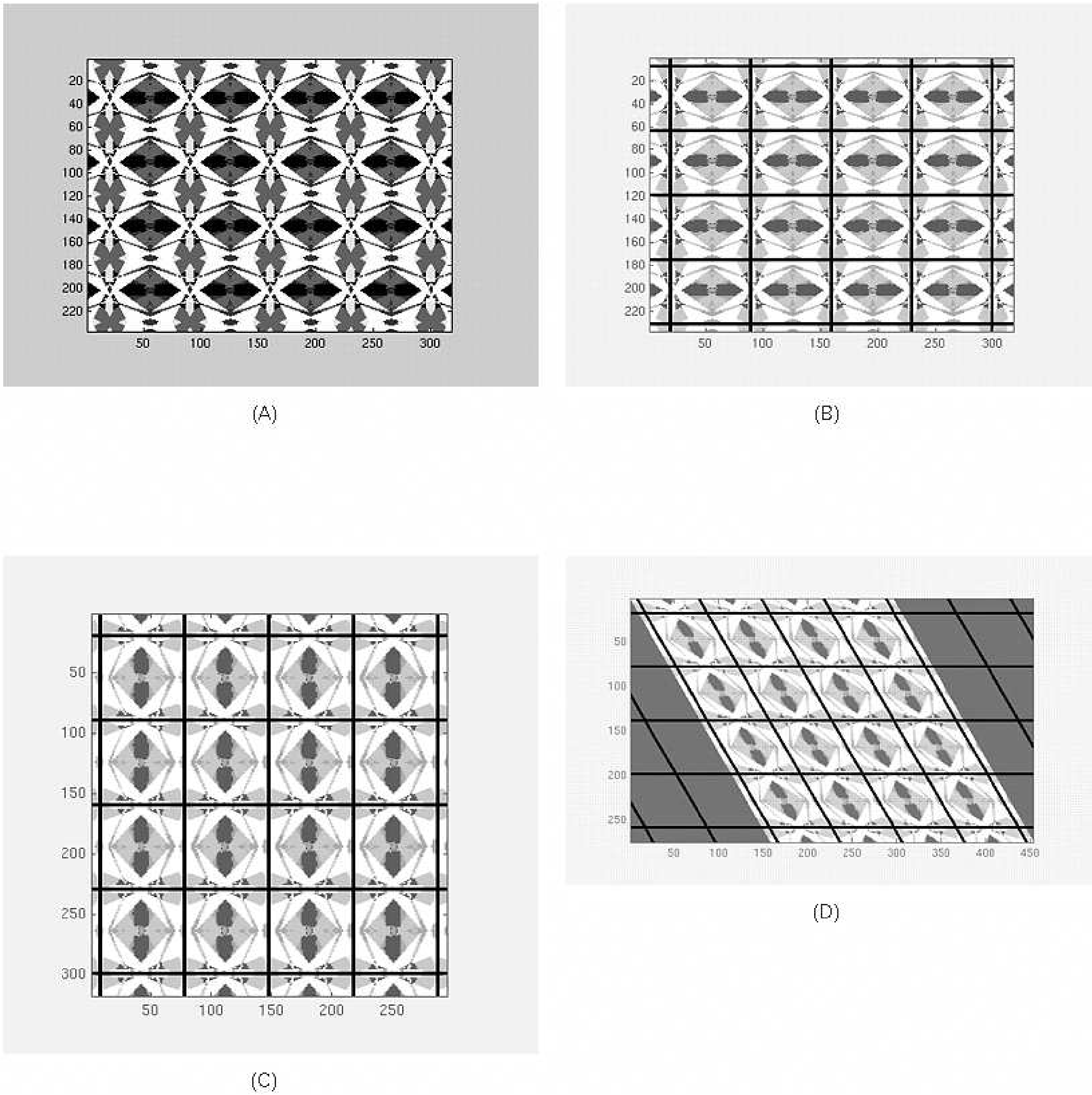


Figure 19: Lattices found for the example of wallpaper group pmm. (A) original pattern. (B) Detected lattice superimposed on pattern. The gray level of the pattern has been lightened to allow easy viewing of the superimposed lattice. (C) Lattice detected after warping the pattern into a square grid structure. (D) Detected lattice after warping the pattern into a hexagonal lattice structure.

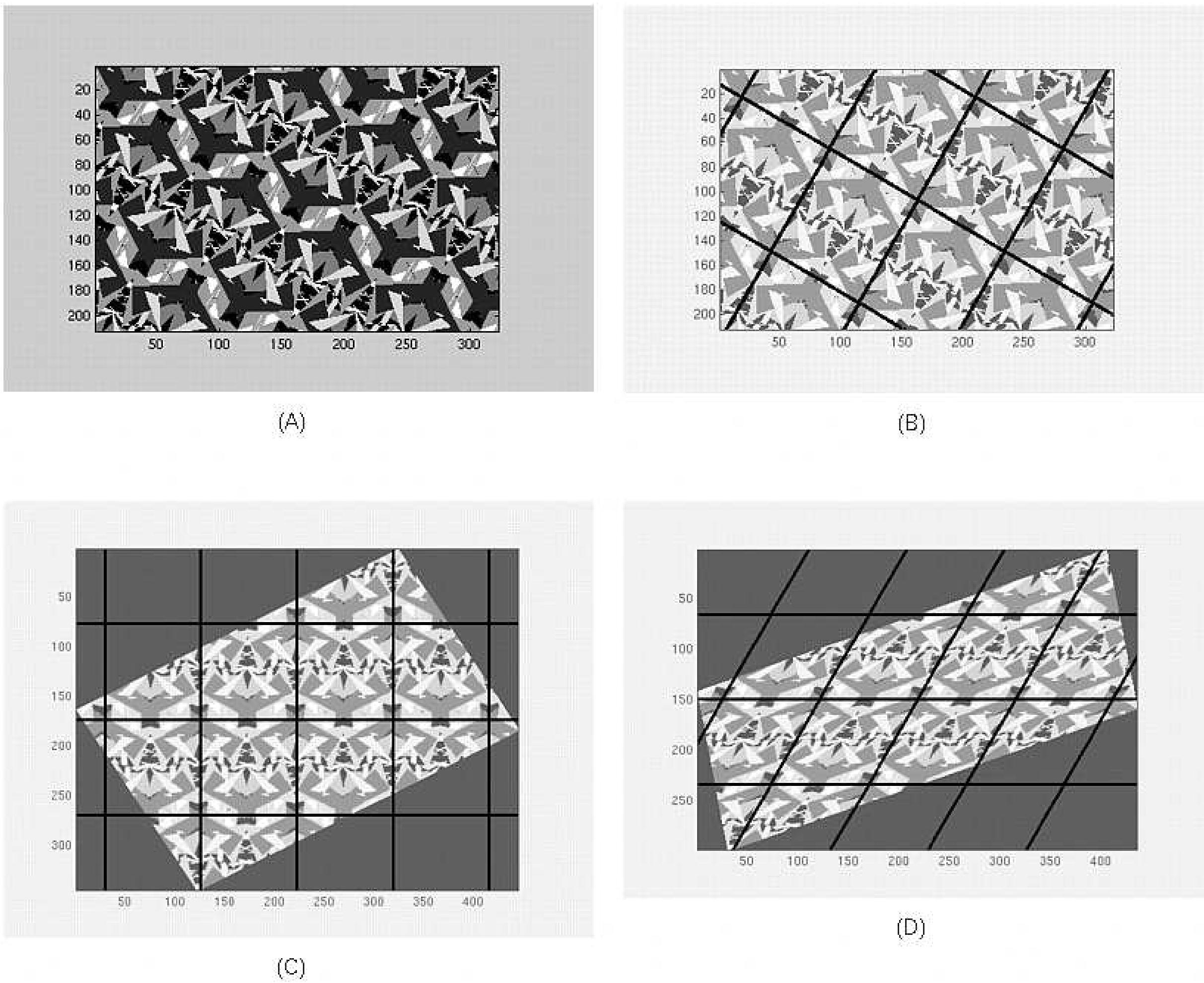


Figure 20: Lattices found for the example of wallpaper group pmg. (A) original pattern. (B) Detected lattice superimposed on pattern. The gray level of the pattern has been lightened to allow easy viewing of the superimposed lattice. (C) Lattice detected after warping the pattern into a square grid structure. (D) Detected lattice after warping the pattern into a hexagonal lattice structure.

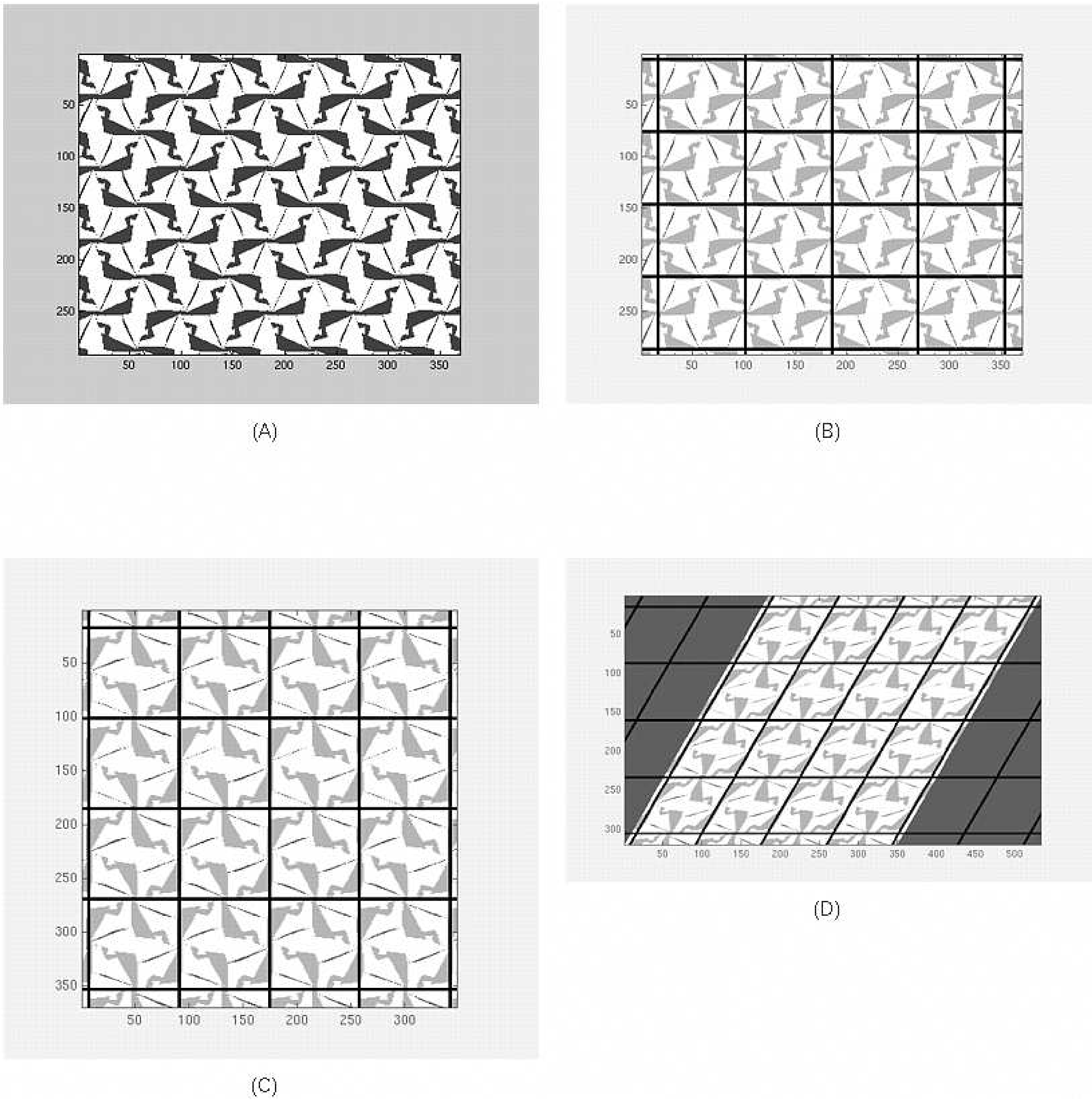


Figure 21: Lattices found for the example of wallpaper group pgg. (A) original pattern. (B) Detected lattice superimposed on pattern. The gray level of the pattern has been lightened to allow easy viewing of the superimposed lattice. (C) Lattice detected after warping the pattern into a square grid structure. (D) Detected lattice after warping the pattern into a hexagonal lattice structure.

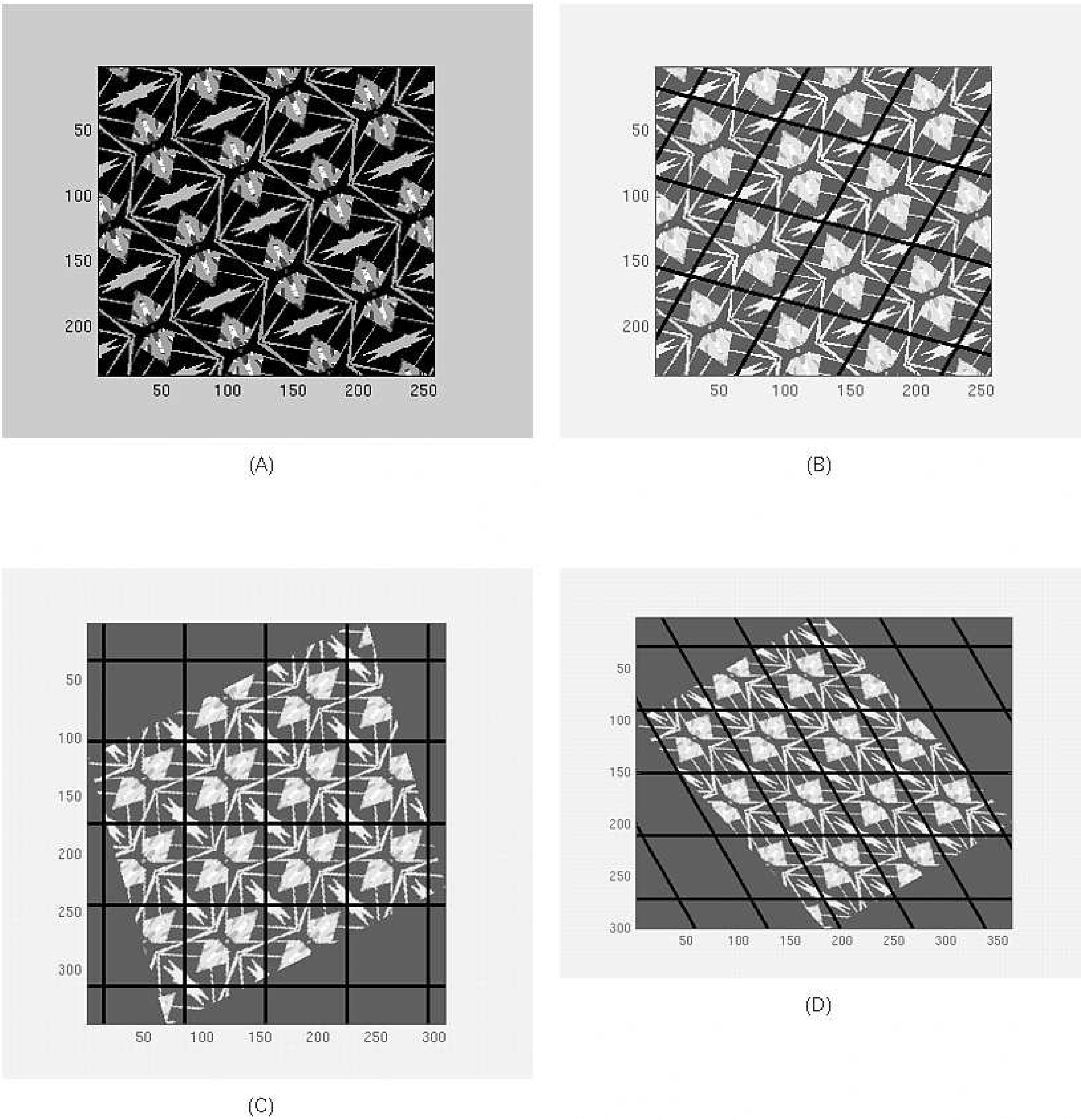


Figure 22: Lattices found for the example of wallpaper group cmm. (A) original pattern. (B) Detected lattice superimposed on pattern. The gray level of the pattern has been lightened to allow easy viewing of the superimposed lattice. (C) Lattice detected after warping the pattern into a square grid structure. (D) Detected lattice after warping the pattern into a hexagonal lattice structure.

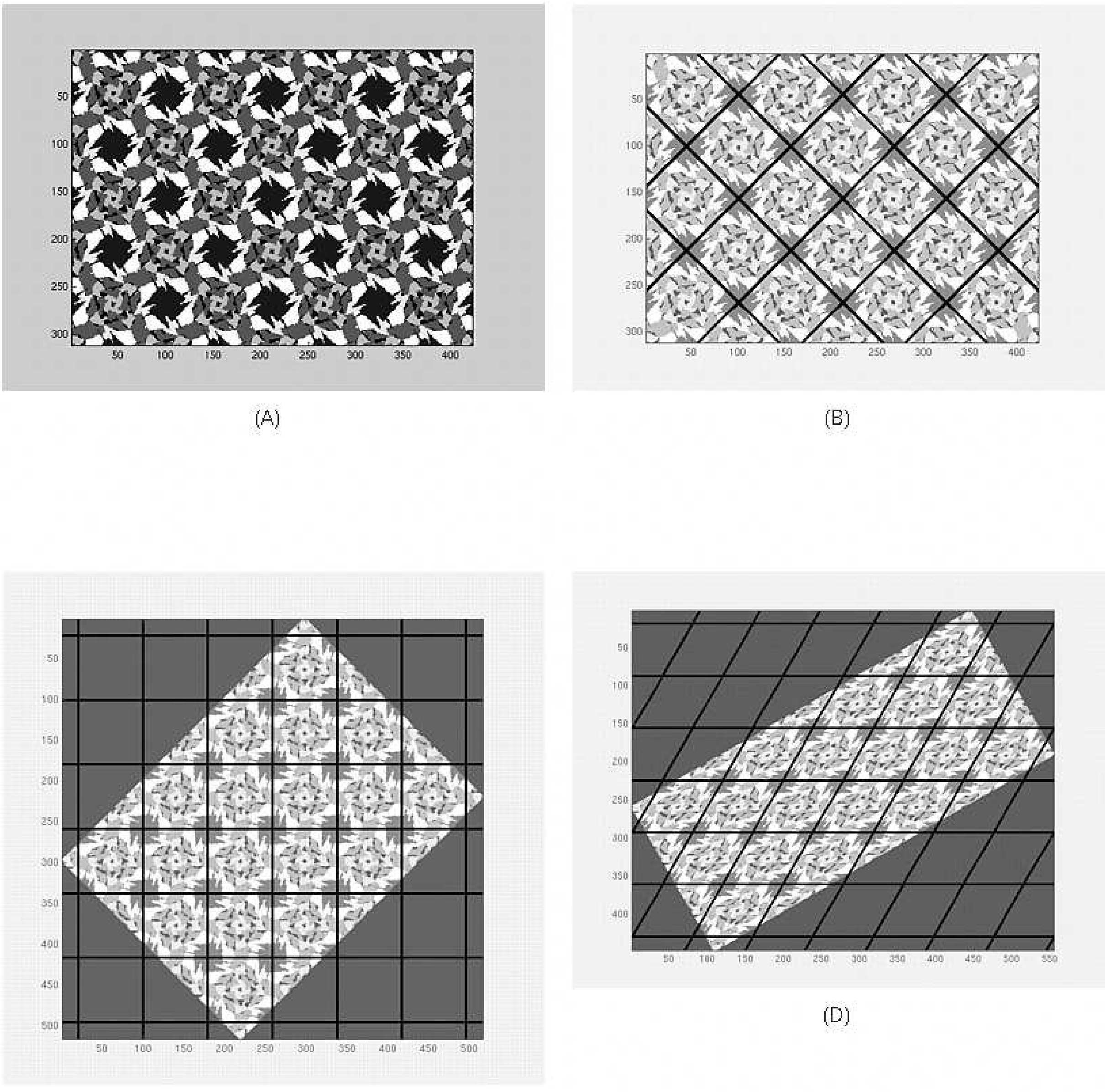


Figure 23: Lattices found for the example of wallpaper group p4. (A) original pattern. (B) Detected lattice superimposed on pattern. The gray level of the pattern has been lightened to allow easy viewing of the superimposed lattice. (C) Lattice detected after warping the pattern into a square grid structure. (D) Detected lattice after warping the pattern into a hexagonal lattice structure.

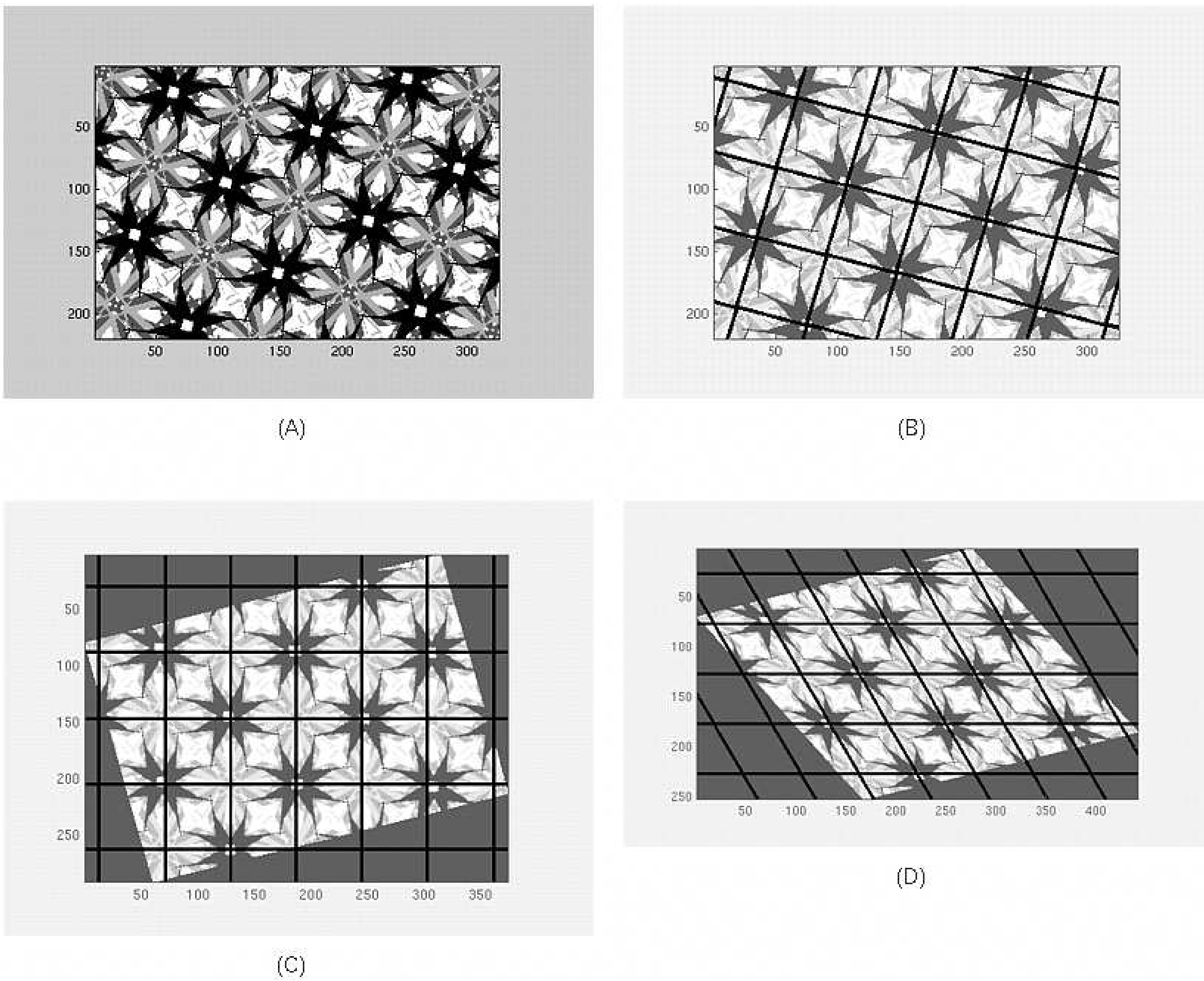


Figure 24: Lattices found for the example of wallpaper group p4m. (A) original pattern. (B) Detected lattice superimposed on pattern. The gray level of the pattern has been lightened to allow easy viewing of the superimposed lattice. (C) Lattice detected after warping the pattern into a square grid structure. (D) Detected lattice after warping the pattern into a hexagonal lattice structure.

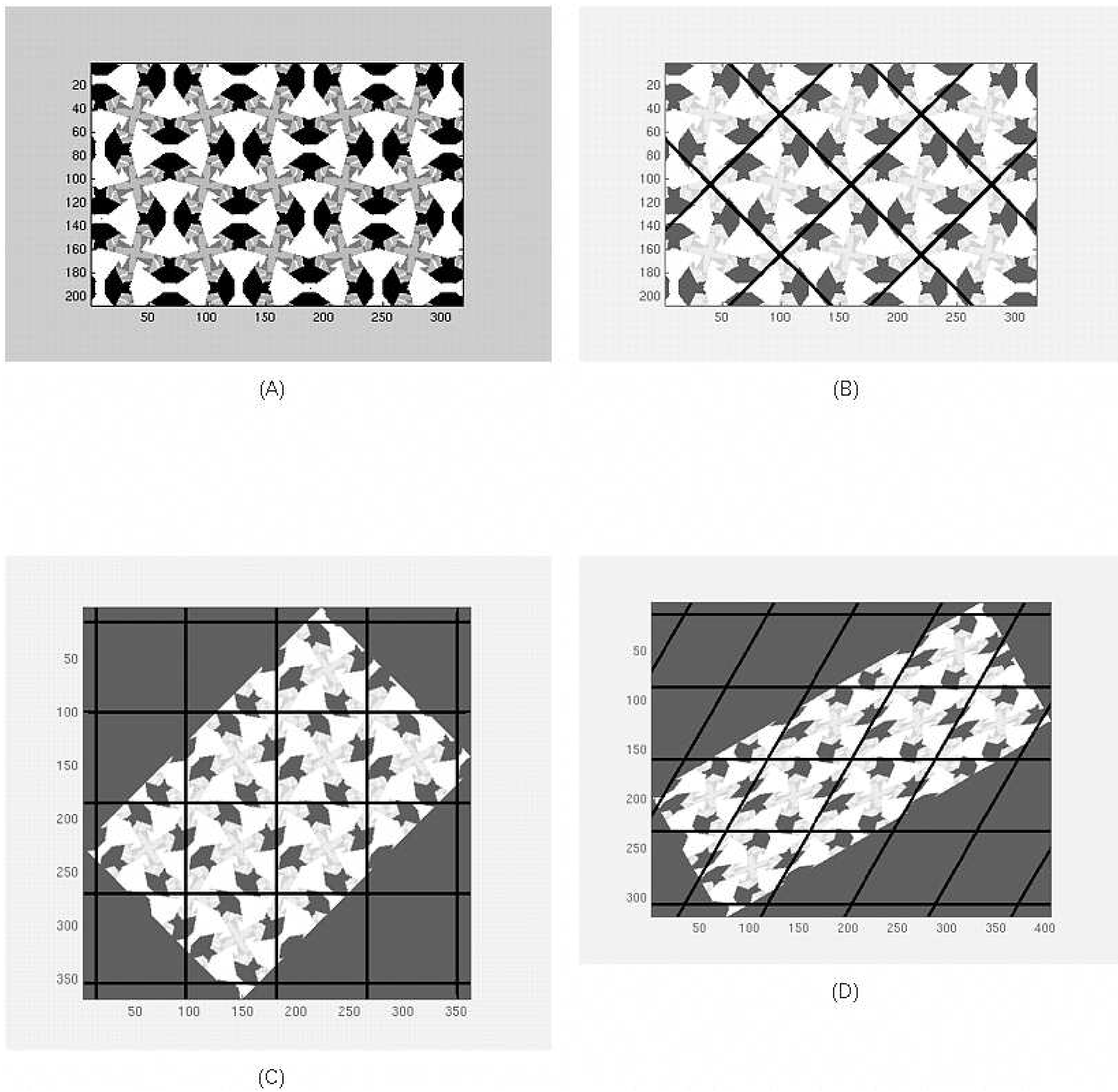


Figure 25: Lattices found for the example of wallpaper group p4g. (A) original pattern. (B) Detected lattice superimposed on pattern. The gray level of the pattern has been lightened to allow easy viewing of the superimposed lattice. (C) Lattice detected after warping the pattern into a square grid structure. (D) Detected lattice after warping the pattern into a hexagonal lattice structure.

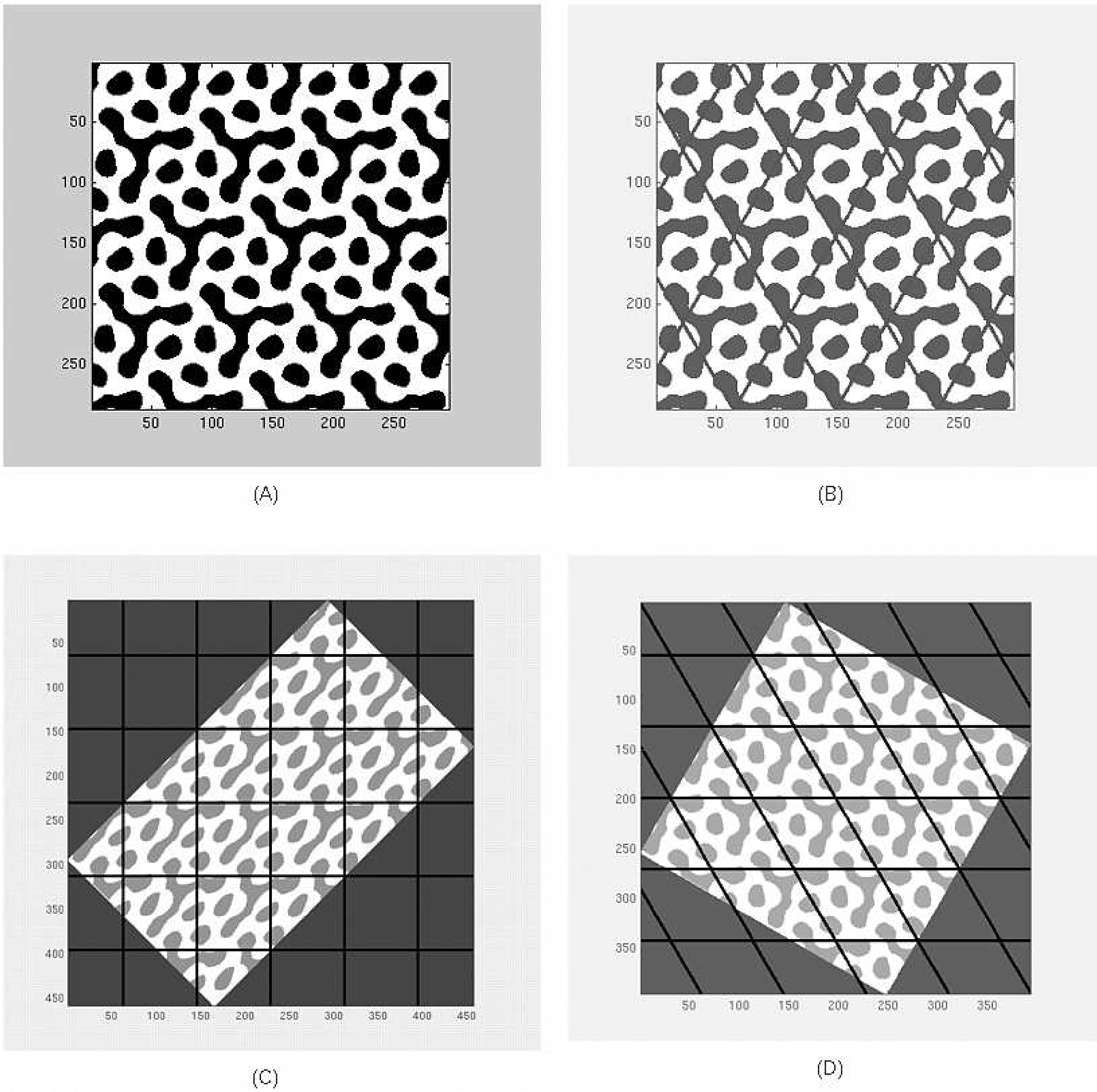


Figure 26: Lattices found for the example of wallpaper group p3. (A) original pattern. (B) Detected lattice superimposed on pattern. The gray level of the pattern has been lightened to allow easy viewing of the superimposed lattice. (C) Lattice detected after warping the pattern into a square grid structure. (D) Detected lattice after warping the pattern into a hexagonal lattice structure.

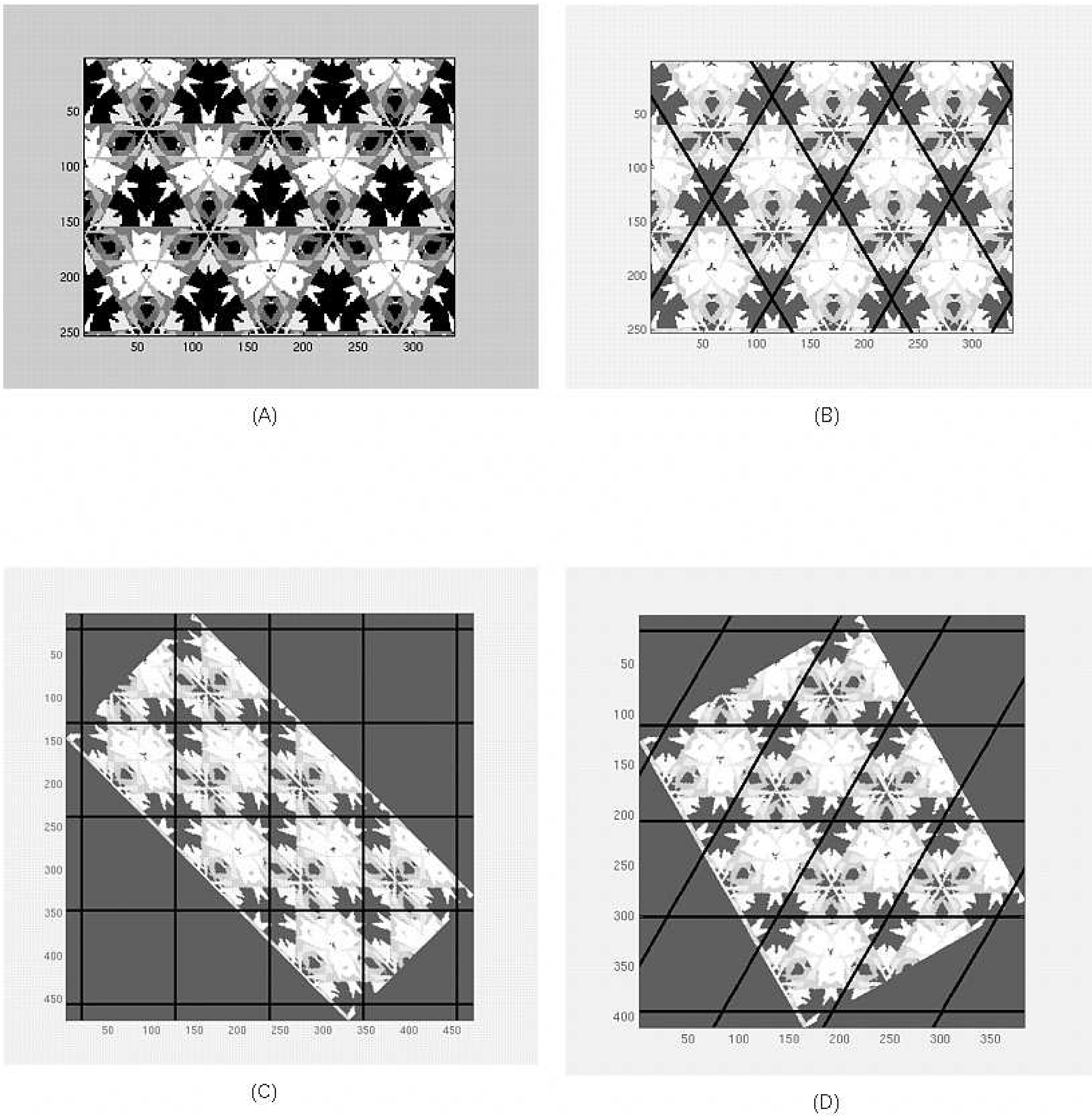


Figure 27: Lattices found for the example of wallpaper group p3m1. (A) original pattern. (B) Detected lattice superimposed on pattern. The gray level of the pattern has been lightened to allow easy viewing of the superimposed lattice. (C) Lattice detected after warping the pattern into a square grid structure. (D) Detected lattice after warping the pattern into a hexagonal lattice structure.

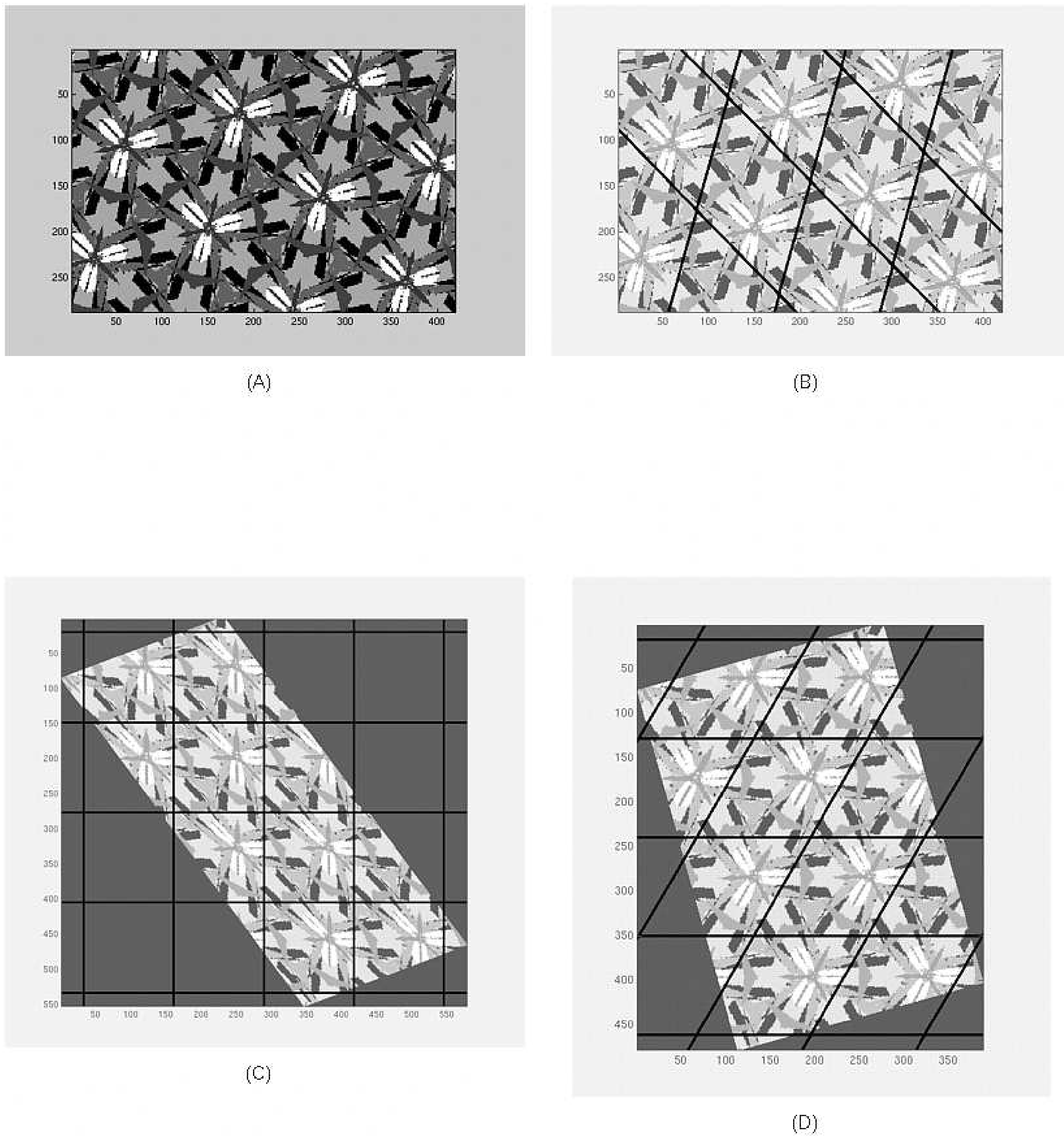


Figure 28: Lattices found for the example of wallpaper group p31m. (A) original pattern. (B) Detected lattice superimposed on pattern. The gray level of the pattern has been lightened to allow easy viewing of the superimposed lattice. (C) Lattice detected after warping the pattern into a square grid structure. (D) Detected lattice after warping the pattern into a hexagonal lattice structure.

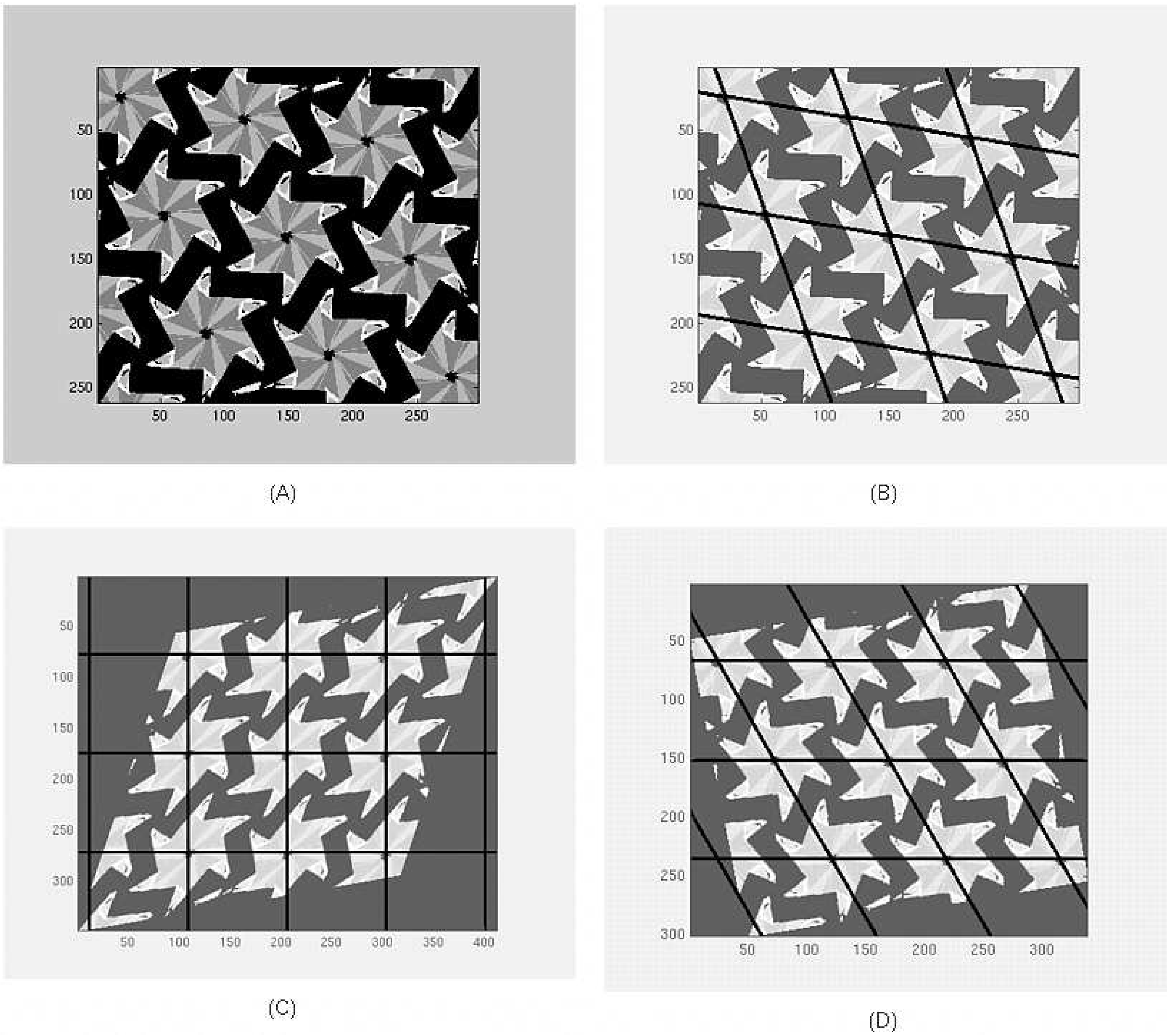


Figure 29: Lattices found for the example of wallpaper group p6. (A) original pattern. (B) Detected lattice superimposed on pattern. The gray level of the pattern has been lightened to allow easy viewing of the superimposed lattice. (C) Lattice detected after warping the pattern into a square grid structure. (D) Detected lattice after warping the pattern into a hexagonal lattice structure.

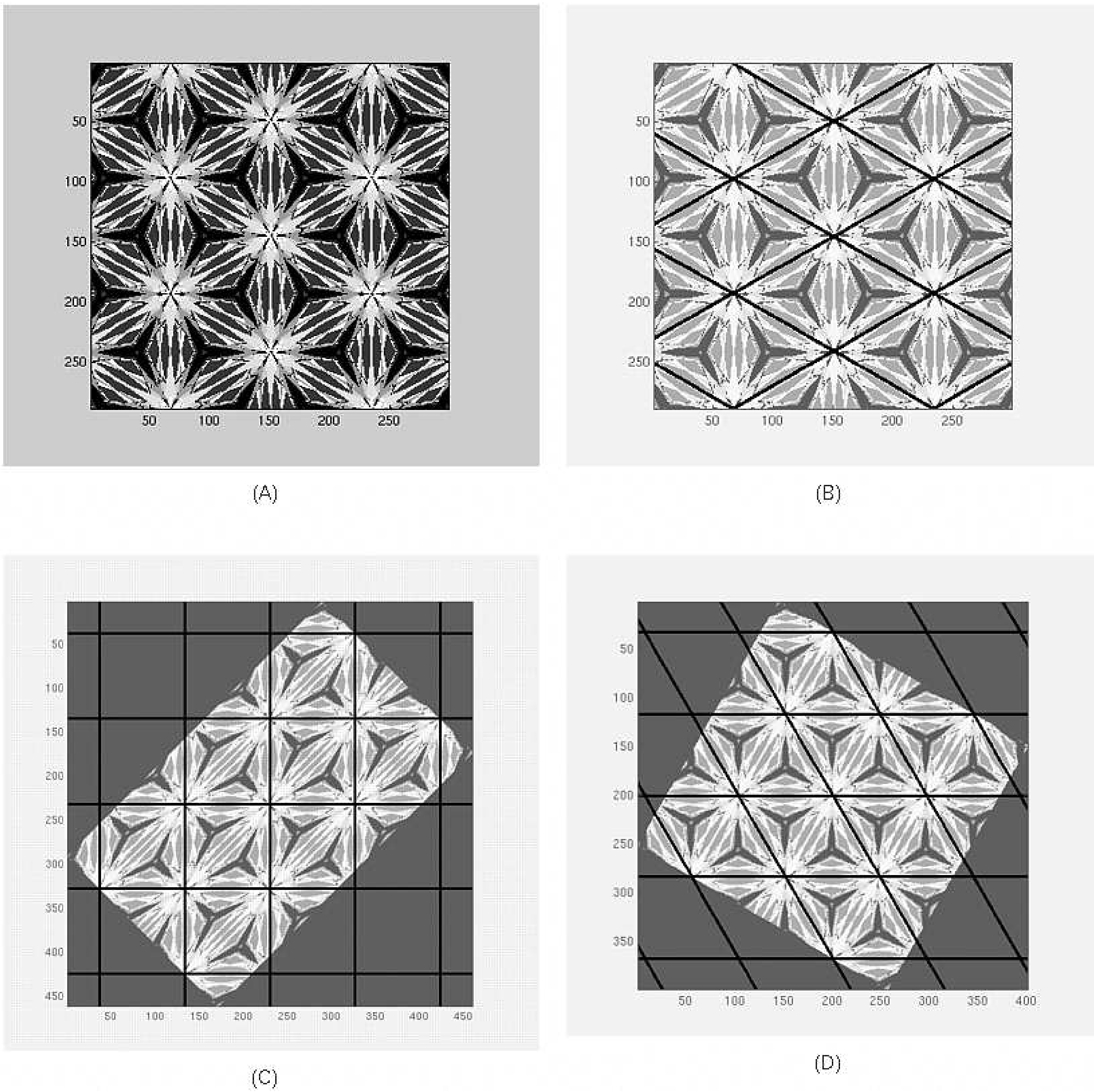


Figure 30: Lattices found for the example of wallpaper group p6m. (A) original pattern. (B) Detected lattice superimposed on pattern. The gray level of the pattern has been lightened to allow easy viewing of the superimposed lattice. (C) Lattice detected after warping the pattern into a square grid structure. (D) Detected lattice after warping the pattern into a hexagonal lattice structure.

pattern	transform	rot180	rot120	rot90	rot60	T1 refl	T2 refl	D1 refl	D2 refl
p3	none	0.254	**0.082**	0.230	0.253	0.153	0.235	0.275	0.233
	square	0.125	0.127	0.145	0.135	0.140	0.143	0.098	0.116
	hexag	0.135	0.171	0.216	0.232	0.129	0.227	0.258	0.221

Table 5: The row marked “none” is correct. However, the row marked “hexag”, which should give the same results, fails to find the 3-fold rotational symmetry.

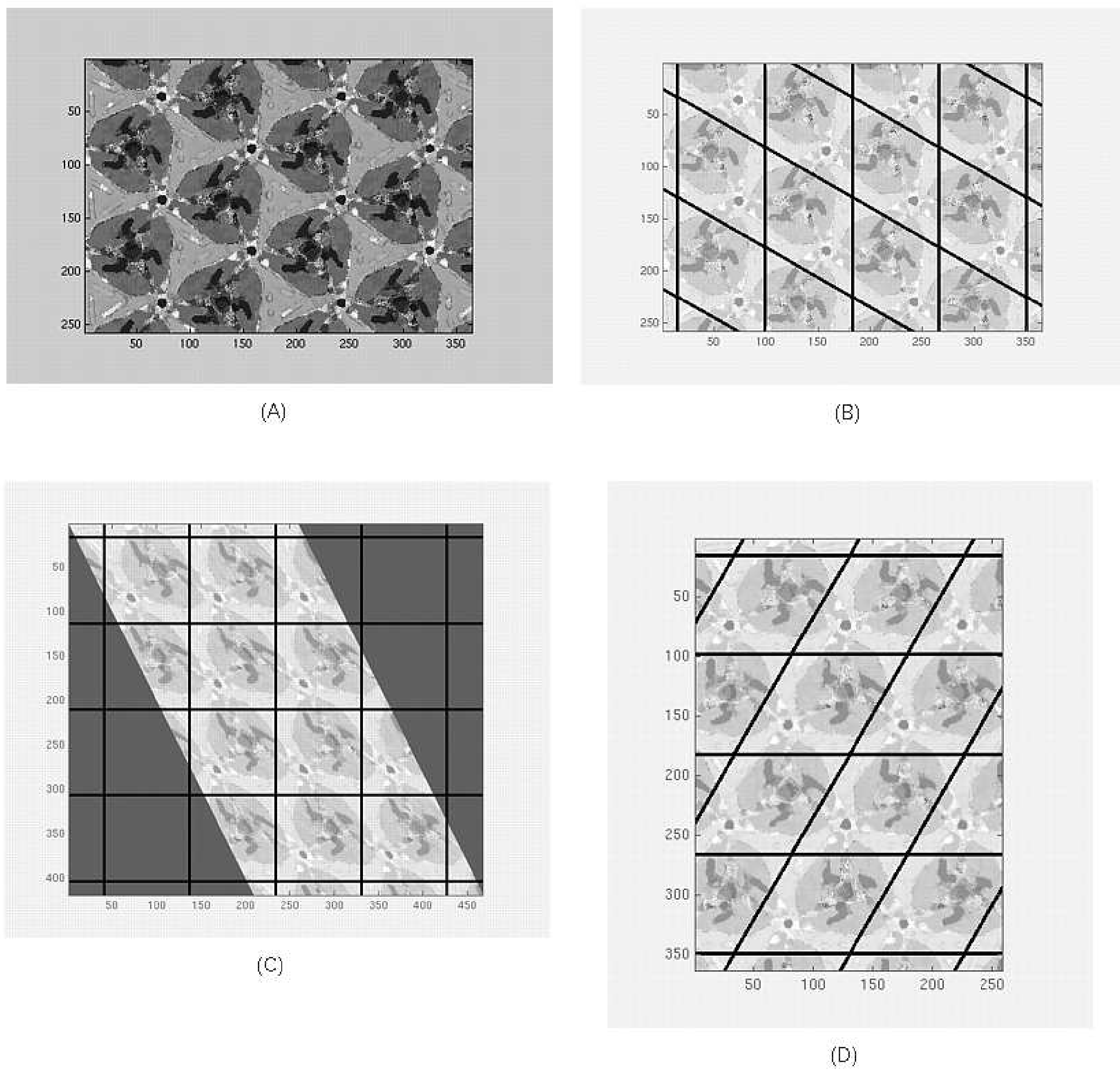


Figure 31: Lattices found for a pattern wallpaper group p3, for which subsequent analysis on the hexagonally warped image does not work correctly.

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