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The Computational Structure of Waves on Drums for Sound Synthesis

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We discuss the geometric content of the wave equation in two dimensions and their implications for numerical methods. Wavefronts and wakes are natural parts of the evolution of point disturbances and yield a rich geometric structure. We discuss how this structure is preserved or destroyed. In particular wavefronts form sharp edged solutions called cusps under repeated reflection. We illustrate the formation and evolution of these cusps using a basic ray-based method and discuss the implication of these cusps for meshed methods. Finally we give an outlook for future algorithms with respect to preservation of geometric structure of the wave equation in two dimensions on arbitrary closed domains as well as some sound examples rendered with the basic ray method.

1 Introduction

This work is part of a larger program trying understand if we can find efficient computational structure in the 2-dimensional wave-equation[1]. A number of related idea within this program are published or in consideration for publication elsewhere [7, 11, 10] and discuss geometric features of wave fronts rendered with the algorithm to used here as well. In [7] I described the basic algorithm and related the results of the algorithm to known features of ray dynamics on a circular domain. In [11] the domain was altered to an ellipse and a stadium (two semi-circles connected with straight lines) to give an argument relating the observed wavefronts to the whispering gallery phenomenon. In [10] the geometric content is used for a first synthesis of sound using this method using a basic dissipation model.

The purpose of this paper is to give more detail of the implication of the chosen method compared to fixed grids and give some indication about wakes in the solution and the relationship to the present method. Hence my focus here will not primarily be on the synthesis aspect of the problem, but on the numerical part of is, yet this immediately relates to practical aspects of synthesis.

We have a good and rather complete understanding of the 1-dimensional case. In particular we have an easily understandable temporal dynamical behavior in form solution space of the wave equation, when written as traveling waves (d'Alembert's original solution). This form allows for a computational structure, called Digital Waveguides [19] to be utilized that is, in a sense, better than a naive estimation of a meshed method to integrate the wave equation. By requiring only constant, rather than order of spatial samples, waveguides are optimal with respect to spatial integration. The second main advantage is that the traveling waves are not, or only mildly disturbed by propagation. This allows for copy-propagation rather than arithmetic propagation. Hence rounding errors are avoided or kept limited to few operations thus giving waveguides highly desirable numerical properties as well. At the same time waveguides are exact with respect to the continuous equations on sampling points.

We are still far from having the same properties for the 2-dimensional case. A number of methods have been employed to simulate drums. These are all predominantly meshed methods. These include spring-based methods [5], standard finite differencing methods, and waveguide meshes [22, 13] and hybrids between waveguide and finite differencing methods [4, 16]. All these share that the dynamics is discretized on mesh points and different directions of propagation are coupled at these mesh points. The difference either consists in the motivation or implementation. The mesh itself is responsible for inexactness in the solution known as mesh dispersion and various attempts have been proposed to reduce this effect for various situations [17, 14].

Another approach is using functional transforms[21]. This is really a modal methods, in which the modes-strengths are calculated from the equation once transform functions have been picked. As these transform functions usually constitute a support that ranges over the whole domain of solution, local aspects of the problem become approximated by truncated infinite series of these transform functions.

Banded waveguides have also been proposed for 2-dimensional structures such as drums and cymbals [12]. This is a modal method which tries to retain some notion of spatial information. However this information is asymptotic and hence incomplete. Secondly, it is very difficult to recover the meaningful spatial positions of various waveguide bands crossing. Recently a related method, which employs details about the circular symmetry have been proposed [23] as well as hybrids with waveguide meshes[18].

[1] Throughout this paper dimensions refer to spatial dimensions of the domain. The additional temporal dimension is implied.

2 Structure of the Wave Equation

The wave equation in two dimensions without external forces and dissipation reads:

$$\frac{\partial^2 y}{\partial t^2} - c^2 \left(\frac{\partial^2 y}{\partial x_1{}^2} + \frac{\partial^2 y}{\partial x_2{}^2} \right) = 0 \quad (1)$$

The initial conditions are $f(\cdot)$ for initial displacement and $g(\cdot)$ for initial velocity.

The fundamental solution of the wave equation in the plane without boundaries is [15, 6]:

$$f(x,t) = \delta(x,t) \quad (2)$$

$$y(x,t) = \frac{H(\pm ct - |x|)ct}{2\pi\sqrt{|x|}((ct)^2 - |x|^2)^{\frac{3}{2}}} \quad (3)$$

$$g(x,t) = \delta(x,t) \quad (4)$$

$$y(x,t) = \frac{H(\pm ct - |x|)}{2\pi c\sqrt{(ct)^2 - |x|^2}} \quad (5)$$

where $\delta(\cdot)$ is the Dirac-delta distribution, or impulse and $H(\cdot)$ is the Heaviside distribution, or step function.

In both cases we have:

$$y(x,t) = 0 \quad \text{if} \quad |x|^2 \geqq (ct)^2. \quad (6)$$

These equations have two parts to them. One is the support of the solution. This part is encoded in the Heaviside function of equations (2–5) and equation (6). The other is the functional shape created by an impulse through propagation inside this support.

The first part encodes what we will call the *wavefront*. Here we will discuss this part of the problem. The second part will be called the *wake*. Here we will only give a brief discussion about wakes. A full treatment of wakes is still future work. The omission of wakes places this work in the traditional field of geometric optics and acoustics.

2.1 Wavefronts and Rays

Wavefronts are the point of first arrival of a disturbance in response to an impulse (or a convolution of impulses).

The argument of the Heaviside functions in (3) and (5) forms a circle for constant time. The radius of the circle expands linearly with time. Hence the wavefront inhabit a cone with the tip at the point of the impulsive excitation. Any section of this cone will give a wavefront at a certain time, which is a line embedded in the plane. Point of the wavefront propagate along straight lines called *rays*. Hence one can think of rays as the direction of propagation of waves and wavefronts as the points of arrival on them.

The crux of the proposed method is to maintain the support of wavefronts on rays. Hence the position will be accurate, up to numerical accuracy on such rays and there is no numerical dispersion by construction. This method is in addition excitation-point centric, hence solves the problem of banded waveguide-like methods in this respect.

2.2 Wakes

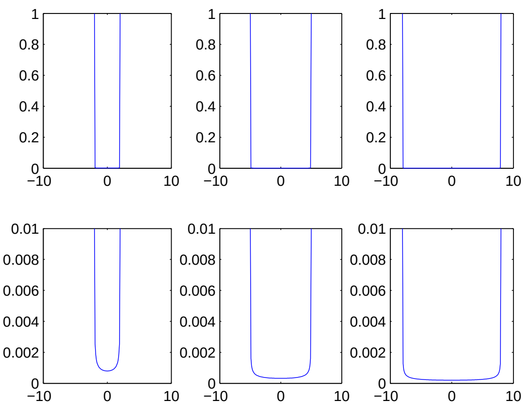

Figure 1: The wake of a radial cross-section of an impulsive velocity excitation at a point at various instances in time aver the excitation. Top: At unit amplitude. Bottom: At scaled amplitude of 0.01.

Wakes are the content of a wave field after the point of first arrival, the wavefront, has passed. These are described by the equations (2) and (4). Detailed discussion about properties of these wakes are not easily found in the literature, while these equations are often derived.

The study of the existence and basic properties of wakes goes back to Petrovsky and has somewhat later been deepened by Atiyah, Bott and Gårding [2, 3]. If a wavefront does not create a wake, they call it a lacuna. It is known since Volterra, that the wave equation in even spatial dimensions creates wakes, whereas in odd spatial dimensions greater or equal three it doesn't. The one-dimensional wave equation constitutes a special case, as a step function is the correct response to velocity excitations hence there is a "wake-like" influence after the impulsive propagation (for a related discussion with respect to 1-D waveguides we refer to [9, 20, 8]).

For example it is noteworthy, that these equations have singular points in their solution. These are at $|x|^2 = (ct)^2$

for both cases, and at $|x| = 0$ for displacement impulses of equation (2).

Another open problem is the proper discretization of these functions. The singularities can in fact easily be made finite by truncating them. At a position away from it. However, it is not immediate, what best to chose as truncation point.

The shape of the wake of a velocity excitation following equation (4) is depicted in Figure 1. The bottom part of the figure shows a greatly magnified view of the curve. The displacement excitation is overall very similar, except that the wake is rapidly pulled towards infinity approaching the origin of the excitation. It is due to this observation of rapid decrease of the wake, that its effect has currently been omitted.

3 Numerical Implications: Rays versus Meshes

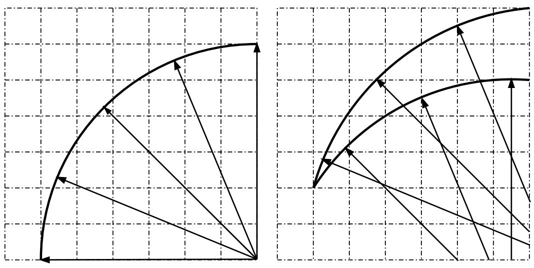

Figure 2: The rays compared to a uniform rectilinear mesh for a circular arc (left) and a cusp (right) (sketch).

There are a number of immediate differences between the rays supported wavefront approach described here and comparable simulations using meshes. For one, meshes only give accurate spatial sampling along the directions of the mesh grids without additional effort. Spatial position is correct up to numerical precision for the ray method. This can be seen on the left in Figure 2.

Additionally, ray methods will be able to resolve spatial information that would be averaged in the meshed case. See the right side of Figure 2. It shows a cusp, which occurs for very frequently for wavefronts under reflection. Close to the cusp point, the lines are close and fall within the same spatial grid point. Hence the solution is really multi-valued with respect to those grid points. A mesh method traditionally does not allow for such multi-valued solutions and hence fails to correctly account for such situations. Such sharp edges are averaged and hence suffer from diffusion.

Finally rays preserve the structure of such singularities as the cusp, even if it is not accurately spatially represented. Cusps form as crossings of neighboring rays. This crossing behavior is preserved in a discrete form in these simulations and hence are cusp evolutions (for detailed discussion of cusp formation and evolution compare [1, 7]).

Figure 3 shows examples of wavefronts rendered with this method. A subset of rays are displayed for illustrative purposes. The locus of intersection of neighboring rays and form so called caustics. Details of caustics rendered with this method can be found in [7].

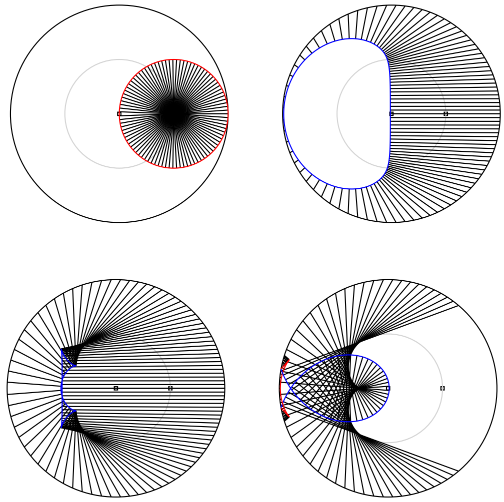

Figure 3: Wavefronts supported on rays rendered with the present method before the first reflection (top left), just the first reflection (top right), forming cusps after the first reflection (bottom left) and after partial completion of the second reflection (bottom).

3.1 Preserved Structure

The ray supported method has a number of properties which are nice with respect to preserving meaningful structures in the simulation despite numerical approximation of the problem. Essentially rays form correct traces of the solution up to accuracy of angle and wavefront position for correct positions on these traces up to numerical accuracy of distance.

In addition rays also encode further information about the solution. The distance of neighboring rays gives a measure of wave front intensity between them which is accurate up to ray angle. Hence the density local bundle of such rays carries immediate information about the energy density. By comparison, in a meshed method the local energy has to be retrieved by inspecting the content of mesh elements exhaustively while at the same time suffering the inaccuracies of dispersive errors of the mesh itself. This advantageous property is conveniently preserved through multivalued solution points in the vicinity of singular points like cusps, which are not immediately

Source	$f_1 : f_0$	$f_2 : f_0$	$f_3 : f_0$	$f_4 : f_0$
Simulation	2.7857	3.6429	4.6429	6.5000
Recording	2.1333	3.7333	4.8667	6.2000
Theory	2.294	3.598	4.903	6.208

Table 1: Comparison of modes between simulation, recording and theory for a strike at the center of the drum.

treated as such in meshed methods.

The ray method is inherently excitation-point dependent. The construction originates at excitation points and traces the solution from that point. This may or may nor be an advantage. If complex, distributed excitation patterns are present, this construction needs to sum over the full excitation, whereas complex excitations are immediately and easily handled by meshed methods. Here the ray supported method shows a difference to 1-dimensional waveguides. In the waveguide case, this problem doesn't occur, because the dimensionality of rays and the domain coincide. Hence different excitation points will always fall on the same "ray" for waveguides. This is a rather pathological exception in 2-dimensions and is one further reason for the increased complexity of this case.

There may be ways around this particular short-coming. The ray construction remains valid under extended excitation shapes, yet recovering these rays then falls into the realm of contact transformations and its utility to find parallel curves[1]. This generalization has yet to be tested in practice.

4 Towards Synthesis

This approach can already be used to render approximate solutions of the wave equation in the plane. The approximate character is introduced by omission of the wakes in the simulation.

This simulation is then assumed to be the source of the radiated field. Comparing the results of this simulation with measurement of a drum struck in the middle and theoretical predictions are found in Table 1 (from [11]). Hence we do get an approximate result compared to theory yet retaining the qualitative nature of the spectrum. Details of this simulation method and its implementation can be found in the forthcoming paper [11].

5 Conclusion

The geometry of wavefronts contains rich information about the solution of the wave equation in two dimensions. By employing a ray based method to support wavefronts, short-comings from meshed methods can be avoided and structure of the solution space is preserved. This way delicate features, such as cusps which form by waves reflecting off the boundary can be simulated without being destroyed by mesh dispersion and the temporal behavior of the solution remains immediately visible.

This approach is still lacking the addition of the wake part of the solution, which is to be included in future work.

References

[1] V. I. Arnold. *Singularities of Caustics and Wave Fronts*. Kluwer Academic Publishers, Dordrecht, Netherlands, 1990.

[2] M. F. Atiyah, R. Bott, and L. Gårding. Lacunas for hyperbolic differential operators with constant coefficients I. *Acta Math.*, 124:109–189, 1970.

[3] M. F. Atiyah, R. Bott, and L. Gårding. Lacunas for hyperbolic differential operators with constant coefficients II. *Acta Math.*, 131:145–206, 1973.

[4] S. D. Bilbao. *Wave and Scattering Methods for the Numerical Integration of Partial Differential Equations*. PhD thesis, Stanford University, May 2001.

[5] C. Cadoz, A. Luciani, and J.-L. Florens. CORDIS-ANIMA: A Modeling and Simulation System for Sound and Image Synthesis - The General Formalism. *Computer Music Journal*, 17(1):19–29, Spring 1993.

[6] Y. V. Egorov, A. I. Komech, and M. A. Shubin. *Elements of the Modern Theory of Partial Differential Equations*. Springer, Berlin, 1999.

[7] G. Essl. Computation of Wave Fronts on a Disk I: Numerical Experiments. Submitted to Electronic Notes in Theoretical Computer Science, Proceedings of MFCSIT'04, September 2004.

[8] G. Essl. Elementary Integration Methods for Velocity Excitations in Displacement Digital Waveguides. Unpublished manuscript. Preprint `http://arxiv.org/abs/physics/0407102`., July 19 2004.

[9] G. Essl. Velocity excitations and impulse responses of strings - Aspects of continuous and discrete models . Unpublished manuscript. Preprint `http://arxiv.org/abs/physics/0401065`., January 13 2004.

[10] G. Essl. Towards the Synthesis of Wavefront Evolution in 2-D. To appear in the Proceedings of the International Computer Music Conference, 2005.

[11] G. Essl. "Whispering" waves and Bate's ridges in a virtual wineglass and elsewhere. To appear in the Acoustics Research Letters Online, 2005.

[12] G. Essl, S. Serafin, P. R. Cook, and J. O. Smith. Theory of Banded Waveguides. *Computer Music Journal*, 28(1):37–50, 2004.

[13] F. Fontana and D. Rocchesso. Physical Modeling of Membranes for Percussion Instruments. *Acustica united with acta acustica*, 84(3):529–542, 1998.

[14] F. Fontana and D. Rocchesso. Signal-Theoretic Characterization of Waveguide Mesh Geometries for Models of Two-Dimensional Wave Propagation in Elastic Media. *IEEE Transactions on Speech and Audio Processing*, 9(2):152–161, 2001.

[15] K. F. Graff. *Wave Motion in Elastic Solids*. Dover, New York, 1991.

[16] M. Karjalainen and C. Erkut. Digital Waveguides Versus Finite Difference Structures: Equivalence and Mixed Modeling. *EURASIP Journal on Applied Signal Processing*, 7:978–989, 2004.

[17] L. Savioja and V. Välimäki. Reducing the Dispersion Error in the Digital Waveguide Mesh Using Interpolation and Frequency Warping Techniques. *IEEE Transactions on Speech and Audio Processing*, 8(2):184–194, 2000.

[18] S. Serafin, P. Huang, and J. O. Smith. The Banded Digital Waveguide Mesh. In *Proc. Workshop on Future Directions of Computer Music (Mosart-01)*, Barcelona, Spain, November 2001.

[19] J. O. Smith. Digital Waveguide Modeling of Musical Instruments. Draft of unpublished online manuscript, available at `http://ccrma-www.stanford.edu/~jos/waveguide/`, 2003.

[20] J. O. Smith. On the Equivalence of the Digital Waveguide and FDTD Finite Difference Schemes . unpublished manuscript. Preprint `http://arxiv.org/abs/physics/0407032`, July 9 2004.

[21] L. Trautmann, S. Petrausch, and R. Rabenstein. Physical Modeling of Drums by Transfer Function Methods. In *Proceedings of the International Conference on Acoustics, Speech & Signal Processing (ICASSP)*, volume 5, pages 3385–3388, Salt Lake City, Utah, May 2001. IEEE.

[22] S. A. Van Duyne and J. O. Smith. The 2-D digital waveguide mesh. In *Proceedings of the IEEE Workshop on Applications of Signal Processing to Audio and Acoustics (WASPAA)*, New Paltz, New York, Oct. 1993. IEEE Press.

[23] S. Zambon and D. Rochesso. Space-time simulations of circular membranes by parallel comb filters. In *Proceedings of Understanding and Creating Music*, Caserta, Italy, December 2004.