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Computation of Wave Fronts on a Disk I: Numerical Experiments

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Abstract

This paper discusses numerical experiments of wave front propagation on a flat disk and related results. In particular, notions of extremal rays and wave front caustics are introduced to investigate the dynamic behaviors of wave fronts under repeated reflection. These results are related to earlier work on caustics of ray systems. It is shown that wave fronts require a modified notion of caustic when considered under repeated reflection. We observe specific conditions for formation and annihilation of cusp singularities, self-tangencies and self-intersection of wave fronts.

Keywords: wave fronts, rays, caustic, cusps, evolution, reflection, self-tangencies, self-intersections

1 Introduction

The purpose of this paper is the development of properties of propagating wave fronts in two spatial dimensions under reflection on a disk-shaped domain. To this end we present numerical experiments to collect properties of this case. Detailed comparison with the extensive theory of generic properties of wave fronts in the plane are planned for a future paper.

1.1 Background

In recent years, it has been realized that two-dimensional hyperbolic dynamical systems can be simulated more efficiently if the dynamics is represented in terms of one-dimensional trajectories [8]. Circular arrays become the core of this approach. In addition to a drastic increase in performance, they offer desirable numerical properties that are unknown in conventional finite difference or finite element methods. This approach has, however, a number of limitations. Typically the underlying

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geometric ray constructions necessitate the usual "short-wave" assumption for accuracy of the method. Also it becomes difficult to efficiently reconstruct the full 2-dimensional domain from a dense set of ray paths.

To find ways to address these shortcomings we emphasize wave fronts as the primary object for constructions. These are complementary to rays in geometric optics [1]. Wave fronts are equidistant sets from an initial curve or point. This work is part of a larger program attempting to understand the long-wave range of the dynamical situation better, as well as to understand the transition from the long-wave to the short range regimes with the goal of finding computationally efficient representations. Alternatively one can also phrase these questions as an attempt to find dynamic representations of the zero sets of integer order Bessel function. These too are not fully understood.

1.2 Definitions

The dynamics of ray systems can be conveniently studied using the concept of a mathematical billiard. A billiard is a bounded domain which maps rays to rays under reflection. The general mathematical billiard is defined over a Riemannian manifold [10]. Here, we consider the flat, Euclidean case only. Hence the geodesics are straight lines. Secondly, we will restrict the consideration to a circular boundary of the manifold.

Definition 1.1 Let $p_0, p_1, \dots p_n \in \partial\Omega = \mathbb{R}/\mathbb{Z}$ be points of reflection on the boundary (short *reflection point*) and let $\alpha_0, \alpha_1, \dots \alpha_n \in [0, \pi]$ be the angles of incidence with respect to the tangent of $\partial\Omega$ at $p_0, p_1, \dots p_n$. Then $T_{\partial\Omega} : (p_{n-1}, \alpha_{n-1}) \mapsto (p_n, \alpha_n)$ is the *billiard map*.

Let $D_r(c) = \{x | d(x, c) <= r\}$ (short D) be the closed Disk with $d(\cdot, \cdot)$ the Euclidean metric. We observe that any billiard map for disk domains leaves the angle unchanged for any fixed angle, hence for any T_D we have $\alpha_n = \alpha_{n-1}$.

Definition 1.2 The *reduced billiard map* for the Disk domain $D_r(c)$ is $T_D : p_{n-1} \mapsto p_n$.

Throughout this paper we deal only with reduced billiard maps and hence any billiard map mentioned here will be such. If the definition does not explicitly use the shape of the domain boundary $\partial\Omega$ or the metric d on the domain, statements generalize to the conventional billiard map.

Definition 1.3 Given a sequence of n billiard maps $T^n = T \circ T \circ T \cdots \circ T$ we call n the *order* of the billiard map. Also objects (rays, wave fronts) defined in relation to billiard maps are called of the n-th order if they are generated by an n-th order billiard map.

We note that distance between points on the reduced billiard map remains unchanged for any order. That is, $l_\alpha = d(p_{n-1}, p_n) = d(p_n, p_{n+1})$. This is equivalent to the trivial fact that the length of line segments of circles is unchanged by rotation with respect to the center of the circle.

Definition 1.4 A *billiard ray* $s = p_n - p_{n-1}$ is the oriented line segment between p_n and p_{n-1} under the (reduced) billiard map. An n-th order billiard ray is a ray arrived at by a sequence of (reduced) billiard maps of order n.

Definition 1.5 A *generic ray* (or just *ray*) r is any oriented line segment within the domain Ω. A *ray bundle* R_{q_0} at a point $q_0 \in \Omega$ is the set of all rays passing through q_0 of distinct angles β relative to some fixed line through q_0. The ray bundle R_{q_0} is dense with respect to angle $\beta \in [0, 2\pi)$.

Points q_n will refer to points inside the domain Ω and possibly on the boundary, whereas points p_n refer only to points on the boundary resulting from a billiard map.

Definition 1.6 The *line of symmetry of the ray bundle* is the line containing the center of the disk domain D and the position of the ray bundle q_0.

Definition 1.7 The *ray distance* l_r from a point q_0 to a point q_1 is the accumulated metric $l = d(q_0, p_0) + \sum_{i=0}^{n-1} d(p_i, p_{i+1}) + d(p_n, q_1)$ measured from q_0, then continued on the domain under the billiard map until q_1 is reached. Note that q_1 is supported on the n-th order ray under the map.

There may be different rays leading to the same point q_1 under the billiard map, hence l_r is not unique, but dependends on a specific ray sequence.

Definition 1.8 A *wave front* (or short *front*) is the set of all points of equal ray distance from a ray bundle R_{q_0}.

Note that usually wave fronts, when defined as equidistant sets, are defined with reference to an initial plane curve [1]. For the purpose of this paper we limit the definition to those fronts which do have a common point for some ray distance. The billiard map is invertible [10], hence this includes billiard maps with ray orientations inverted. It is an open question what generic plane curves can be generated by wave fronts from point sources under billiard maps.

These are all the definitions we need to describe the continuous case and for the introduction of the numerical algorithm, which follows next.

2 Numerical Algorithm

The discrete implementation of the continuous model is achieved by replacing the dense ray bundle with a sparse and finite ray bundle. Taking a finite set of size m, we take a discrete subset of $R_{q_0,m} \subset R_{q_0}$. Throughout this work we have used uniform discrete angles $\beta_k = k\beta/m, k \in [0, \ldots, m-1]$. In preliminary work we also used sets that were uniform in the distance of neighboring reflection points. Additionally all scalars are replaced by floating point representations thereof.

This immediately leads to an algorithm for implementing the dynamics of discrete representations of wave fronts:

Algorithm 1 *Step 1: Choose initial wave front point q.*
Step 2: Calculate discrete ray bundle $R_{q_0,k}$.

Step 3: Calculate billiard map angle α and billiard ray length l_α for all rays in the bundle. Calculate oriented intersection points p_0 of all rays with the domain boundary. Set ray order to 0.
Loop (Steps 4-5): Increase wave front distance l
Step 4: If q_1 outside Ω, increase ray order n, calculate p_n from p_{n-1} and α.
Step 5: Calculate ray distances and points q_1 on the wave front.

All results presented here are calculated using this basic method. All figures are rendered with a discrete bundle size of 4000 rays, which reduces to calculation of 2000 rays due to the mirror symmetry of the problem.

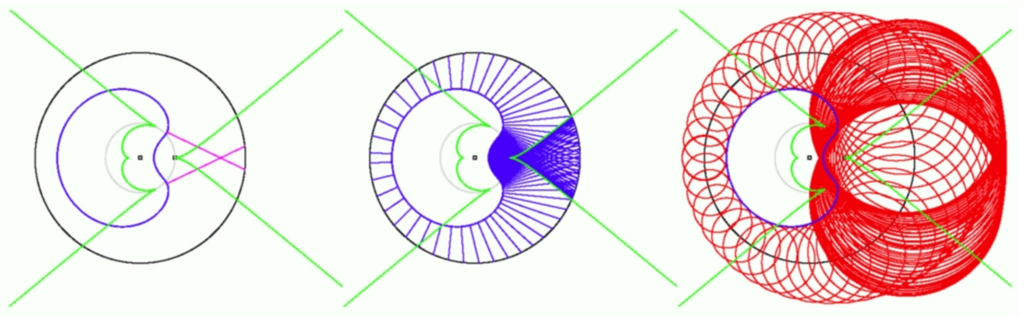

Fig. 1. Left: A third order wave front. Middle: Its supporting ray bundle (only a subset shown). Right: Its Huygens set (only a subset shown). All these are for a ray bundle located at 0.33.

The basic properties of this algorithm are depicted in figure 1, which shows the rendered front, its supporting ray bundle and the related Huygens' set. The Huygens' set is a circle bundle with the points of reflection defining the center of the circle and the associated ray defining the radius. The green curve is the caustic of the ray system to be discussed later. We see the close relationship between the three concepts. The shape of the wave front is supported on the rays and appears as singular envelope of the Huygens' set. The Huygens' set shows additional features, but these will not be discussed any further. Throughout the rest of our discussion, the first two pictures will be used.

Additionally we calculate other objects to facilitate the study of wave fronts. They will also help relate the simulations to prior knowledge and results from billiard maps. These objects will be discussed next.

3 Caustics and Extremal Rays

A number of geometric constructions help explain the behavior of wave fronts. In particular, the relationship of the caustic of families of rays has long been known to be the trace of cusps of the wave front. We discuss prior work by Holditch, and refine the definition of a caustic to account for the discrepancy in ray order arising for wave fronts.

It turns out that many properties of wave fronts and the caustic curves can be more conveniently explained using certain exceptional rays. We will call these rays extremal, for they correspond to minima and maxima of possible reflection angles of a ray bundle.

First we introduce various notions of caustics.

3.1 Holditch's Ray Caustic and Wave front Caustic

A caustic can be defined in various ways. For our purposes, the definitions with respect to sets of rays and wave fronts are relevant [2].

Definition 3.1 (compare [5]) The *caustic of a ray system* is the envelope of a set of rays, i.e. if R_t is the set of rays parametrized by t, i.e. any one-parameter family of lines $R_t = f(x(t))$, $x(t) \in \mathbb{R}^n$. In our case the caustic is the set of points in the plane formed by the envelope, or $C_R = \{x(t) \in \mathbb{R}^2 : \partial R_t/\partial t = 0\}$. If R_t is generated by an n-th order billiard map we call it the *ray caustic by n-th order reflection.*

Definition 3.2 (compare [11]) The *caustic of a wave front* is the locus of critical points traced out by a wave front. If the wave front is generated by iterated billiard maps, we call it the *caustic of a wave front by reflections.*

Holditch derived the n-th order caustic of reflection of rays on a disk billiard [9]. This is rarely-cited work, even though it contains many interesting and relevant results not found elsewhere. We will hence call this type of caustic the *Holditch caustic.*

Theorem 3.3 *(Holditch [9]) The ray caustic of the n-th order billiard map is a curve $f : [0, 2\pi) \mapsto \mathbb{R}^2$ with*

$$x = -(-1)^n a \sin\theta \cdot \left(sinP + \frac{\cos P}{2n \tan\alpha - \tan\theta} \right) \tag{1}$$

$$y = (-1)^n a \sin\theta \cdot \left(\frac{sinP}{2n \tan\alpha - \tan\theta} + \cos P \right) \tag{2}$$

with $\theta \in [0, 2\pi)$. a is the distance $d(0, q_0)$ from the ray bundle origin q_0 to the center of the disk. Furthermore, we have auxiliary equations $\sin\alpha = \frac{a\sin\theta}{b}$, where b is the radius of the disk domain, and $P = 2n\alpha - \theta$. The x coordinate coincides with the line of symmetry of the ray bundle.

Many interesting properties of this caustic are already derived by Holditch. These include the asymptotes, position of cusps and the non-existence of inflections on the curve away from the cusps. He also shows that all cusps are indeed semi-cubical [9]. Holditch's paper is remarkable also for it is the only source I am aware of which fully treats the ray caustic by reflection for all orders and all positions of an initial ray bundle. Most sources only deal with first order reflections [7,6,5]. Ucke and Engelhardt rederive the special case of the bundle at infinity, which corresponds to parallel rays [12]. All of these cases are included in Holditch's treatment.

Here we will need one result, which is the position of the cusps away from the line of symmetry of the ray bundle. These cusps are, using our definitions, placed at the middle point of the maximal ray between two reflections for all orders. We will see that this corresponds to the point where the maximal ray touches its "circle caustic". Experimentally we observe, that this property persists even with the modified definition of a wave front caustic, to be defined below. A proof of this is difficult due to the current lack of a closed expression of this caustic.

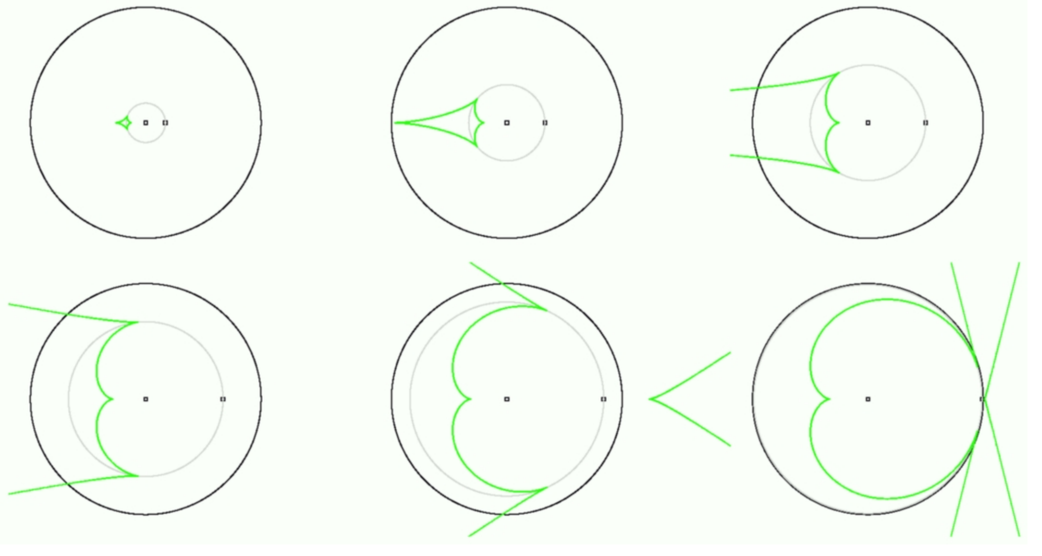

Fig. 2. The figures show, left to right, top to bottom, the first order Holditch caustic (green) for ray bundles on a unit disk at 0.17, 0.33, 0.5, 0.67, 0.84 and 0.99.

A number of instances of the rather well-known first order caustic of ray systems using Holditch's result can be seen in figure 2. The Holditch caustic vanishes at the center of the disk if the position of the ray bundle coincides with the center of the disk. In this case wave fronts under reflection do not form cusps but collapse once per reflection to a point at the origin. Hence one can say that the ray caustic and the critical set of wave front are the same for all orders of reflection. This is a highly exceptional situation [3].

It has been noted by Bruce, Giblin and Gibson that ray caustics by reflection may differ from experimental observation due to additional reflections [6]. However they don't develop the problem or give explicit examples.

In our case, the intuition can be rather simply stated. Generically, rays have different lengths. Given equal distance, some will reflect more frequently than others depending on the length of the segments formed by rays between reflections at the disk boundary. This creates a discrepancy in the order of reflection between different rays of one ray bundle. The correct relationship between orders of these rays has to be picked to properly define the caustic. The Holditch caustic is thus not the proper caustic for wave fronts as it calculates envelopes of rays of the same order of reflection. To develop a proper definition we will need the following

Proposition 3.4 *The order of a wave front with a ray distance l (see definition 1.7) and a line segment length l_α generated by the ray between to billiard map points* *is*

$$n = \begin{cases} 0 & \textit{if } l < d(q_0, p_0), \\ 1 & \textit{if } l < d(q_0, p_0) + d(p1, q_1), \\ 1 + \frac{l-(d(q_0,p_0)+d(p1,q_1))}{l_\alpha} & \textit{otherwise.} \end{cases} \tag{3}$$

This is an immediate consequence of definition 1.7.

Theorem 3.5 *A wave front caustic by reflections differs from n-th order ray caustics for sufficiently large n for all ray bundles except the one at the origin of the disk domain.*

Proof (Sketch). Case (a) Assume the ray bundle at the origin of the disk domain. The angle of reflection is the same for all rays of the bundle and hence the segment generated by the ray is equal for all rays and so is its wave front order. Case (b) Assume the ray bundle away from the origin inside the disk domain. By definition 1.7 the equidistance on a ray l of a wave front grows linearly with the length of the ray between two reflections. By definition 3.4 the order of the ray for this distance remains linear. As two distances between rays are different by our assumption, the difference in length between reflections will accumulate linearly and hence for sufficiently large distance will exceed the diameter of the disk, which is the maximum length for two rays to stay within the same order of reflection. □

We immediately get the correct definition of a wave front caustic:

Definition 3.6 The *wave front caustic* is the envelope of the family of rays supporting a wave front generated by a ray bundle on a domain.

Hence the correct order of reflection of rays is defined by the equidistance. This definition generalizes for other domain shapes and higher dimensions. In our case results from proposition 3.4 apply.

The wave front caustic is only a piecewise connected curve and hence doesn't easily allow a closed form expression like Holditch's caustic. For low orders the Holditch caustic and the wave front caustic usually coincide. They start to differ only as the distance of the wave front increases.

Due to the lack of a closed form solution, we calculate a discrete approximation of the wave front caustic by finding intersections between neighboring rays in the ray bundle. Hence the depiction of the wave front caustic as seen in some figures is only an approximate trace, as the ray bundle is discrete.

Finally, we define another object which correctly traces the order of the wave front. It is motivated by the observation that every ray has a unique circle caustic, a well-studied object of circle billiards [10]. Each ray supporting a wave front has a unique point touching its circle caustic.

Definition 3.7 (compare [10]) The *circle caustic* of a ray is the circle sharing its center with the center of the disk touching the ray at one point. Hence the ray is tangent to its circle caustic.

Definition 3.8 The *circle caustic set of a wave front* is the set of all points of rays supporting the wave front which touch the rays' circle caustics. This is the mid-point of each ray between two reflections.

This is of course not a caustic, as it does not correspond to the trace of wave front cusps or focal sets of rays.

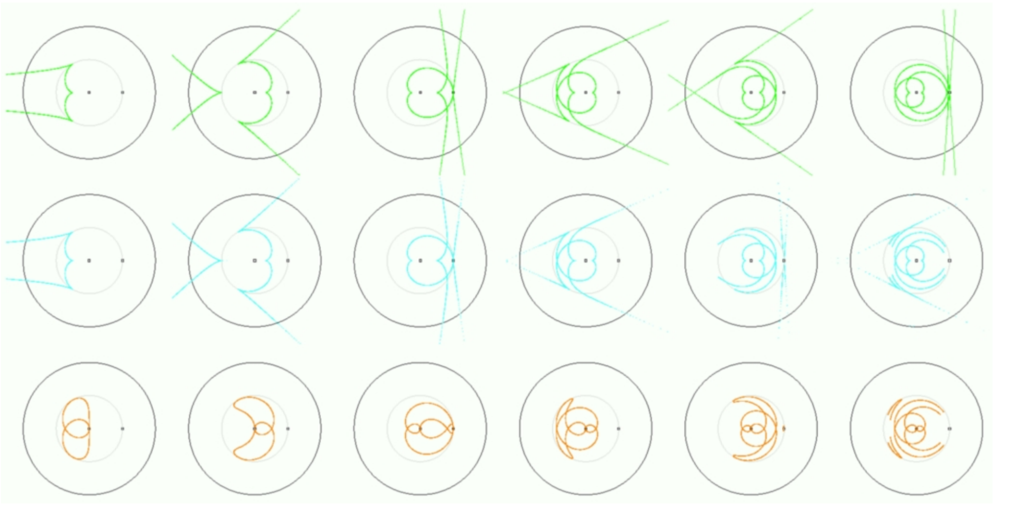

Fig. 3. The first six orders of a ray bundle at located at 0.5 of the unit disk D. The top row shows the Holditch caustic, the middle row the wave front caustic and the bottom row the circle caustic set.

A comparison of Holditch caustic, wave front caustic and circle caustic set for the same ray bundle can be seen in figure 3. Observe how for higher orders the Holditch caustic differs from the wave front caustic. Both the wave front caustic and the circle caustic set become disconnected at points of differing ray order of the wave front set.

3.2 Extremal Rays

Next we introduce the notion of extremal rays. Extremal rays have a number of interesting properties. They will be of interest with respect to properties of rays of a ray bundle and with respect to positions of cusps of the wave front caustic and cusp formation and annihilation on wave fronts.

Definition 3.9 A ray is called *minimal* if it is supported on the line of symmetry of the ray bundle.

Definition 3.10 A ray is called *maximal* if it is an element of the ray bundle that is normal to the line of symmetry of the ray bundle.

We will call either *extremal.* This attribution is justified by the angle of reflection of the billiard rays.

Lemma 3.11 *Maximal rays generate the largest angle of reflection of the ray bundle.*

Proof. The maximal ray is by definition tangent to its circle caustic. As all rays other rays of a chosen ray bundle containing this maximal ray have a different angle with respect the line of symmetry of the ray bundle, those rays will intersect the circle caustic of the maximal ray and hence have a circle caustic of small radius. We observe that the bisector angle between tangents to this circle caustic and the radius of the disk D is smaller than the same of the maximal ray. □

Lemma 3.12 *Minimal rays generate the smallest angle of reflection of all rays of the ray bundle. This angle is identical to* 0*.*

The proof of this lemma is trivial given that we define the angle of reflection on the interval $[0, \pi)$.

Lemma 3.13 *There are two minimal rays per dense ray bundle over* $[0, 2\pi)$ *with respect to the line of symmetry of the ray bundle. These are at degree* 0 *and* π*.*

The following properties of the circle caustic set are immediate from properties of the minimal and maximal ray.

Proposition 3.14 *The circle caustic set of a wave front always contains (a) two points at the origin, one from each minimal rays, and (b) the cusp points of the wave front caustic.*

The mid-point of the minimal ray, and hence the touching point with its vanishing circle caustic, is the center of the domain. The mid-points of maximal rays touching its circle caustic is, by Holditch's result, the cusp point.

4 Experimental Results

We now discuss properties of wave fronts as observed in the numerical simulation. In a few select cases we will also sketch proofs of these properties. The main theoretical result shows the non-existance of so-called dangerous self-tangencies, as introduced by Arnold, for the disk domain.

4.1 Wave Front Eversion Properties

If the wave front is considered an oriented set, then eversion is the process of changing the orientation of the set via front evolution (compare Arnold [3]).

We classify two types of eversions from experimental observation:

(i) Complete Eversion: (a) If the first order (wave front and ray) caustic is completely contained within the disk D. (b) If the wave front caustic intersects the boundary of D but has disconnected components. Alternatively this means that two parts of the wave front caustic with four sets of asymptotes do not self-intersect.

(ii) Incomplete Eversions: New cusps form before old cusps are canceled. Hence the number of cusps persists or increases. Open question: What is the relationship between wave front caustic self-intersection and the number of persistent cusps?

We conjecture that completion of eversion is only possible when the caustic does not self-intersect.

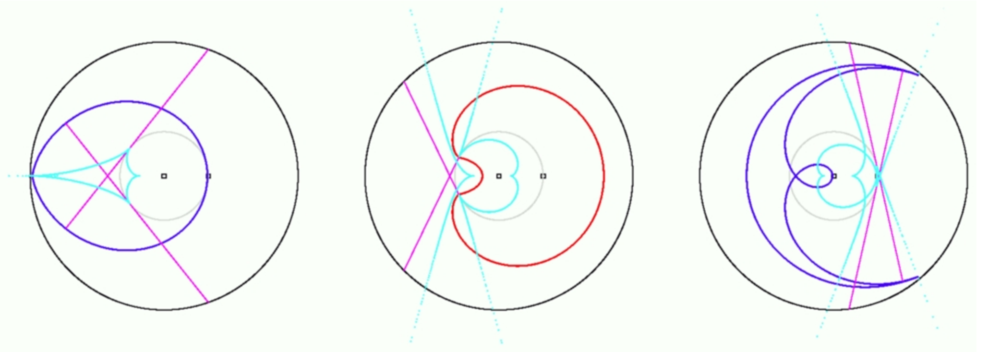

Fig. 4. The figures show a complete eversion of the first order wave front of a ray bundle at 0.33, the fourth order complete eversion of the same and an incomplete eversion at when the order has reached five. The front is red (blue) if the order of the front is even (odd). Note that the wave front caustic (light blue) self-intersects when the eversion is incomplete. The maximal rays (purple) are also shown.

Two complete eversions, one without reflection, one with reflections, as well as an incomplete eversion are shown in figure 4. The observation of the difference between the latter two instances justifies the conjecture of self-intersecting caustics as condition for incomplete eversions. If the caustic self-intersects, new cusps form before old ones can completely annihilate, hence creating the persistence. Details of cusp formation and annihilation will be discussed next.

4.2 Cusp formation, cusp annihilation and cusp motion

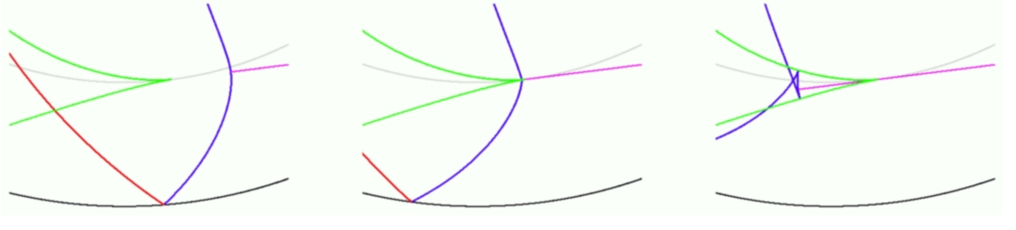

Fig. 5. The figures show the cusp formation on the wave front (blue) of first order rays of a ray bundle at 0.75. The maximal ray (purple), its circle caustic (gray) and a cusp of the Holditch caustic of the first order (green) coincide. The first front (red) not yet reflected at the boundary of the disk (black) is also visible.

Cusp points on the wave front will trace out the wave front caustic. There are three situations:

(i) Cusp formation: Cusps are formed at the intersection of the maximal ray, the circle caustic of the maximal ray, the wave front's circle caustic set and the cusp of the wave front caustic. If the order of reflection of the maximal ray is chosen, this coincides with the cusp of the Holditch caustic. Two examples are depicted in figures 5 and 6. Cusp formation happens once per order of the maximal ray.

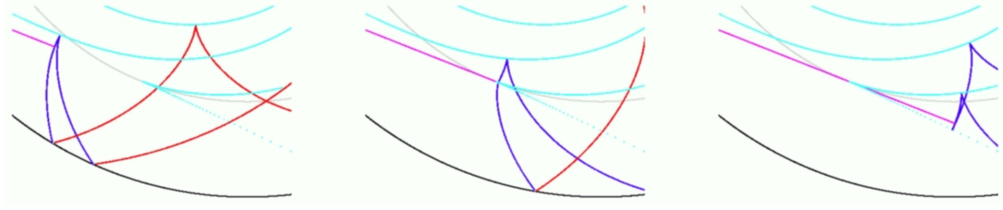

Fig. 6. The figures show the cusp formation on the wave front (blue) of third order rays of a ray bundle at 0.75. The maximal ray (purple), its circle caustic (gray) and a cusp of the wave front caustic of the first order (light blue) coincide. The first front (red) not yet reflected at the boundary of the disk (black) is also visible.

(ii) Cusp annihilation: A pair of cusps cancel at cusps of the wave front caustic which lie on the minimal ray trajectory. See figure 7. We observe that cusp annihilations occur exactly once per order on the minimal ray.

(iii) Cusp motion: Cusps will move along smooth sections of the wave front caustic.

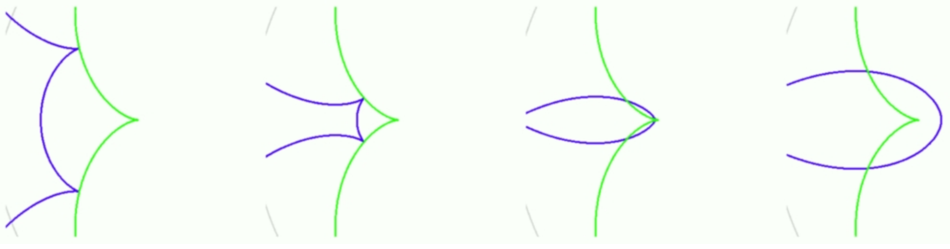

Fig. 7. The figures show the cusp annihilation of the wave front (blue) of first order rays of a ray bundle at 0.55. The cusps trace out the Holditch caustic of the first order (green) and cancel at the caustic's cusp. Part of the caustic circle (gray) is also visible.

As we observe only these three specific scenarios, we conclude, that the wave front caustic is smooth and free of inflections everywhere, except at two symmetric positions on the maximal ray, and one or two positions on the minimal ray. Holditch has already shown that two consecutive orders of his caustic are connected at the intersection points of the caustic curve with the domain boundary ∂D [9]. This still holds as can be seen in figure 8 which also shows the reflection of the associated cusp on the wave front.

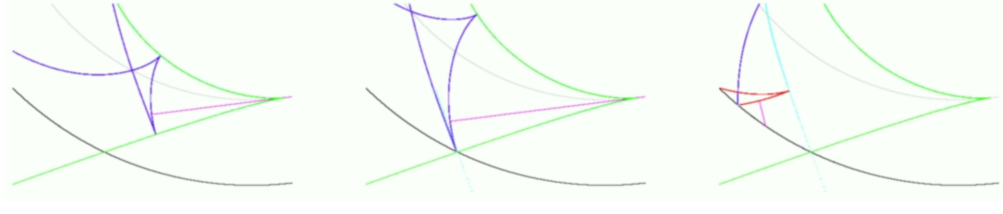

Fig. 8. The reflection of a cusp at the boundary of the disk. The cusp on the unreflected wave front (blue) traces out the Holditch caustic (green) and reflects at the boundary. The wave front caustic (light blue) shows the trace of the cusp of reflected wave front (red). The maximal ray (purple) and its circle caustic (gray) can also be seen.

Proposition 4.1 *The number of persistent cusps is exactly twice (due to symmetry) the difference in the order of the minimum and the maximum ray)*

It is worthwhile mentioning that smooth branches of the wave front caustic will self-intersect. This happens in two ways. First, due to the symmetry of the dynamics along the line of the minimal ray. Smooth paths will cross a symmetric copy of the same equidistance when crossing the locus of the trajectory of the minimal ray front. Hence we get a non-generic situation of cusps coinciding. Yet these are distinct cusps which can easily be resolved by lifting the curve by reasonably chosen parameter, for example, an ordered labeling of the supporting rays, or a curve parametrization. Additionally other intersections become possible at higher orders, which will not lie on the minimal ray locus.

4.3 Self-tangencies of Wave Fronts

A self-tangency of two bounded segments of the wave front occurs when these segments touch and their tangents locally coincide, up to sign. If the direction of the tangents both fronts have the same sign then these are called *dangerous self-tangencies* [3].

Lemma 4.2 *The supporting ray of a wave front locus is unique up to ray orientation. A ray supports one front of equal orientation for extremal rays and two fronts otherwise, which have a distinct locus.*

By this lemma, if rays of the same billiard map coincide locally for some equidistance the orientation of these rays will be opposite. The core of the proof is then to show that rays of equal orientation will not have the same locus on the domain.

Proof (Sketch). Case (a): Rays are extremal. The minimal ray coincides with the line of symmetry of the ray bundle, hence there are two possible unique rays with opposite orientation. The maximal ray is tangent to the line of symmetry of the ray bundle. It hence also has only two possible rays with opposite orientation. Case (b): Rays are not extremal. Any other angle of reflection occurs four times for a ray bundle, one in each quadrant formed by the maximal rays and the minimal rays centered at the ray bundle position. Two pairs of rays will have the same orientation, yet different distances and hence will not coincide. □

Lemma 4.3 *The disk domain does not allow dangerous self-tangencies.*

Proof. With the previous lemma it is sufficient to note that wave fronts are supported on rays, a dangerous self-tangency implies coinciding rays. □

The following result links self-intersections of the circle caustic set of a wave front to points of self-tangency.

Proposition 4.4 *Self-intersection points of the circle caustic set on the line of symmetry of the ray bundle which are not at the center of origin correspond to points of self-tangency of the wave front.*

The dual self-tangency occurs on the circle caustic of the maximal ray. It is simply the point of equal distance of the minimal ray reflecting in opposite directions from the position of the ray bundle under the billiard map. For odd order, this is opposite the ray bundle position. For even order it is on the ray bundle position.

Observation The normal through the self-intersection point of the circle caustic set of a self-tangency forms a double tangent with the wave front caustic under the conditions of matching order of supporting rays. □

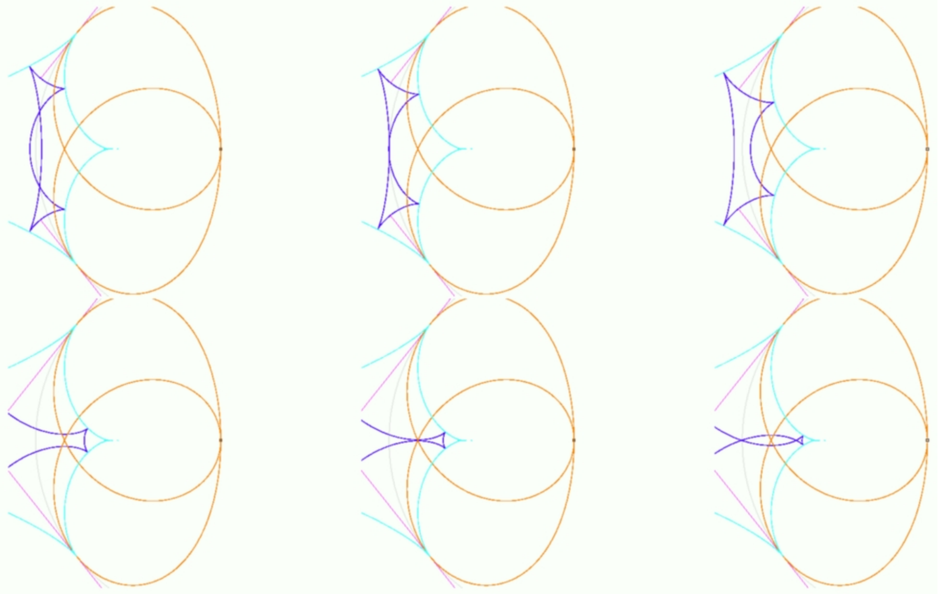

Fig. 9. Example of the first order wave front (blue) eversion of a ray bundle at 0.33. The equidistance of the wave front increases from left to right. The top row shows the self-tangent crossing at the equal-distant point of the minimal ray on the circle caustic (gray) of the maximal ray (purple). The bottom row shows the self-tangent crossing at the self-intersection of the circle caustic set (orange). The wave front caustic (light blue) can also be seen.

All these can be observed in figure 9. We also observe a minimum of two self-tangencies per complete inversion.

Self-tangencies appear as a singular moment of resolution (also called perestroika) of regular self-intersection of wave fronts of the same order [3].

Finally we note the observation of a rather remarkable and more general relationship between self-intersections of wave fronts segments of the same order. We get the following result:

Observation. Self-intersections away from line of symmetry of the ray bundle following the creation of a pair of wave front cusps lie on the circle caustic of the maximal ray of this bundle. This continues until the regular intersection turns into a self-tangency followed by the disconnection of the self-intersection. This self-tangency is exactly the self-tangency on the minimal ray. □

In addition other self-intersections of wave fronts of the same order are also observed. Specifically, the symmetry of all constructions leads to self-intersections on the line of symmetry of the ray bundle. Additionally we observe self-intersections away from this line and away from the circle caustic of the maximal ray. It is still open how this can be explained or if these cases constitute a full classification of self-intersection of equal order wave fronts. Self-intersections of mixed order are observed in abundance.

4.4 Wave Front Folds at the Boundary

When the wave front comes in contact with the domain boundary, segments of the wave front already reflected and segments not yet reflected from fold. These folds propagate along the boundary as the wave front propagates and reflects along it. A pair of fold is created when yet unreflected segment of a wave front reaches the boundary as depicted in figure 10. The same behavior can also be seen when cusps reflect, forming two pairs of symmetric folds (figure 8).

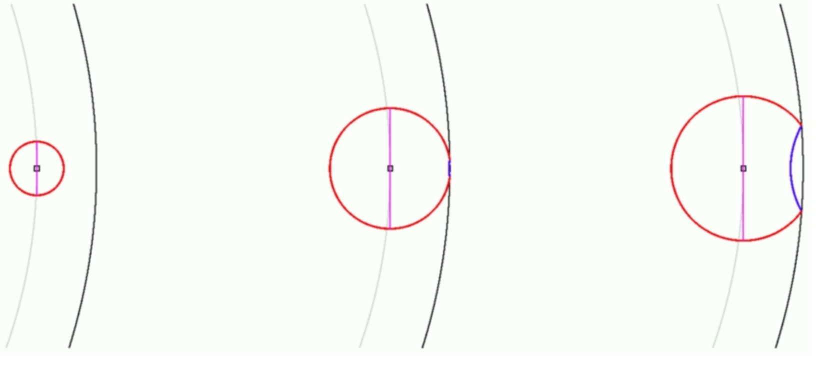

Fig. 10. Wave front fold creation for a ray bundle at 0.9.

We observe that the annihilation of a symmetric pair of folds only happens at the line of symmetry of the ray bundle. An example is depicted in figure 11. It should be pointed out that reaching this line is only necessary but not sufficient for folds to annihilate. We observe that folds created by regular wave front segments are annihilated as one pair per order of the wave front, whereas folds created by cusps are annihilated as two pairs per order of the wave front.

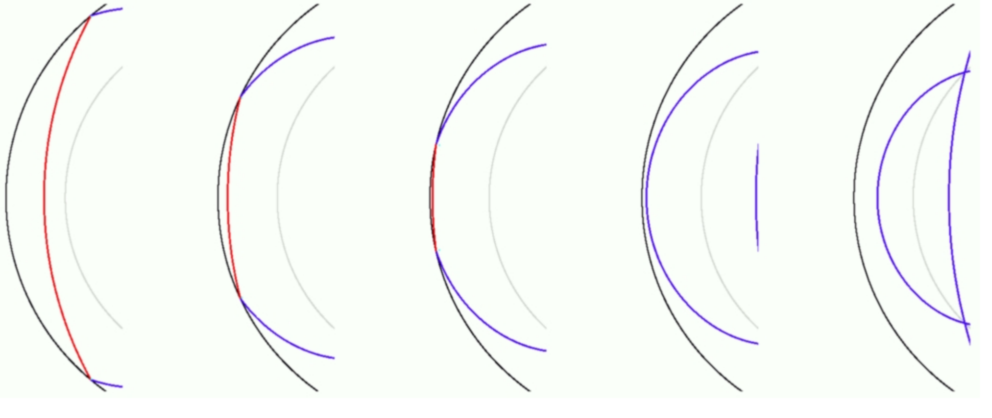

Fig. 11. Wave front fold annihilation for a ray bundle at 0.75.

4.5 Whispering Gallery Wave front

It's long been observed that sources close to curved walls in buildings can be heard clearly over larger distances for recipients close to the same wall than otherwise. This phenomenon is called *whispering gallery* and was studied by Rayleigh, Airy, Sabine, Raman and others. Bate offers a discussion of the observations up until the late 1930s [4]. The phenomena has since also been studied for electromagnetic and optical cavities.

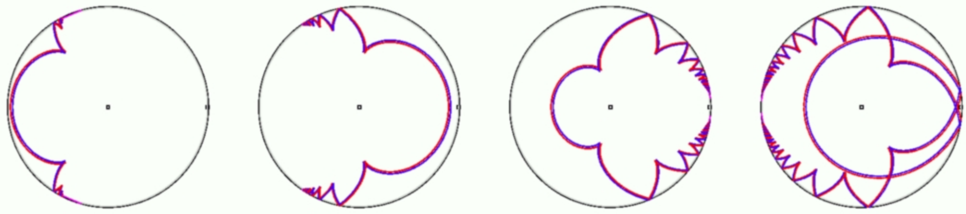

Fig. 12. Second, third, forth and fifth order wave fronts (red, blue) of a ray bundle at .99.

The formation of the early orders of wave fronts of a source close to the boundary of the disk can be seen in figure 12. Observe that the wave fronts stay close and form a repeating pattern through reflections. The number of cusps gradually increases due to frequent reflections of the maximal ray. It is particularly interesting to note, that the wave fronts cover the area close to the center more sparsely than closer to the boundary of the disk. This behavior persists even for higher order of reflections and for sources not as close to the boundary. This will be illustrated next.

4.6 High Order Wave Fronts by Reflection

Figure 13 shows the wave fronts for higher orders of reflection for positions very close to the center, at half-radius and close to the boundary. Observe how the center area is less dense than the outer areas for the latter two cases.

5 Conclusions

In this paper we discuss a numerical method and experimental results for wave fronts on flat disk domains. The simulation uses purely geometric methods. All calculations are based on a finite pencil of supporting rays centered at a point of excitation and any further calculation uses only elementary geometric intersections and interval rescaling.

We observe a number of known properties of wave fronts. We also verify the relationship between the caustic of ray geometric optics as derived by Holditch in the 19th century and the evolution of cusp singularities for low orders. Based on observations we form conjectures and sketch proofs for the condition of the formation and annihilation of cusps, on properties of wave front eversion and on self-intersections and self-tangencies.

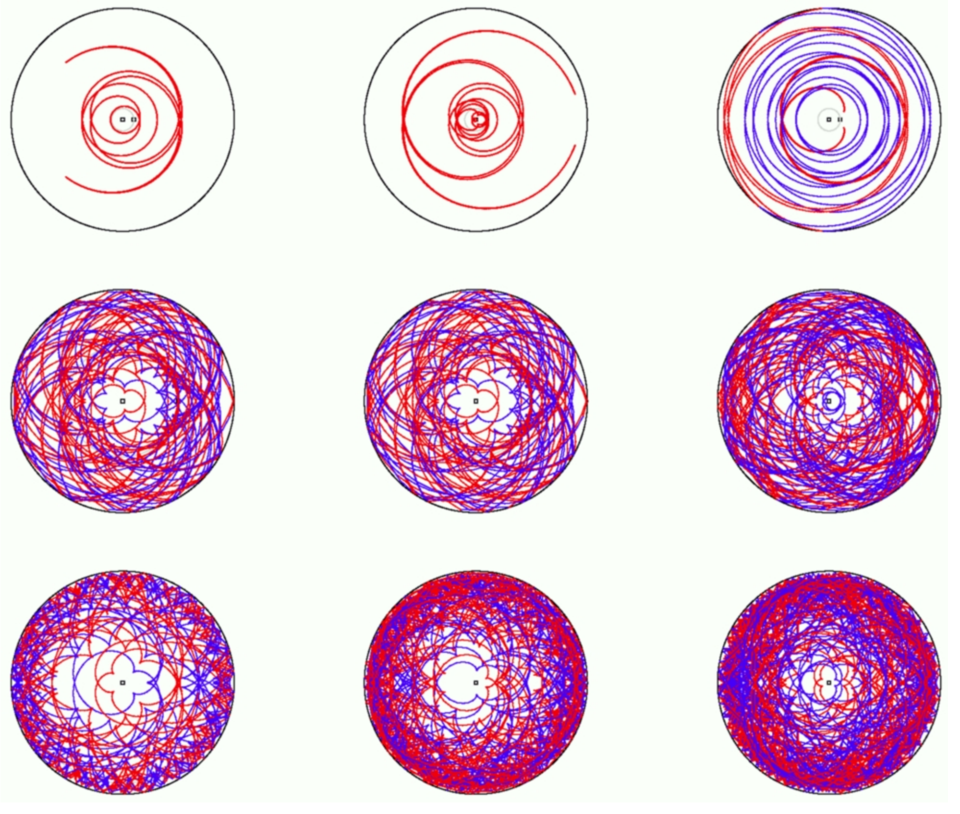

Fig. 13. Higher order wave fronts. Top row: Ray bundle at 0.1. Middle row: Ray bundle at 0.5. Bottom row: Ray bundle at 0.9. The front is red (blue) if the order of the front is even (odd). From left to right, the equidistances are 100, 200 and 300.

A companion paper is planned giving more detailed theoretical discussions which could not be provided here given space limitations. It also should include relating this work to the theory of Legendrian links and knots and contact geometry.

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